Rational Hypergeometric Functions in two Variables

Eduardo Cattani, U. of Massachusetts

Alicia Dickenstein, U. of Buenos Aires

Fernando Rodriguez-Villegas, U. of Texas

Rational A-Hypergeometric Functions

We denote as usual:

$$A = [a_1|a_2|\cdots|a_n] \in \mathbb{Z}^{d\times n}$$

and assume that:

- $\text{span}_{\mathbb{Z}}\{a_1,\ldots,a_n\} = \mathbb{Z}^d$.
- $(1,\ldots,1) \in \text{rowspan}_{\mathbb{Q}}(A)$.

We set $\dim(A) := d-1$; $\text{codim}(A) := n-d$.

Given $\beta \in \mathbb{C}^d$ we define

$$\begin{aligned} H_A(\beta) &= \langle \partial^u - \partial^v : u,v \in \mathbb{N}^n, A\cdot u = A\cdot v; \\ &\quad \sum_i a_{ij}x_i\partial_i - \beta_j; j = 1,\ldots,d\rangle \\ &\subset \mathbb{C}\langle x_1,\ldots,x_n,\partial_1,\ldots,\partial_n\rangle. \end{aligned}$$

We are interested in *rational hypergeometric functions* $F(x)$, i.e. rational functions with:

$$H_A(\beta)\cdot F = 0.$$

Definition: A rational function is said to be *stable* if it is not annihilated by any partial derivative operator ∂^u.

- Which configurations A admit stable rational hypergeometric functions?
- Characterize all such functions.

Conjecture [CDS]: If A admits a stable RHF, then A is affinely equivalent to an *essential Cayley configuration*, i.e. $d = 2r + 1$ and there exist configurations $A_0, \ldots, A_r \in \mathbb{Z}^r$ such that

$$A = \{e_0\} \times A_0 \cup \cdots \cup \{e_r\} \times A_r \subset \mathbb{Z}^{r+1} \times \mathbb{Z}^r,$$

and $\dim(\sum_{i \in I} A_i) \geq |I|$ for all proper subsets $I \subset \{0, \ldots, r\}$.

It holds if:

$\dim(A) \leq 3$ or if $\operatorname{codim}(A) = 1$ [CDS, 2001].

Some codimension two cases [CD, 2004].

Theorem: The conjecture holds for codimension-two configurations.

Key ingredients of the proof

- We understand hypergeometric Laurent series.

- A diagonal series of a two-variable rational hypergeometric series must be algebraic (Furstenberg, 1967) and is a classical univariate hypergeometric function.

- We are able to transform the diagonal to an algebraic univariate hypergeometric function of the type classified by Beukers and Heckman (1989).

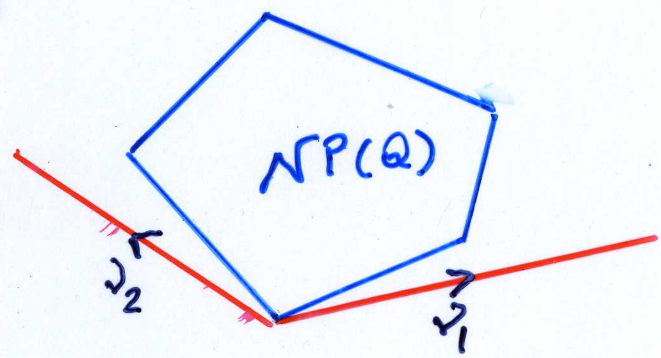
Laurent Expansions

Let $F(x) = P(x)/Q(x)$ be a stable RHF of degree β and let $\operatorname{codim}(A) = 2$. Then $\mathcal{NP}(Q)$ is two-dimensional and

F has a Laurent series expansion supported in a region of the form:

$$\gamma_0 + \mathbb{N} \cdot \nu_1 + \mathbb{N} \cdot \nu_2,$$

where ν_1, ν_2 are a $\mathbb{Z}$-basis of $\ker_{\mathbb{Z}}(A)$.



The series converges in:

$$\mathcal{U} = \{x \in (\mathbb{C}^*)^n : |x^{\nu_1}| < \varepsilon, |x^{\nu_2}| < \varepsilon.\}$$

Hypergeometric Series

Let $M_\beta = \{v \in \mathbb{Z}^n : A \cdot v = \beta\} = v_0 + \ker_{\mathbb{Z}}(A)$.

For $v \in M_\beta$ set: $\mathrm{nsupp}(v) = \{i : v_i < 0\}$ and

$N_v := \{u \in \ker_{\mathbb{Z}}(A) : \mathrm{nsupp}(u+v) = \mathrm{nsupp}(v)\}$.

A subset $I \subset \{1, \ldots, n\}$ is called *minimal* iff:

- $\Sigma(I) := \{v \in M_\beta : \mathrm{nsupp}(v) = I\} \neq \emptyset$.
- For $J \subset I, J \neq I$, $\Sigma(J) = \emptyset$.

For v in a minimal region $\Sigma(I)$ the formal series

$$\Phi_\Sigma(x) := \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} \cdot x^{u+v}$$

is hypergeometric, where $u = u_+ - u_-$, and

$$[a]_w = \prod_{i:w_i \neq 0} \prod_{k=1}^{w_i} (a_i - k + 1), \quad a \in \mathbb{C}^n, w \in \mathbb{N}^n.$$

Gale Duality

Via the $\mathbb{Z}$-basis ν_1, ν_2 we identify $\ker_{\mathbb{Z}}(A) \cong \mathbb{Z}^2$

$$M_\beta \cong v_0 + \mathbb{Z}^2$$

The intersection of the coordinate hyperplanes with $M_\beta \otimes \mathbb{R} \cong \mathbb{R}^2$ defines an oriented line arrangement in $\mathbb{R}^2$ and the minimal regions are cells of the arrangement.

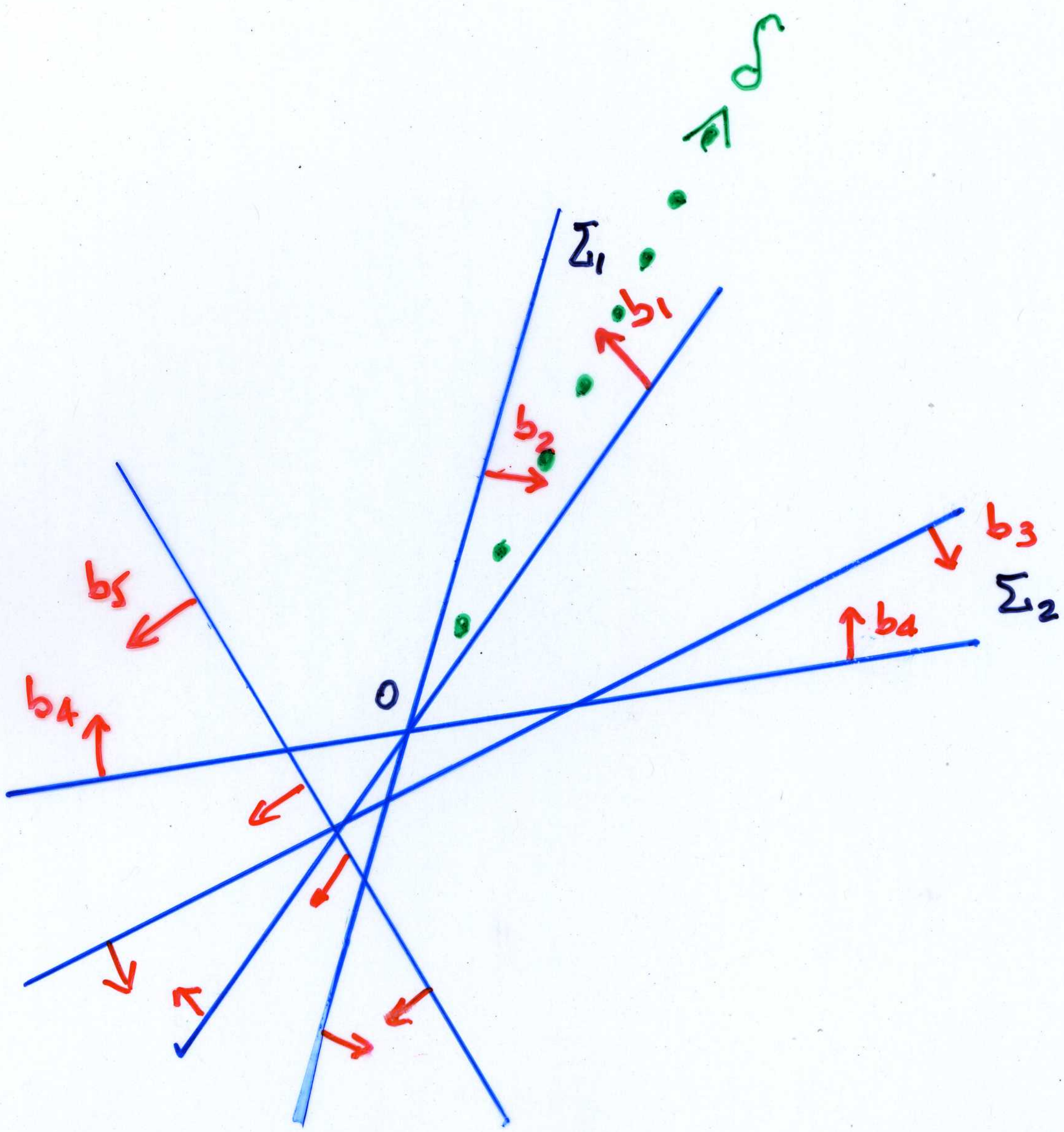
The arrangement consists of lines normal to the vectors

$$b_i := ((\nu_1)_i, (\nu_2)_i)$$

and their position is determined by $\beta \in \mathbb{Z}^n$.

Theorem: For an appropriate choice of v_0, the RHF $F(x)$ is a linear combination $\sum_j \lambda_j \Phi_{\Sigma_j}(x)$, where the minimal regions Σ_j are contained (up to translation) in the first quadrant.

Taking derivatives we may assume that one of the minimal regions is a cone with vertex at the origin.



The diagonal series corresponding to a minimal region Σ_I and to a direction $\delta \in \mathbb{Z}^2$ is of the form:

$$\xi_\delta(t) = \sum_{k=0}^{\infty} \frac{\prod_{i\in I}(p_i k + c_i)!}{\prod_{j\notin I}(q_j k)!} t^k,$$

where $p_i = -\langle b_i, \delta\rangle$, $q_j = \langle b_j, \delta\rangle$, and $c_i \in \mathbb{N}$.

Theorem [Furstenberg (1967)]: $\xi_\delta(t)$, being a diagonal series of a rational bivariate series, is algebraic.

Theorem: The function

$$\hat{\xi}_\delta(t) = \sum_{k=0}^{\infty} \frac{\prod_{i\in I}(p_i k)!}{\prod_{j\notin I}(q_j k)!} t^k, \qquad (1)$$

is also algebraic.

The proof uses the fact that these functions are hypergeometric and monodromy results by Beukers-Heckman (1989) and Levelt (1961).

Observation: Pairs $\{b, -b\}$ are "lost".

Factorial Ratios and Hypergeometric Series

Algebraic functions of the form (1) have been classified (Beukers and Heckman, Rodriguez-Villegas, Bober):

- $p_1 = a + b$, $q_1 = a, q_2 = b$, $\gcd(a, b) = 1$.

- $p_1 = 2a, p_2 = b$; $q_1 = a$, $q_2 = 2b, q_3 = a - b$, $\gcd(a, b) = 1$, $a > b$.

- $p_1 = 2a, p_2 = 2b$; $q_1 = a$, $q_2 = b, q_3 = a + b$, $\gcd(a, b) = 1$.

- Fifty-two exceptional cases, including

$$\sum \frac{(30k)!k!}{(15k)!(10k)!(6k)!} t^k \text{ (Chebyshev).}$$

Punch line

Each direction δ in the minimal region gives rise to a different one-variable, algebraic hypergeometric function. Thus, infinitely many functions of the first three types may appear. However, the second and third types involve a Gale configuration with 5 vectors and this is not possible. Hence, we must have three relevant b-vectors and for infinitely many δ's,

$$\langle b_1, \delta \rangle = -(a+b) = -\langle b_2, \delta \rangle - \langle b_3, \delta \rangle$$

Hence

$$b_1 + b_2 + b_3 = 0$$

This is easily seen to be the Gale dual of a Cayley configuration.