# CONNECTIONS BETWEEN D-MODULES, LOCAL SYSTEMS AND MULTIPLIER IDEALS

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## ① Application (Fermat coverings)

$X$ = proj. nonsing. var. $/ \mathbb{C}$

$D$ = hypersurface

$U = X - D$

$H_1(U, \mathbb{Z}) \twoheadrightarrow H_1(U, \mathbb{Z}/N\cdot\mathbb{Z})$ gives

$$\begin{tikzcd} U_N \arrow[r, hook] \arrow[d, "g_N"'] & X_N \arrow[d] \\ U \arrow[r, hook] & X \end{tikzcd}$$

s.t. $U_N$ = unramified $/ U$

$X_N$ = normal, ramified $/ D$

$\mathbb{Z}/N$ - Galois coverings.

__THM__:

(a) $b_i(U_N)$ = quasi-poly. in $N$

(Sarnak-Adams);

(b) $h^{0,q}(\widetilde{X_N}) = h^{q,0}(\widetilde{X_N})$ = qpoly

($n=2$: E. Hironaka, Sakuma)

intersection theory of curves

(c) $h^{p,q}(U_N, g_N^* \mathcal{W})$ = qpoly

if $\mathcal{W}$ = unitary local system on $\mathcal{U}$

s.t. $\begin{cases} \mathcal{W} = \mathcal{W}'|_{\mathcal{U}}, \ \mathcal{W}' \text{ on } X \\ \quad \text{or} \\ H_1(X,\mathbb{Z}) = 0 \end{cases}$

__Problem__: Find coefs. of the qpolys.

## ② Local Relation

"multiplier ideals =

= $F^{top}$ on certain D-modules"

$D = \{f=0\}$

$X \xrightarrow{f} \mathbb{A}^1$ regular fc.

**Def:**

(a) $$\mathcal{J}(\alpha\cdot D) := \left\{ h \in \mathcal{O} \,\middle|\, \int \frac{|h|^2}{|f|^{2\alpha}} < \infty \right\}$$

(b) $D_i \leftrightarrow f_i$, $\alpha_i > 0$, $\alpha\cdot D = \sum \alpha_i D_i$

$$\mathcal{J}(\alpha\cdot D) := \left\{ h \in \mathcal{O} \,\middle|\, \int \frac{|h|^2}{\prod |f_i|^{2\alpha_i}} < \infty \right\}.$$

Graph embedding :

$$i_f : X \hookrightarrow X \times \mathbb{A}^1_t$$

$$x \longmapsto (x, f(x))$$

$$i_{f+} \mathcal{O}_X = \mathcal{O}_X[\partial_t] \qquad D_{X\times\mathbb{A}^1}\text{-mod}$$

$$V^\alpha \mathcal{O}_X := \left(V^\alpha i_{f+} \mathcal{O}_X\right) \cap \left(\mathcal{O}_X \otimes 1\right)$$

$$= F^{top}\left(V^\alpha i_{f+} \mathcal{O}_X\right)$$

<u>THM</u> (B.-M. Saito) : $\alpha > 0$

$$\mathcal{J}((\alpha - \varepsilon)\cdot D) = V^\alpha \mathcal{O}_X$$

$0 < \varepsilon \ll 1$ .

## ③ "Global" relation

(a) Why this is a $\mathcal{D}$-modules talk:

$$\mathrm{Perv}(X) \longleftrightarrow M_{reg+hol}(D_X)$$

Simple objects:

$$i_* \, IC_{\overline{V}} \, \mathcal{L}[d] \qquad \text{s.t.}$$

$$\left\{ \begin{array}{l} V = \text{smooth irred. loc. closed} \\ d = \dim V \\ V \overset{j}{\hookrightarrow} \overline{V} \overset{i}{\hookrightarrow} X \\ \mathcal{L} = \text{finite rank irred. local sys. on } V \end{array} \right.$$

Note: $\quad IC_{\overline{V}} \mathcal{L} = j_* \mathcal{L}$

if $\mathcal{L} = $ unitary

(b) Results:

<u>THM</u> (known for $D =$ SNC div.)

$X, U, D = \bigcup_{i \in S} D_i$ as in 1

$$\{\text{unitary loc. sys. rank } 1 \text{ on } U\}$$

$$\|$$

$$\mathrm{Hom}(H_1(U, \mathbb{Z}), S^1)$$

$$\|$$

$$\mathrm{Pic}^\tau(X, D) := \Big\{ (L, \alpha) \in \mathrm{Pic}(X) \times [0,1)^S \;\Big|\; c_1(L) = \sum \alpha_i \cdot [D_i] \in H^2(X, \mathbb{R}) \Big\}$$

<u>THM</u>: Let $\nu \leftrightarrow (L, \alpha)$.

$$\dim F^{top} H^{n-q}(U, \nu) = \dim H^q(X, \omega_X \otimes L \otimes \mathcal{J}(\alpha \cdot D))$$

(e) How to apply it to 1 :

<u>THM</u>: $V_i^q := \{(L,\alpha) \in \mathrm{Pic}^\tau(X, D) \mid h^q(\omega_X \otimes L \otimes \mathcal{J}(\alpha D)) \geq i\}$

$= \bigcup_{\text{finite}} P \times T$

rational convex polytope

torsion translated subtorus of $\mathrm{Pic}^\tau(X)$

<u>1 (b)</u>: $h^{0,q}(\widetilde{X}_N) = \sum_{i \geq 1} \# \underbrace{V_i^{n-q}[N]}_{N\text{-torsion}}$

Boils down to :

$\#[\mathbb{Z}^n \cap N \cdot P] = q\text{poly}$

(E. Ehrhart)