Lyubeznik numbers of projective schemes

Wenliang Zhang

April 6th, 2008

- For any Noetherian commutative ring R and an ideal I of R, one can define a functor Γ_I as

  $\Gamma_I(M) = \{x \in M | I^n x = 0 \text{ for some integer } n\}$

  for any R-module M.

- $H_I^i(M) = \mathcal{R}^i \Gamma_I(M)$.

- also $H_I^i(M) = \varinjlim_n \mathrm{Ext}_R^i(A/I^n, M)$.

- If $(R, \mathfrak{m})$ is a regular local ring containing a field, the following properties are known (Huneke-Sharp; Lyubeznik)

  1. $\mathrm{Ass}_R(H^i_I(R))$ is finite for all i;

  2. the Bass numbers of $H^i_I(R)$ are finite for all i.

  3. $H^i_{\mathfrak{m}}(H^j_I(R))$ are injective.

  Remark: to prove this result in char. 0, one has to use D-module theory!

- If A is a local ring containing a field k and admits a surjection $R \to A$ where $(R, \mathfrak{m})$ is a n-dim regular local ring containing k, then one can define the Lyubeznik numbers

$$\lambda_{i,j}(A) := \dim_k(\mathrm{Ext}^i_R(R/\mathfrak{m}, H^{n-j}_I(R))).$$

- $\lambda_{i,j}(A)$ do NOT depend on the choice of $R \to A$ (Lyubeznik'93).

- If A is a local ring containing a field k, then one can define (due to Lyubeznik'93)

$$\lambda_{i,j}(A) := \lambda_{i,j}(\widehat{A}).$$

- $\lambda_{i,j}(A)$ are finite (cf. 2nd slide).

- $H^i_{\mathfrak{m}}(H^{n-j}_I(R)) \cong \mathsf{E}^{\lambda_{i,j}(A)}$ (due to Lyubeznik)

- By the highest Lyubeznik number, we mean $\lambda_{d,d}(A)$, $d = \dim(A)$.

Let X be a projective scheme over a field k (assume $k = \bar{k}$). Given an embedding $\eta : X \to \mathbb{P}^n_k$, one can write $X = \text{Porj}(k[x_0, \ldots, x_n]/I)$, where I is a homogeneous ideal. Let $A = (k[x_0, \ldots, x_n]/I)_{(x_0,\ldots,x_n)}$. Then one can consider the Lyubeznik numbers of A.

In 2007, it is proven (by myself) that the highest Lyubeznik number of A is a numerical invariant of X, i.e., it depends only on X itself, but NOT on the embedding, which provides supporting evidence to a positive answer to the following question

Question: With notations as above, is it true that all $\lambda_{i,j}(A)$ depend only on X, but not on the embedding?

Why interesting?
Short Answer: connection with topology!
Example (essentially due to Garcia-Lopéz and Sabbah): Let X be a smooth complex projective variety. Then

$$\lambda_{0,j+1}(A) = b_j(X),$$

where $b_j(X)$ is the j-th Betti number of X, and other $\lambda_{i,j}(A)$ can be determined by $\lambda_{0,j}(A)$s.

Remark. If the variety X in the above example is singular, then we can not say anything about those numbers. However, if char$(k) = p > 0$, then we have the following

Main Theorem When char(k) $= p > 0$, each $\lambda_{i,j}(A)$ can only achieve finitely many possible values for all choices of embeddings.

The proof of this result is based on (or inspired by) Lyubeznik's F-module theory.

Before we can outline the proof, let's introduce some notations.

• $R = k[x_0, \ldots, x_n]$, I is the defining ideal of the projective scheme X. Since field extentions do not change Lyubeznik numbers, we assume $k = \bar{k}$. Let

$$\mathcal{M} = \mathrm{Ext}_R^{n+1-i}(\mathrm{Ext}_R^{n+1-j}(R/I, R), R).$$

• Let $\{L_i, \theta_{ij}\}$ be an inverse system of R-modules and assume that L_i are graded and all θ_{ij} are degree-preserving. Then define ${}^*\varprojlim_i L_i$ as follows

$$({}^*\varprojlim_i L_i)_l = \varprojlim_i (L_i)_l$$

$\mathcal{M}$ is a very interesting object.

• $\mathcal{M}$ is naturally graded and its degree-0 piece only depend on X but not on the embedding.

<u>Reason</u>. When $i \geq 2$,

$\mathcal{M}_0 \cong \mathrm{Hom}_k(H^{i-1}(X, \mathcal{E}xt^{n+1-j}(\mathcal{O}_X, \omega_{\mathbb{P}^n})), k)$

where $\mathcal{E}xt^{n+1-j}(\mathcal{O}_X, \omega_{\mathbb{P}^n})$ depends only on X since it is the (-j)-th cohomology sheaf of the dualizing complex on X. The proof of the case $i \leq 1$ is done by considering some exact sequences, which will be skipped here.

• There is a natural action of Frobenius (or a p-linear endomorphism) on $\mathcal{M}$.

$$\begin{aligned}\mathcal{M} &\xrightarrow{\alpha} R^{(1)} \otimes_R \mathcal{M} \\ &\xrightarrow{\beta} \mathrm{Ext}_R^{n+1-i}(\mathrm{Ext}_R^{n+1-j}(R/I^{[p]}, R), R) \\ &\xrightarrow{\gamma} \mathcal{M}\end{aligned}$$

where $\alpha(m) = 1 \otimes m$, β is the natural isomorphism, and γ is induced by $R/I^{[p]} \to R/I$. Then the action of Frobenius $f : \mathcal{M} \to \mathcal{M}$ is defined to be $\gamma \circ \beta \circ \alpha$, noticing that β and γ are R-linear and α is p-linear.

An important feature of f: $\deg(f(m)) = p \deg(m)$, for all homogeneous $m \in \mathcal{M}$.

Once we have such an action of Frobenius on $\mathcal{M}$, we can consider

$$\mathcal{M}_s := \bigcap_e (f^e(\mathcal{M}))$$

called the stable part of $\mathcal{M}$.

Theorem

1. $\mathcal{M}_s \subseteq \mathcal{M}_0$ and is a finite-dimensional k-space.

2. $\dim_k(\mathcal{M}_s) = \lambda_{i,j}(A)$

The first part of the above theorem is fairly easy. To prove the second part, let $\mathcal{N}$ be the R-submodule of $\mathcal{M}$ genearted by $\mathcal{M}_s$, and then prove the following:

1. ${}^*\varprojlim_e F^e(\mathcal{N}) \cong {}^*\varprojlim_e F^e(\mathcal{M})$

2. ${}^*\varprojlim_e F^e(\mathcal{N}) \cong R^{\dim_k(\mathcal{M}_s)}$

3. ${}^*\varprojlim_e F^e(\mathcal{M}) \cong R^{\lambda_{i,j}(A)}$

where F is the Frobenius on R.

From what we have seen, one can notice that actually we are very close to a complete solution. Namely, if we can show that this action of Frobenius restricted to $\mathcal{M}_0$ does not depend on the embedding, then it follows that $\lambda_{i,j}(A)$ do not depend on the embedding. We believe this should be the case and we pose it here as a conjecture:

Conjecture. With notations as above, the action of Frobenius $f : \mathcal{M} \to \mathcal{M}$ restricted to $\mathcal{M}_0$ does not depend on the embedding.