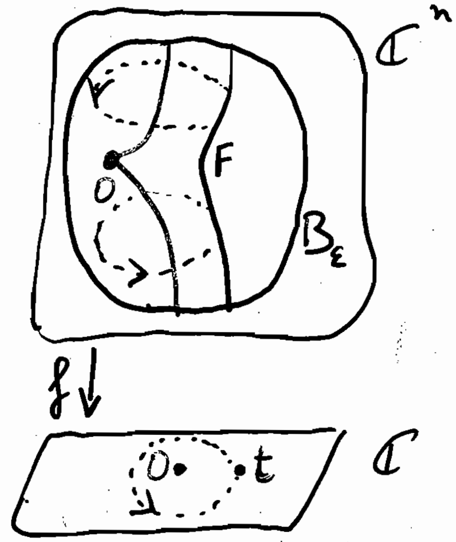
$f \in \mathbb{C}[x_1, \dots, x_n]$, $f(0) = 0$ singularity germ

I three invariants

① Monodromy



Milnor fibre F of f at 0 is $\{f = t\} \cap B_\varepsilon$

$\boxed{H^q(F) \circlearrowleft \text{Monodromy action}}$

} eigenvalues
} Jordan blocks (size $\leq n$)

② b-function = Bernstein-Sato polynomial $\boxed{b(s)}$

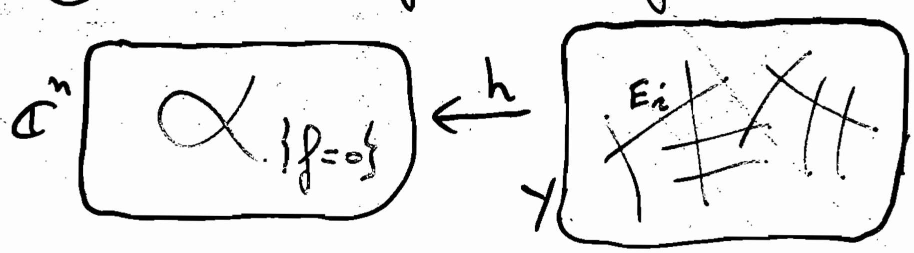
⇝ monic of minimal degree s.t.

$$P \cdot f^{s+1} = b(s) f^s$$

P ↖ differential operator on germ $(\mathbb{C}^n, 0)$

} roots
} their multiplicities ($\leq n$)

③ (local) topological zeta function



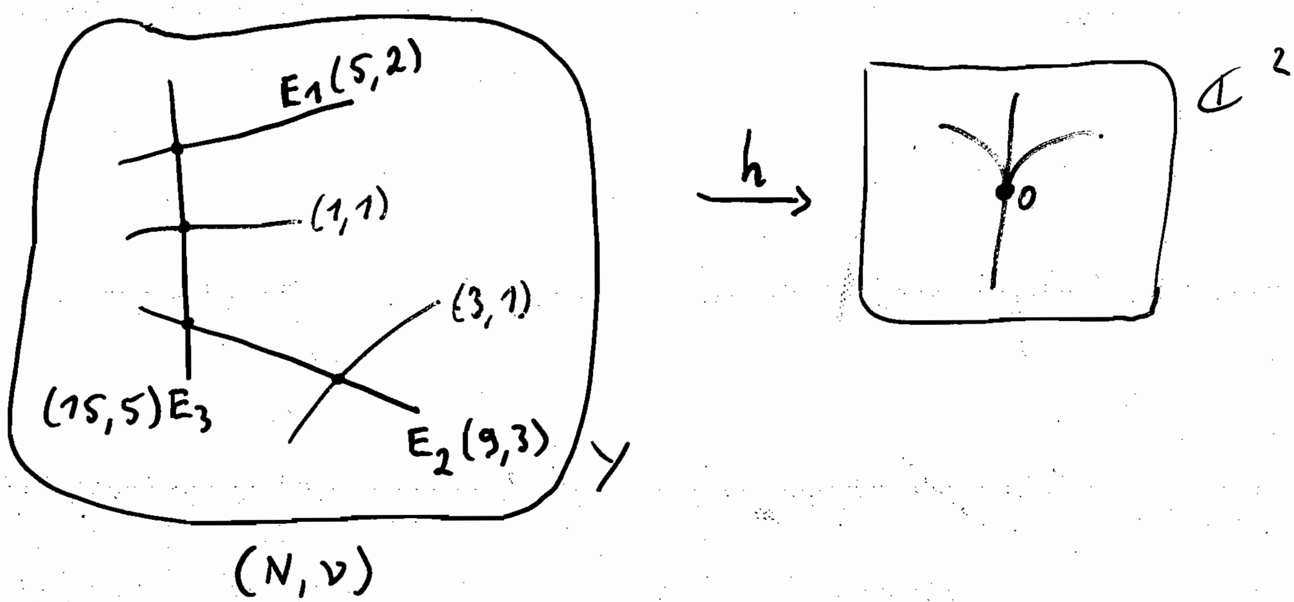
- $h^{-1}\{f=0\} = \bigcup_{i \in T} E_i$
- $E_I^\circ := \left(\bigcap_{i \in I} E_i\right) \setminus \left(\bigcup_{\ell \notin I} E_\ell\right)$ for $I \subset T$

- for $i \in T$: $\begin{cases} N_i \text{ is multiplicity of } E_i \text{ in } \operatorname{div}(h^* f) \\ \nu_i - 1 \quad \text{"} \quad \text{"} \quad \text{"} \quad \operatorname{div}(h^* dx_1 \wedge \dots \wedge dx_n) \end{cases}$

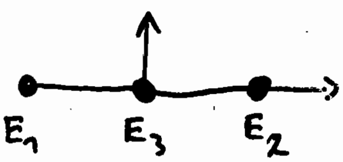
$$\boxed{Z(s) = Z_{top,0}(f; s) := \sum_{I \subset T} \chi\left(E_I^\circ \cap h^{-1}\{0\}\right) \prod_{i \in I} \frac{1}{\nu_i + s N_i}}$$

} poles
} their orders ($\leq n$)

example : $f = x^3(y^3+x^2)$



dual picture:



$$Z(s) = Z_{top,0}(f;s) = \text{sum of } 7 \text{ terms}$$
$$= \cdots$$
$$= \frac{5+13s+6s^2}{5(1+3s)^2(1+s)}$$

Poles: $-1/3$ of order 2

-1 of order 1

II Conjectures

- (A) s_0 is pole of $Z(s) \Rightarrow s_0$ is root of $b(s)$
  (A') s_0 is pole of order $k \Rightarrow s_0$ is root of multiplicity $\geq k$
  (B) s_0 is pole of $Z(s) \Rightarrow \exp(2\pi i s_0)$ is monodromy eigenvalue at some point of $\{f=0\}$

<u>Note</u>:
- (A) $\Rightarrow$ (B)
- for ISOLATED singularity: multiplicity of root of $b(s)$ is related to size of monodromy Jordan block [Varchenko]

- [Loeser '88] $n=2$: (A) for arbitrary f
  (A') for REDUCED f (i.e. ISOLATED singul.); so

$$\boxed{\begin{array}{l} s_0 \text{ pole of order } 2 \Rightarrow s_0 \text{ root of multiplicity } 2 \\ \qquad\qquad\qquad \searrow \qquad\qquad\qquad \Updownarrow \\ \qquad\qquad\qquad\qquad \exp(2\pi i s_0) \text{ has Jordan block of size } 2 \\ (\text{exclude } f = x \cdot y) \end{array}}$$

- s_0 pole of order n $\overset{?}{\rightsquigarrow}$ s_0 root of multiplicity n
  $\Leftrightarrow$ $\qquad \overset{?}{\rightsquigarrow}$ $\exp(2\pi i s_0)$ has Jordan block of size n
  $\exists\, n$ intersecting E_i with $s_0 = -\nu_i/N_i$

- concentrate on $s_0 = -\operatorname{lct}(f) = -\min_{i \in T} \nu_i/N_i$
  - $-\operatorname{lct}(f)$ is largest root of $b(s)$
  - Conjecture [V.]: $Z(s)$ has AT MOST ONE pole of order n, and in this case it is $-\operatorname{lct}(f)$

  (OK for $n=2$ [V.] and for f nondegenerate [Laeremans, V.])

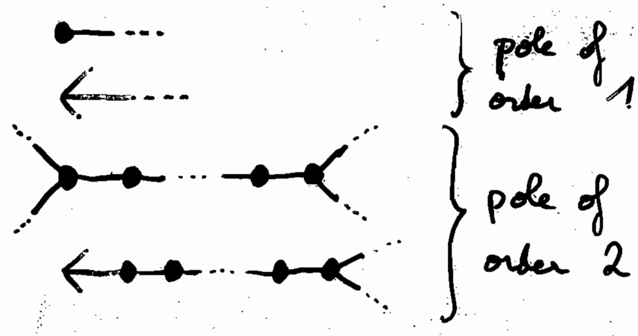
- any pole of order n is of the form $-1/N$ $(N \in \mathbb{Z}_{>0})$ [L., V.]

III results for curves ($n=2$)

- $f = \prod_{j \in J} f_j^{N_j}$ (exclude $f \approx x^N y^{N'}$)

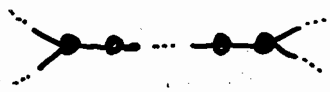
- structure of dual resolution graph w.r.t. decorations ν_i / N_i:

  4 possibilities for $\mathcal{M} := \{ E_i \mid \nu_i / N_i = \mathrm{lct}(f) \}$

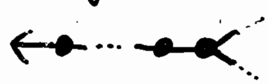


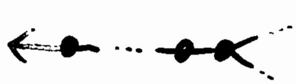
- **Theorem** Suppose $s_0 = -\mathrm{lct}(f) = -1/N$ is pole of order 2. Then

  $\exp(2\pi i s_0)$ has Jordan block of size 2

  $\Leftrightarrow \mathcal{M}$ is  $\Leftrightarrow N \neq N_j \quad \forall j \in J$

  $\exp(2\pi i s_0)$ has only Jordan blocks of size 1

  $\Leftrightarrow \mathcal{M}$ is  $\Leftrightarrow N = N_j$ for some $j \in J$

- fact: $\mathcal{M}$ is  $\Leftrightarrow \begin{cases} f \approx x^N g(x,y) \\ \text{intersection number of } \{x=0\} \text{ and } \{g=0\} \text{ at origin is } N \end{cases}$

- **Theorem** If $f = x^N g(x,y)$ with ↗ and g weighted homogeneous and reduced, then $s_0 = -1/N$ is root of $b(s)$ of multiplicity 2.

- Note: so: 'dichotomy' for size of Jordan blocks on $H^1(F) \circlearrowleft$ Mon.

  BUT: on nearby cycle complex $R\Psi_f \mathbb{C}$ 'morally' $\exp(2\pi i(-1/N))$ ALWAYS has a Jordan block of size 2

IV results for arbitrary n

- for monodromy eigenvalue λ, consider

$$\underbrace{Gr^W_{2n-2}}_{\text{graded quotient for Weight filtration in MHS}} H^{n-1}(F)_\lambda \quad] \rightsquigarrow \lambda\text{- generalized eigenspace}$$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}} \quad s_0 = -\mathrm{lct}(f) = -1/N \text{ is pole of order } n \text{ of } Z(s) \\ \quad \Rightarrow Gr^W_{2n-2} H^{n-1}(F)_{\exp(2\pi i s_0)} \neq 0\end{array}}$$

- notation: 1) Monodromy M on (shifted) perverse sheaf $R\psi_f \mathbb{C}$ decomposes as $M = M_s \cdot M_u$

$\rightsquigarrow$ operator $(\log M_u)^n = 0$ on $R\psi_f\mathbb{C}$

support of $(\log M_u)^{n-1}(R\psi_f\mathbb{C})$ has dimension 0 or is ϕ

2) $R\psi_f\mathbb{C} \xleftarrow[\text{subobject}]{} R\psi_{f,\lambda}\mathbb{C} := \mathrm{Ker}(M_s - \lambda)$

$$\boxed{\begin{array}{l}\underline{\text{Corollary 1}} \quad s_0 = -\mathrm{lct}(f) \text{ is pole of order } n \text{ of } Z(s) \\ \quad \Rightarrow (\log M_u)^{n-1} \neq 0 \text{ on } R\psi_{f,\exp(2\pi i s_0)}\mathbb{C} \\ \text{`morally' } \exp(2\pi i s_0) \text{ has Jordan block of size } n \text{ for } R\psi_f\mathbb{C} \circlearrowleft M\end{array}}$$

- $$\boxed{\begin{array}{l}\underline{\text{Corollary 2}} \quad \text{ISOLATED singularity. Then} \\ \quad s_0 = -\mathrm{lct}(f) \text{ is pole of order } n \text{ of } Z(s) \\ \quad \Rightarrow s_0 \text{ is root of } b(s) \text{ of multiplicity } n\end{array}}$$