MATH 416
Homework 5 Solutions

May 29, 2014

4.20 Let R_1, R_2, and R_3 denote the events of winning on the first, second, and third spin respectively.

(a) $P(X>0) = P(R_1 \cup R_1^c R_2 R_3) = P(R_1) + P(R_1^c) P(R_2) P(R_3) = \frac{18}{38} + \frac{20}{38} \times \frac{18}{38} \times \frac{18}{38} = \mathbf{\frac{4059}{6859}} \approx 0.5918$.

(c) In order to find $E[X]$, we first notice that the only possible values of X are 1 (first win or first loss and two wins), -1 (loss-win-loss and loss-loss-win), and -3 (loss-loss-loss). $P(X=-1) = 2 \times \frac{18}{38} \times \frac{20}{38} \times \frac{20}{38} = \frac{1800}{6859}$, $P(X=-3) = \frac{20}{38} \times \frac{20}{38} \times \frac{20}{38} = \frac{1000}{6859}$. Then $E[X] = 1 \times P(X=1) + (-1) \times P(X=-1) + (-3) \times P(X=-3) = \frac{4059}{6859} - \frac{1800}{6859} - \frac{3000}{6859} = -\mathbf{\frac{741}{6859}} \approx -0.1080$. So the gambler would lose on average 11 cents.

4.21(b) First, we'll find the probability mass function for X and Y:

$$\begin{aligned} p_X &= \left\{ \left(40, \frac{40}{148}\right), \left(33, \frac{33}{148}\right), \left(25, \frac{25}{148}\right), \left(50, \frac{50}{148}\right) \right\}, \\ p_Y &= \left\{ \left(40, \frac{1}{4}\right), \left(33, \frac{1}{4}\right), \left(25, \frac{1}{4}\right), \left(50, \frac{1}{4}\right) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} E[X] &= p_X(40) \times 40 + p_X(33) \times 33 + p_X(25) \times 25 + p_X(50) \times 50 = \mathbf{\frac{5814}{148}} \approx 39.28; \\ E[Y] &= p_Y(40) \times 40 + p_Y(33) \times 33 + p_Y(25) \times 25 + p_Y(50) \times 50 = \mathbf{\frac{148}{4}} = \mathbf{37}. \end{aligned}$$

4.25 Let H_1 and H_2 correspond to the events that coin 1 and 2 lands heads, respectively.

(a) $P(X=1) = P(H_1 H_2^c \cup H_1^c H_2) = P(H_1) P(H_2^c) + P(H_1^c) P(H_2) = 0.6 \times 0.3 + 0.4 \times 0.7 = \mathbf{0.46}$.

(b) $E[X] = 0 \times P(X=0) + 1 \times P(X=1) + 2 \times P(X=2) = 0.46 + 2P(H_1 H_2) = 0.46 + 2 \times 0.7 \times 0.6 = \mathbf{1.3}$.

4.27 Let S denote the amount the company would charge the customer. In this case, the company earns $S - A$ in the event E (with probability p) and S in the event E^c (with probability $1-p$). Let X denote the profit of the company; then

$$\begin{aligned} P(X = S - A) &= p; \\ P(X = S) &= 1 - p; \\ E[X] &= p(S - A) + (1-p) S = S - pA. \end{aligned}$$

The expected profit is $0.1A = S - pA$; thus $S = (\mathbf{0.1} + \mathbf{p})\,\mathbf{A}$.

Th 4.2 Let $Y = e^X$; note that since $e^X > 0$, for $y \le 0$, $F_Y(y) = 0$. Otherwise $F_Y(y) = P(Y \le y) = P\left(e^X \le y\right) = P(X \le \ln y) = F_X(\ln y)$.

5.1 (a)

$$1 = \int_{-\infty}^{\infty} f(x)\ dx = \int_{-1}^{1} c\left(1 - x^2\right)\ dx = c\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{4}{3}c \Longrightarrow c = \mathbf{\frac{3}{4}}.$$

(b)

$$F(x) = \int_{-\infty}^{x} f(u)\ du = \begin{cases} 0 & : x < -1, \\ \frac{3}{4}\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{x} = \frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2} & : x \in [-1, 1], \\ 1 & : x > 1. \end{cases}$$

5.2 First, we need to find C:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{\infty} Cxe^{-0.5x}\, dx = -C\left((2x+4)\, e^{-0.5x}\right)\Big|_{0}^{\infty} = 4C \Longrightarrow C = \frac{1}{4}. \\ P(X \geq 5) &= \int_{5}^{\infty} f(x)\ dx = \int_{5}^{\infty} \frac{1}{4}xe^{-0.5x}\, dx = -\frac{1}{4}\left((2x+4)\, e^{-0.5x}\right)\Big|_{5}^{\infty} = \frac{14e^{-2.5}}{4} \\ &= \mathbf{3.5e^{-2.5}}. \end{aligned}$$

5.4 (a)

$$P(X > 20) = \int_{20}^{\infty} f(x)\ dx = -\frac{10}{x}\Big|_{20}^{\infty} = \mathbf{\frac{1}{2}}.$$

(b)

$$F(x) = \int_{-\infty}^{x} f(u)\ du = \begin{cases} 0 & : x \leq 10, \\ -\frac{10}{u}\Big|_{10}^{x} = 1 - \frac{10}{x} & : x > 10. \end{cases}$$

(c) Assume that the events that different devices are functioning at least 15 hours are independent. First, we find the probability p that a device will function for at least 15 hours:

$$p = P(X \geq 15) = \int_{15}^{\infty} f(x)\ dx = -\frac{10}{x}\Big|_{15}^{\infty} = \frac{2}{3}.$$

To find the probability that at least three devices will function for at least 15 hours, we will use the binomial distribution with $p = \frac{2}{3}$ and $N = 6$ (with Y denoting the number of devices functioning for at least 15 hours):

$$P(Y \geq 3) = \sum_{i=3}^{6} \binom{6}{i} \frac{2^i}{3^6} = \frac{1}{3^6}\left(20 \times 8 + 15 \times 16 + 6 \times 32 + 64\right) = \frac{3696}{3^6} = \mathbf{\frac{656}{729}}.$$