Solutions 1 (to homework 1).

(1) Let C be the region defined by $-1 < x_1 < 1$ and $-1 < x_2 < 1$. This is a square without its boundary. Suppose we try to maximize $z = x_1 + x_2$ over C, which is linear. The optimal thing would be $x_1 = x_2 = 1$, but we can't do that because the corner is not part of C. So z is bounded on C (it is never bigger or equal to 1), but there is no optimal point in C.

(2) Take the region $-1 \leq x_1 \leq 1$ and $-1 \leq x_2 \leq 1$, the square *with* its boundary. If $z = 1 - x_1^2 - x_2^2$ then z has a critical point at (0,0) which turns out to be a maximum. But, the maximum is not on the boundary. This could happen because z is not linear.

(3) The max is at $x_1 = 4, x_2 = 5, z = 37$. It is easy to see that this point satisfies the constraints. The other corners of the feasible region are $(2, 6), (0, 6), (4, 0), (0, 0)$. None of the others get $z = 37$.

(4) The corners of the feasible region are $(9, 0), (45/7, 30/7), (0, 15/2)$. The best z occurs at both the second and the third point.

Solutions 2 (for homework 2)**.**

(1) If we let x_1 and x_2 stand for the wheat and the corn, we get an initial problem

$$\begin{aligned} x_1 + x_2 &\leq 100 \\ 2x_1 + x_2 &\leq 150 \\ 5x_1 + 10x_2 &\leq 800 \\ x_1, x_2 &\geq 0 \\ z &= 80x_1 + 60x_2 \to \max \end{aligned}$$

With slack, that is

$$\begin{aligned} x_3 &= 100 - x_1 - x_2 \\ x_4 &= 150 - 2x_1 - x_2 \\ x_5 &= 800 - 5x_1 - 10x_2 \\ x_1, x_2 &\geq 0 \\ z &= 80x_1 + 60x_2 \to \max \end{aligned}$$

Now x_1 in, $x_1 \leq 100, 75, 160$, so x_4 out:

$$\begin{aligned} x_3 &= 25 - x_2/2 + x_4/2 \\ x_1 &= 75 - x_2/2 - x_4/2 \\ x_5 &= 425 - 15/2x_2 + 5/2x_4 \\ x_1, x_2 &\geq 0 \\ z &= 6000 + 20x_2 - 40x_4 \to \max \end{aligned}$$

Now x_2 in, x_3 out:

$$\begin{aligned} x_2 &= 50 - 2x_3 + x_4 \\ x_1 &= 50 + x_3 - x_4 \\ x_5 &= 50 + 15x_3 - 5x_4 \\ x_1, x_2 &\geq 0 \\ z &= 7000 - 20x_3 - 20x_4 \to \max \end{aligned}$$

This is optimal, Jones should use half his land for wheat and the other for corn. He'll make \$ 7000, and \$ 50 of his capital will be unspent.

(2) Let x_1, x_2 be the steak/potato amounts. Then we have

$$\begin{aligned} 5x_1 + 15x_2 &\geq 50 \\ 20x_1 + 5x_2 &\geq 40 \\ 15x_1 + 2x_2 &\leq 60 \\ x_1, x_2 &\geq 0 \\ 4x_1 + 2x_2 &\to \min \end{aligned}$$

With slack that is

$$\begin{aligned}
5x_1 + 15x_2 &= 50 + x_3 \\
20x_1 + 5x_2 &= 40 + x_4 \\
15x_1 + 2x_2 + x_5 &= 60 \\
x_1, x_2 &\geq 0 \\
4x_1 + 2x_2 &\to \min
\end{aligned}$$

The initial meal schedule calls for $x_1 = 0, x_2 = 30$ and hence $x_3 = 400, x_4 = 110, x_5 = 0$. So the first dictionary is

$$\begin{aligned}
x_2 &= 30 - 15x_1/2 - x_5/2 \\
x_3 &= 400 - 215x_1/2 - 15x_5/2 \\
x_4 &= 110 - 35x_1/2 - 5x_5/2 \\
x_i &\geq 0 \\
z &= 60 - 11x_1 - x_5 \to \min
\end{aligned}$$

Since this is written as a min-problem, we look for variables that make this z *smaller*, for example x_1. Then $x_1 \leq 4, 160/43, 44/7$. So $x_1 = 160/43$ and x_3 exits. On the other hand, we could introduce x_5, which has restrictions $x_5 \leq 60, 44, 160/3$. So $x_5 = 44$ and x_3 exits. This is a much nicer fraction than 160/43, so we use this second substitution:

$$\begin{aligned}
x_2 &= 8 - 4x_1 + x_4/5 \\
x_3 &= 70 - 55x_1 + 3x_4 \\
x_5 &= 44 - 7x_1 - 2x_4/5 \\
x_i &\geq 0 \\
z &= 16 - 4x_1 + 2x_4/5 \to \min
\end{aligned}$$

Now we have to bring in x_1, subject to $x_1 \leq 44/7, 8/4, 70/55$. So $x_1 = 14/11$ and x_3 exits:

$$\begin{aligned}
x_1 &= 14/11 - x_3/55 + 3x_4/55 \\
x_2 &= 32/11 + 4x_3/55 - x_4/55 \\
x_5 &= 386/11 + 7x_3/55 - 43x_4/55 \\
x_i &\geq 0 \\
z &= 120/11 + 4x_3/55 + 10x_4/55 \to \min
\end{aligned}$$

This is now optimal. Hence one ought to eat 14/11 steaks and 32/11 potatoes per day for a price of about \$ 10.91. One eats exactly the required amounts of carbohydrates and protein, and stays way under the limit for fat.

Solutions 3 (to homework 3).

(1) The first dictionary is

$$\begin{aligned} x_4 &= -8 + x_1 + 4x_2 + 2x_3 + x_0 \\ x_5 &= -6 + 3x_1 + 2x_2 + x_0 \\ z &= -x_0 \end{aligned}$$

x_0 in, x_4 out:

$$\begin{aligned} x_0 &= 8 - x_1 - 4x_2 - 2x_3 + x_4 \\ x_5 &= -6 + 3x_1 + 2x_2 + (8 - x_1 - 4x_2 - 2x_3 + x_4) \\ &= 2 + 2x_1 - 2x_2 - 2x_3 + x_4 \\ z &= -8 + x_1 + 4x_2 + 2x_3 - x_4 \end{aligned}$$

this is feasible. x_1 in, x_0 out:

$$\begin{aligned} x_1 &= 8 - 4x_2 - 2x_3 + x_4 - x_0 \\ x_5 &= 2 - 2x_2 - 2x_3 + x_4 + 2(8 - 4x_2 - 2x_3 + x_4 - x_0) \\ &= 18 - 10x_2 - 6x_6 + 3x_4 - 2x_0 \\ z &= -8 + 4x_2 + 2x_3 - x_4 + (8 - 4x_2 - 2x_3 + x_4 - x_0) \\ &= -x_0 \end{aligned}$$

This is optimal. So $x_1 = 8, x_5 = 18, x_4 = 0, x_2 = 0, x_3 = 0$ are the initial solution for the original problem:

$$\begin{aligned} x_1 &= 8 - 4x_2 - 2x_3 + x_4 \\ x_5 &= 18 - 10x_2 - 6x_6 + 3x_4 \\ z &= 2(8 - 4x_2 - 2x_3 + x_4) + 3x_2 + x_3 \\ &= 16 - 5x_2 - 3x_3 \end{aligned}$$

Next, x_4 in, but no candidate for leaving the basis. Means: problem unbounded.

(2) Auxiliary problem is

$$\begin{aligned} x_3 &= -10 + 2x_1 + x_2 + x_0 \\ x_4 &= 6 + 3x_1 - 2x_2 \\ z &= -x_0 \end{aligned}$$

This is infeasible. x_0 in, x_3 out:

$$\begin{aligned} x_0 &= 10 - 2x_1 - x_2 + x_3 \\ x_4 &= 6 + 3x_1 - 2x_2 \\ z &= -10 + 2x_1 + x_2 - x_3 \end{aligned}$$

This is now feasible. x_1 in, x_3 out:

$$\begin{aligned} x_1 &= 5 - x_2/2 + x_3/2 - x_0/2 \\ x_4 &= 6 - 2x_2 + 3(5 - x_2/2 + x_3/2 - x_0/2) \\ &= 21 - 7x_2/2 + 3x_3/2 - 3x_0/2 \\ z &= -x_0 \end{aligned}$$

This is optimal. So the start solution for the original problem is $x_1 = 5, x_4 = 6, x_2 = x_3 = 0$:

$$\begin{aligned} x_1 &= 5 - x_2/2 + x_3/2 \\ x_4 &= 21 - 7x_2/2 + 3x_3/2 \\ z &= 2x_2 - 3(5 - x_2/2 + x_3/2) \\ &= -15 - x_2/2 - 3x_3/2 \end{aligned}$$

This is optimal and the optimal z as stated in the problem is 15 (recall, that we turned it around to make it a max!)

Solutions 4 (to homework 4)**.**

(1) Given is

$$
\begin{array}{rcl}
x_1 + x_2 + x_3 & \leq & 480 \\
x_4 + x_5 + x_6 & \leq & 400 \\
x_7 + x_8 + x_9 & \leq & 230 \\
x_2 + x_5 + x_8 & \leq & 420 \\
x_3 + x_6 + x_9 & \leq & 250 \\
x_i & \geq & 0 \\
8x_1 + 14x_4 + 11x_3 + 4x_4 + 12x_5 + 7x_6 + 4x_7 + 13x_8 + 9x_9 & \rightarrow & \max
\end{array}
$$

The proposed optimal solution is $\vec{x}^* = (440, 0, 40, 0, 400, 0, 0, 20, 210)$.
By (5.22) we must have

$$
\begin{array}{rcl}
y_1^* & = & 8 \\
y_1^* + y_5^* & = & 11 \\
y_2^* + y_4^* & = & 12 \\
y_3^* + y_4^* & = & 13 \\
y_3^* + y_5^* & = & 9
\end{array}
$$

which solves to $\vec{y}^* = (8, 5, 6, 7, 3)$. By (5.23) the following need to be satisfied:

$$
\begin{array}{rcl}
y_1^* + y_4^* & \geq & 14 \\
y_2^* & \geq & 4 \\
y_2^* + y_5^* & \geq & 7 \\
y_3^* & \geq & 4
\end{array}
$$

Plugging in the numbers one sees that all inequalities hold. Hence $\vec{x}^*$ ($\vec{y}^*$) is the optimal primal (dual) solution by Theorem 5.3 in the book.

(2) An example would be

$$
\begin{array}{rcl}
x_1 - x_2 & \leq & -2 \\
-x_1 + x_2 & \leq & 1 \\
x_1 - x_2 & \rightarrow & \max
\end{array}
$$

with dual

$$
\begin{array}{rcl}
y_1 - y_2 & \geq & 1 \\
-y_1 + y_2 & \geq & -1 \\
-2y_1 + y_2 & \rightarrow & \min
\end{array}
$$

but there are many many more.

(3)

$$\begin{aligned} x_1 - x_2 &\leq -2 \\ -x_1 + x_2 &\leq 1 \\ 2x_1 - x_2 &\to \max \end{aligned}$$

with dual

$$\begin{aligned} y_1 - y_2 &\geq 2 \\ -y_1 + y_2 &\geq -1 \\ -2y_1 + y_2 &\to \min \end{aligned}$$

but there are many many more.

(4) What I had in mind was an example of the sort

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ x_1 &\leq 1 \\ x_1 + x_2 &\to \max \end{aligned}$$

This has as optimal solutions all points on the line segment from $(1, 1)$ to $(0, 2)$.

Its dual is

$$\begin{aligned} y_1 + y_2 &\geq 1 \\ y_2 &\geq 1 \\ y_1 + 2y_2 &\to \min \end{aligned}$$

This has its optimum in the point $(0, 1)$. Note that this point is degenerate (the meeting point of more than 2 lines). In fact, this is typical:

Proposition 0.1. *If the primal has infinitely many optimal solutions, the dual has degenerate optimal solutions.*

Proof. It is easy to see (I think) that if you have an LP with infinitely many optimal solutions, then this is kind of an accident: if you change the objective function coefficients all just slightly, most of the previously optimal solutions won't be optimal any more.

This is to say that in the given problem a bunch of optimal dictionaries exist, but in the disturbed one only one (or a few) are still optimal.

Now each optimal primal dictionary comes from a basis, and by the complimentary basis/slackness theorem corresponds to a basis for the dual. For the given LP, there are a bunch of dual optimal dictionaries then of which only one (or a few) survive as optimal in the disturbed problem. Comparing the optimal dictionaries of different disturbances we see that they may be *very* different.

Let's now inspect the possibility of a non-degenerate optimal dual solution for the given LP. If we wiggle in the objective function for the primal, we change slightly the right hand side of the constraints in the dual. If an optimal solution is non-degenerate, such a change will have only mild changes for the y-variables as consequence. But, we know that depending on the changes in $\vec{c}$ we administer the changes in the y-variables are quite dramatic: switching dictionaries completely changes the values of many variables!

The conclusion is that the dual system cannot have non-degenerate optimal solutions.(That is, all optimal solutions are degenerate.)

Solutions 5 (to homework 5)**.**

(1) From the solution to Exercise 5.4 (last assignment) we know that the optimal solution is

$$\vec{x}^* = (440, 0, 40, 0, 400, 0, 0, 20, 210).$$

You also found the optimal dual solution $\vec{y}^* = (8, 5, 6, 7, 3)$. The primal solution will be optimal as long as the dual is. (Recall that the dual and primal have the same optimal value.) The dual solution (assuming that the price for fresh bellies grows by the number x to $4+x$) stays optimal until the condition that relates to the fresh belly price is violates. The constraint in question is $y_2 \geq 4 + x$ (this is just the 4-th column of the given LP, with modified belly price). Since $y_2^* = 5$, x larger than 1 will cause the optimal dual solution to become infeasible. That means that the optimal value for the dual will rise and hence so will the optimal value for the primal.

To answer the second part, the price of fresh picnics is also 4 bucks. The variable that talks about fresh picnics is x_7. The 7-th equation of the dual with price $4+y$ for the fresh picnics is $y_3 \geq 4+y$. This will be violated if y exceeds 2.

(2) "Degenerate" in 2 dimensions means that at least 3 lines meet in a point. (Because 2 are needed to define a point, and degeneracy speaks about accidents.)

Now, the non-negativity conditions give 2 lines, and the 2 constraints 2 more. So let's say we make $(1, 0)$ our optimum. It is on the line $x_2 = 0$, so we need two more that go through there. For example, take $x_1 + x_2 = 1$ and $x_1 + 4x_2 = 1$. The feasible region is a triangle with corners $(0, 0)$, $(0, 1)$, $(1, 0)$. The LP should be therefore

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_1 + 4x_2 &\leq 1 \\ x_i &\geq 0 \end{aligned}$$

It remains to find an objective function that picks $(1, 0)$ as optimum and not either of the 2 other corners. One that works is $5x_1 + x_2 \to$ max.

What is the dual system then?

$$\begin{aligned} y_1 + y_2 &\geq 5 \\ y_1 + 4y_2 &\geq 1 \\ y_i &\geq 0 \\ y_1 + y_2 &\to \min \end{aligned}$$

The point of this example is that if you increase the right hand side of *just one* of the primal constraints, nothing happens to the optimal solution. This is because whichever you relax, the other makes sure the optimum does not move.

The dual optimal solution is (easy check) the line segment from $(0, 5)$ to $(5, 0)$. So, no matter what optimal y-solution you take, one of the values (usually both) is positive. Hence, Theorem 5.5 suggests that the marginal value of at least one primal constraint is positive. So the primal optimum should grow if one primal constraint is relaxed. The fact that it does not is not a violation of Theorem 5.5, because our problem is degenerate, and such problems are not considered in Theorem 5.5.

(3) Complementary slackness says that if a primal constraint is an inequality then the marginal value of the resource is zero. Conversely, if a marginal value is nonzero, the corresponding constraint must be maxed out (an equality). The economic implication is that one should try to trade some of the unused resources for buying resources corresponding to maxed out constraints.

Solutions 6 (to homework 6).

(1) Given is $\vec{x}^T = (0,0,0,0,1,0,7,5)$ with $\vec{c}^T = (1,2,-1,-,1,2,1,-3,1)$ and $z = 14$.

(a) $\vec{c}_B^T = (2,-3,1)$, $B = \begin{pmatrix} -1,0,1 \\ -2,0,-1 \\ 2,-1,1 \end{pmatrix}$. This gives $\vec{v}^T = (1/3, 2, 8/3)$.
Then $\vec{v}^T A_N = (5, -13/3, -4, 5/3, -1)$. We pick $j = 2$. Then $\vec{a}^T = (-1,-2,0)$ and $\vec{d}^T = (1,2,0)$. It turns out that t can go up to 1, and $i = 5$.
The new $\vec{x}^T$ is then $(0,1,0,0,0,0,5,5)$. It has $z = -8$.

(b) This time, $B = \begin{pmatrix} -1,0,1 \\ -2,0,-1 \\ 0,-1,1 \end{pmatrix}$, $\vec{c}_B^T = (2,-3,1)$. Then $\vec{v}^T = (1/3, 4/3, 7/3)$ and $\vec{v}^T A_N = (13/3, -10/3, 1, 5/3, 1/3)$. We pick $j = 3$. So $\vec{a}^T = (1,-1,-1)$ and $\vec{d}^T = (0,2,1)$. t may go up to 2.5 and therefore $i = 7$.
The new $\vec{x}^T$ is $(0,1,5/2,0,0,0,0,5/2)$ with $z = 2$.

(c) Now $B = \begin{pmatrix} -1,1,1 \\ -2,-1,-1 \\ 0,-1,1 \end{pmatrix}$ and $\vec{c}_B^T = (2,-1,1)$. Then $\vec{v}^T = (-1/3, 1/3, 4/3)$ and $\vec{v}^T A_N = (1, 2/3, 7/3, 0, -4/3)$. We pick $j = 6$. So $\vec{a}^T = (2,-2,1)$ and $\vec{d}^T = (0, 1/2, 3/2)$. Then t is $5/3$, $i = 8$.
The new $\vec{x}$ is then $(0,1,5/3,0,0,5/3,0,0)$. It has $z = 2$.

(d) Now $B = \begin{pmatrix} -1 & 1 & 2 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ and $\vec{c}_B^T = (2,-1,1)$. Then $\vec{v}^T = (-2/3, -2/3, 1)$ and $\vec{v}^T A_N = (-1, 0, 4, -1, 1)$. We pick $j = 1$. So $\vec{a}^T = (2,1,1)$ and $\vec{d}^T = (-1, -1/3, 2/3)$. Then $t = 5/2$, $i = 6$.
The new $\vec{x}$ is $(5/2, 7/2, 5/2, 0, 0, 0, 0, 0)$. It has $z = 7$.

(e) Now $B = \big(2,-1,1;1,-2,-1;1,0,-1\big)$ and $\vec{c}_B^T = (1,2,-1)$. Then $\vec{v}^T = (0,-1,2)$ and $\vec{v}^T A_N = (-1,6,4,-2,3)$. No component of this vector is smaller than the corresponding component of $\vec{c}^T$. Hence we have an optimal solution in our hands.

(2) If $a_1x_1 + \ldots + a_nx_n \le b$, then since all a_i are *positive* and x_i may *not be negative*, we get individual estimates $a_ix_i \le b$ or $x_1 \le b/a_i$. Now x_i is bounded from below by 0, from above by b/a_i, and it must be an integer. There are only finitely many possibilities for x_i, say m_i is the number of these possibilities. Then let $\vec{x}^*$ be a feasible solution. The component x_i^* must be using one of the m_i possible integer values between 0 and b/a_i. Hence there are $m_1 \cdot m_2 \cdots m_n$ possibilities for picking values in a feasible solution, which is a finite number. (*Note:* the actual number of feasible solutions is likely to

be *much smaller* than $m_1 \cdots m_n$. We only found this product to be an upper bound.)

(3) The point is that the displayed equations with the boxes are all linear combinations of those given initially. Imagine you introduce slack:

$$\begin{aligned} 2x_1 + 2x_2 + x_3/2 + x_4 &= 2 \\ -4x_1 - 2x_2 - 3x_3/2 + x_5 &= 3 \\ x_1 + 2x_2 + x_3/2 + x_6 &= 1 \\ 6x_1 + x_2 + 2x_3 &= z \end{aligned}$$

Now make a linear combination of those equalities with coefficients L_4, L_5, L_6, L_z, all real numbers. It is quite obvious (I think) that the result is an expression that has L_4 copies of x_4, L_5 copies of x_5, L_6 copies of x_6 and L_z copies of z.

One can therefore read off the coefficients that were used to make any of the 4 equalities with boxes from the numbers in front of x_4, x_5, x_6 and z. For example, the first equality says you must have used minus one times the x_4-constraint, none of the x_5-constraint, and one copy of the x_6-constraint. This implies that the whole equation looks like

$$x_1 = 1 + 0x_1 + 0x_2 + 0x_3 - 1x_4 + 0x_5 + 1x_6.$$

Similarly, the other ones come out as

$$\begin{aligned} x_1 &= 1 + 0x_1 + 0x_2 + 0x_3 - 1x_4 + 0x_5 + 1x_6 \\ x_3 &= 0 + 0x_1 - 4x_2 + 0x_3 + 2x_4 + 0x_5 - 4x_6 \\ x_5 &= 7 + 0x_1 - 4x_2 + 0x_3 - 1x_4 + 0x_5 - 2x_6 \\ z &= 6 - 0x_1 - 0x_2 - 0x_3 - 2x_4 - 0x_5 - 2x_6 \end{aligned}$$

Solutions 7. (1) The number $z(k)$ is the best value we can fit into a knapsack of volume k. Suppose that the knapsack has volume at least equal to the smallest item. That means $k \geq \min_j(a_j)$. (Note: this is given as hypothesis in the problem.) Of course this means that $z(k)$ is not zero, because we could put the at least the smallest item into the knapsack.

Then imagine that a copy of one of the items in the optimal knapsack of volume k is actually in a side pocket all by itself that it fills completely. Let's say this is the t-th item. This means that the rest of the knapsack has space equal to $k - a_t$. The best value of a backpack with this volume is $z(k - a_t)$. This means that the value for the original backpack of volume k is $z(k-a_t)+c_t$ (recall that we have an item t in the side pocket!).

This proves that $z(k) = z(k - a_t) + c_t$. The problem is that we have no clue which item item t is, because a priori we have no idea what items are in the best backpack of volume k at all. The only things we know for sure are a) the backpack is not empty, and b) the only items in there are those of volume at most k. Consider all expressions $c_i + z(k - a_i)$ where i runs through those indices whose items fit in the volume-k-backpack. Each of them stands for a different way of packing the k-th knapsack. The optimal way of packing the knapsack will therefore correspond to the maximum of these numbers.

(2) First order them by efficiency:

$$\begin{aligned} 27x_1 + 34x_2 + 41x_3 + 31x_4 + 33x_5 + 23x_6 &= 168 \\ 59x_1 + 74x_2 + 89x_3 + 67x_4 + 71x_5 + 23x_6 &\to \max \\ x_i &\in \mathbb{N} \end{aligned}$$

The start solution is $(6, 0, 0, 0, 0, 0)$ with value 354. The next branch is $(5, *, *, *, *, *)$ with potential value 366. Its top leaf is $(5, 0, 0, 0, 1, 0)$ with value 366. The next lower branch is $(4, *, *, *, *, *)$ of potential value 366. Thus we can prune this branch. By the lemma we proved, we can also prune the branches directly underneath this branch. But that prunes the whole tree. So the optimal value is 366.

(3) The algorithm goes like this: Start with $k = \min_j(a_j)$. (Before, the knapsack question is easy: it has to be empty.) Make a chart in which k, a particular index t_k, and $z(k)$ For now the chart contains $\min_j(a_j)$ (for k), the index t for which a_t is minimal, and c_t (for $z(k)$.

Now move to $k + 1$. For each i that has $a_i \leq k + 1$, look up in the chart each $z(k+1-a_i)$. Between these indices, find the index t which maximizes $c_t + z(k+1-a_i)$. Record this index as t_{k+1} in the chart. Record $c_{t_{k+1}} + z(k+1-a_i)$ as $z(k+1)$ in the chart.

Then move to $k+2$ and so on. The flowchart is a bit difficult with this text-setting program...