- Does $x^7 - 33x^4 + 12x^3 + 2x^2 + 3$ have rational roots? If so, find them.

- How many quadrics are there in $\mathbb{Z}/11\mathbb{Z}[x]$? How many are reducible? How many are irreducible? How many reducible cubics are there?

  (For the quadrics, note that reducibility of a quadric is tantamount to both roots being in $\mathbb{Z}/11\mathbb{Z}$. How many degree 2 polynomials have this latter property? The cubics part is more tricky; you need to do the quadric part first and then ask how a factorization could happen.)

- Find the gcd between $f(x) = x^5 + 3x^3 - 5x^2 - 2x + 1$ and $g(x) = x^4 + 5x^3 + 5x^2 + x - 1$ in $\mathbb{Z}/11\mathbb{Z}[x]$. Factor the gcd as much as possible and write it as linear combination of f, g.

- Find the total number of fields contained in $\mathrm{GF}(23, 48)$. What is the longest chain of field inclusions within this set?

- The polynomial $f(x) = x^3 + 2x + 2$ is irreducible in $\mathbb{Z}/3\mathbb{Z}[x]$. Explain why. Then

  $$\frac{\mathbb{Z}/3\mathbb{Z}[x]}{\langle x^3 + 2x + 1\rangle} = \mathrm{Kron}(\mathbb{Z}/3\mathbb{Z}, f)$$

  is a field with 27 elements. Let α be the Kronecker root $\alpha = \bar{x}$ of f in this field.

  Find the other two roots of $f(x)$ in $\mathrm{Kron}(\mathbb{Z}/3\mathbb{Z}, f)$ as linear combinations of $\bar{1}, \alpha, \alpha^2$.

- Let $\mathbb{F}_1 = \mathbb{Q}(\sqrt{2})$, $\mathbb{F}_2 = \mathbb{Q}(\sqrt{5})$, $\mathbb{F} = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. Let $\alpha = \sqrt{2} + \sqrt{5}$. Show that $\mathbb{F}_1 \neq \mathbb{F}_2 \neq \mathbb{F} \neq \mathbb{F}_1$.

  Find $[\mathbb{F} : \mathbb{F}_2]$, $[\mathbb{F} : \mathbb{F}_1]$, $[\mathbb{F} : \mathbb{Q}]$, $[\mathbb{F}_1 : \mathbb{Q}]$ and $[\mathbb{F}_2 : \mathbb{Q}]$.

  Find the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbb{Q}$. Why is the polynomial that constitutes your answer minimal?

  Then show that $\mathbb{F} = \mathbb{Q}(\sqrt{2} + \sqrt{5})$.

- Of the three polynomials $f(x) = x^2 + x + 5$, and $g(x) = x^2 + x + 6$, and $h(x) = x^2 + x + 7$, which one is irreducible modulo 23?

- Of the three polynomials $f(x) = x^8 + x^4 + x^2 + x + 1$, $g(x) = x^8 + x^4 + x^6 + x^2 + 1$ and $h(x) = x^8 + x^6 + x^4 + x + 1$, one will have multiple roots in some field containing $\mathbb{Z}/2\mathbb{Z}$. Which one of f, g, h does?

- Is the polynomial $x^4 - 96x^2 + 4x - 26$ reducible in $\mathbb{Q}[x]$?

- Find the inverse of $11 - 4\sqrt{3}$ in $\mathbb{Q}(\sqrt{3})$. Write it as $a + b\sqrt{3}$ with $a, b \in \mathbb{Q}$. Find the minimal polynomial of $11 - 4\sqrt{3}$ over $\mathbb{Q}$. Find the minimal polynomial of $1 + 9^{1/3}$ over $\mathbb{Q}$. (In particular, explain in both cases why your answer is minimal).

- Why is $\mathbb{F} = \mathbb{Z}/11\mathbb{Z}[x]/(x^3 + 2x + 2)$ a field? Why does it have 1331 elements? What are the minimal polynomials of $\alpha := \bar{x}$ and of $\beta := \bar{x}^2 - \bar{3}$ over $\mathbb{Z}/11\mathbb{Z}$?

- Which fields $\mathbb{F}$ allow a surjective ring morphism $\mathbb{Z}/105\mathbb{Z} \to \mathbb{F}$?
- In $\mathbb{Z}[\sqrt{-15}]$, use the norm function $a + b\sqrt{-15} \mapsto a^2 + 15b^2$ to investigate which of $3 + 4\sqrt{-15}$, $4 + 3\sqrt{-15}$ are not factorizable in $\mathbb{Z}[\sqrt{-15}]$.
- Is $\mathbb{Q}[x]/(x^2 + 3x + 2)$ a domain?
- What is a prime ideal in $\mathbb{Z}[x]$ that is not a maximal ideal?
- Display an infinite field extension $\mathbb{F}$ of $\mathbb{Q}$.
- Describe the splitting field of $x^4 + 5x^3 + 5x^2 + x - 1$ over $\mathbb{Z}/11\mathbb{Z}$. (It can be made very concrete).
- Express the number of reducible quadrics over a finite field with p^e elements in terms of p and e.
- Let β be the coset of $2x + 3$ in $\mathbb{F} = \mathrm{Kron}(\mathbb{Z}/7\mathbb{Z}, x^2 + 5x + 2)$. Compute (with 3 multiplications) its 8-th power. From information obtained in this way, determine its order as element of the cyclic group $U(7, 2)$. How many elements does this group have?
- Discuss the possible orders of elements in $U(7, 2)$. Explain why for any element of $\mathbb{F}$ the order of the element is the same as the order of its Frobenius image. (Starter: what does the Frobenius do? Then: what do you know about orders of k-powers of elements with known order n—look back to the chapter on cyclic groups).
- Suppose $\mathrm{GF}(p, e)$ is a field such that its group of units is simple (has no proper subgroups). What does that tell you about p? How many such fields can you think of? (If you can find more than 51, you probably would get some kind of prize).