Midterm Practice Problems

You should not use a calculator for any of these.

1. The Fibonacci numbers are defined by $f_1 = 0, f_2 = 1, f_{n+1} = f_n + f_{n-1}$ for $n > 0$. Prove that f_{5n} is divisible by 5.

2. Let $S := \mathbb{Q}[x]$ be the set of all polynomials in x with rational coefficients. Is $|S| = |\mathbb{Z}|$?

3. Let $S = \{1, \ldots, 666\}$. Find the number of elements in S that are divisible by 37 but neither by 2 nor by 3. Find also the number of elements in S divisible by 2 but neither be 3 nor 37.

4. Find the gcd between 91663 and 104247. Write the gcd as $a \cdot 91663 + b \cdot 104247$ for suitable integers a, b.

5. Find an integer a such that $a = 12 \bmod 17$ and $a = 3 \bmod 11$.

6. Explain why if p is a prime number, and a any integer not divisible by p, there is an integer b such that $ab - 1$ is a multiple of p.

7. Find a natural number n such that the equation $x^3 = 1 \bmod n$ has more than 3 solutions in $\mathbb{Z}/n\mathbb{Z}$. (Hint: you are looking for n such that $U(n)$ is a product of two or more things such that tow or more of the factors have order divisible by 3).

8. Is 4492388410833576181834 divisible by 99?

9. Describe the symmetry group of "OHO". (Number of elements? Abelian? Cyclic? Normal subgroups?)

10. Describe all elements in the symmetry group of the two-way infinite sequence "...pqpqpq...".

11. Explain why an automorphism of $\mathbb{Z}/15\mathbb{Z}$ must send 1 to one of $1, 2, 4, 7, 8, 11, 13, 14$. Then use FTFGAG to discuss the abstract structure of $\mathrm{Aut}(\mathbb{Z}/15\mathbb{Z})$ and describe how the 8 options above line up with the structural statements you make. (I want you discuss generators.)

12. Is there an n such that $|U(n)| = 25$?

13. How many cyclic groups exist inside $\mathbb{Z}/30\mathbb{Z}$?

14. How many elements does $\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}/6\mathbb{Z}))$ have?

15. Consider the symmetry group G (of physical symmetries) of a cube. Find some way to convince me that it is not cyclic.

16. In a group G you are told that $\mathrm{ord}(a) = 2 = \mathrm{ord}(b)$. Does that imply that $\mathrm{ord}(ab) = 2$?

17. Let G be the group of 2×2 matrices with entries in $\mathbb{Z}/3\mathbb{Z}$, determinant not divisible by 3, and multiplication as group operation. How many elements does G have? (Ask yourself what $a, b, c, d \in \mathbb{Z}/3\mathbb{Z}$ satisfy $ad - bc = 1 \bmod 3$). Let H be the subgroup of upper triangular matrices in G. How many elements does H have?

18. How many subgroups of size 4 are there in $\mathbb{Z}/14\mathbb{Z}$? How many of size 5,6,7?

19. How many elements of order 4,5,6,7 are there in $\mathbb{Z}/14\mathbb{Z}$?

20. What is $\phi(100)$?

21. Explain why $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.

22. Give an example of a group G with subgroup H that is not normal.

23. Discuss all conceivable sizes of subgroups of the symmetry group of a regular pentagon.

24. Explain why a subgroup H of a finite group G with exactly half as many elements as G must be normal.

25. State from memory what a conjugate subgroup is. For $G = \mathbb{Z}/12\mathbb{Z}$ and $H =$the multiples of $\bar{8}$, ow many conjugate subgroups does H have?

26. Suppose $\psi\colon \mathbb{Z}/21\mathbb{Z} \to \mathbb{Z}/35\mathbb{Z}$ is a morphism. Determine all possible values of $\psi(1 \bmod 21)$. Can $\psi(5 \bmod 21)$ be $7 \bmod 35\mathbb{Z}$? Can $\psi(7 \bmod 21)$ be $5 \bmod 35\mathbb{Z}$?

27. Suppose $\psi\colon \mathbb{Z}/21\mathbb{Z} \to \mathbb{Z}/21\mathbb{Z}$ is a morphism. Can we have $\psi(18 \bmod 21\mathbb{Z}) = 15 + 21\mathbb{Z}$? Can we have $\psi(14 + 21\mathbb{Z}) = 7 + 21\mathbb{Z}$? Can we have both at the same time?

28. Let G be the group of 2×2 matrices with entries in $\mathbb{Z}/3\mathbb{Z}$, determinant not divisible by 3, and multiplication as group operation. How many elements does G have? (Ask yourself what $a, b, c, d \in \mathbb{Z}/3\mathbb{Z}$ satisfy $ad - bc = 1 \bmod 3$). Let H be the subgroup of upper triangular matrices in G. Is H normal in G?

29. The determinant is a morphism from the invertible real 2×2 matrices to the nonzero real numbers. Describe the kernel of this map. Is the kernel a normal subgroup?

30. Are the invertible symmetric 2×2 real matrices a subgroup of the group of all 2×2 real matrices?

31. Let us say that a 2×2 real matrix A is *orthogonal* if $A^T A$ is the identity matrix. Show that the set of all orthogonal matrices is a group (note that you must show two things here). Is it normal in the group of all invertible matrices?

32. Suppose 12 people sit on a round table. At a given moment, they all get up and move left by 3 seats. Write the corresponding permutation in cycle notation. What is its order? How many inversions does it have? Is it odd or even?

33. Suppose you have written a permutation in cycle notation. Describe upper and lower bounds for the order of the permutation, in terms of the cycle lengths. Document by example that both extremes can occur.

34. Write down all quotient groups that one can make from the symmetry group of a regular triangle. Repeat with a square and a regular pentagon.

35. Let G' be the group of invertible real matrices of size 2×2 with MULTIPLICATION, and G the group of strictly upper triangular real 2×2 matrices (that have the form $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$) with ADDITION. Consider the recipe $\exp\colon G \to G'$ that sends $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Show that exp is a morphism and find its kernel. What matrices in G' are not in the image of exp?

36. Find the elementary divisors of the group G given as the quotient of $\mathbb{Z}^4$ modulo the $\mathbb{Z}$-rowspan of $\begin{pmatrix} 33 & -12 & 0 & 15 \\ 0 & 6 & 9 & 27 \\ -6 & 12 & 15 & 0 \end{pmatrix}$ and write G in standard form according to FTFGAG.

37. Find the Smith normal form of the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}$.

    Any thoughts?

38. Find the number of elements of all orders in the symmetry group of a regular pentagon. Repeat with the group $\mathbb{Z}/10\mathbb{Z}$.

39. S_4 acts naturally on the vertices of a regular tetrahedron. In particular, it permutes the 6 edges. This makes a morphism ψ from S_4 to S_6. We consider one particular edge e of our tetrahedron. How many elements σ of S_4, when interpreted via ψ as elements of S_6, will have this edge as a fixed point? (Warning. This is a trick question. $\psi(\sigma)$ having e as fixed point is not the same as σ not moving the points on the edge e. Make sure you understand this before you try to solve the problem.)

40. In the previous question, what is the kernel of ψ?

41. Let G be the symmetry group of the regular triangle. Let X be all symmetries of the regular triangle. (So, X is G is you forget that you can multiply things in G). We let G act on X by conjugation. That is, if $\sigma \in G$ and $\pi \in X$ are symmetries of the regular triangle, then $\lambda(\sigma, \pi) = \sigma\pi\sigma^{-1}$.

    Is this action transitive? If not, find all orbits. Discuss in detail the Burnside theorem in this case, by carefully listing all terms that show up in the Burnside theorem.