1. Let $a = (2, 1, -2)$ and $b = (2, -1, -3)$.

(a) Apply the Gram-Schmidt procedure to construct orthonormal vectors q_1 and q_2 in the plane spanned by a and b.

(b) Find the projection of $c = (1, 2, 1)$ onto the plane spanned by a and b.

(c) Find a unit vector q_3 orthogonal to a and b.

2. Let V be the vector space spanned by the functions $\phi_1(x) = 1$, $\phi_2(x) = x$, $\phi_3(x) = x^3$ on the interval $[-1, 1]$. Use the Gram-Schmidt algorithm to find an orthogonal set of functions which span V.

3. Evaluate the determinant

$$\begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & 2 \\ 7 & 8 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix}$$

4. (a) Find a third column so that

$$U = \begin{pmatrix} 1/\sqrt{3} & i/\sqrt{2} & \dots \\ 1/\sqrt{3} & 0 & \dots \\ i/\sqrt{3} & 1/\sqrt{2} & \dots \end{pmatrix}$$

is unitary.

(b) How many solutions does problem (a) have?

5. Find the exponentials of the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

6. In the vector space of all 2×2 matrices, consider the linear operator L that transposes a matrix, that is $L(A) = A^T$.

(a) What are the eigenvalues of L ?

(b) Describe the eigenspace of each eigenvalue and find its dimension.

(c) Is L diagonalizable ?

7. True or false:
(a) Every non-singular matrix is diagonalizable.
(b) If A and B are Hermitian matrices of the same size then $A+B$ is also Hermitian.
(c) If A and B are real symmetric matrices of the same size then AB is also symmetric.
(d) For every square matrix A we have $AA^H = A^H A$.
(e) If A is symmetric and orthogonal then $A^2 = I$.

8. (a) Compute the following determinants:

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

(b) Use the results from (a) to evaluate the 101×101 determinant

$$\begin{vmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ 1 & 0 & \dots & 0 & 0 \end{vmatrix}$$

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9. One solution to $y' = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} y$ is $y = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$. Find another independent solution.

10. The system

$$y_1' = y_1 + 3y_2$$
$$y_2' = 4y_1 + 2y_2$$

is (a) unstable, (b) stable, (c) neutrally stable.