Compressible hydrodynamic flow of liquid crystals in 1-D

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Abstract

We consider the equation modeling the compressible hydrodynamic flow of liquid crystals in one dimension. When the initial density function ρ_0 has a positive lower bound, we obtain the existence and uniqueness of global classical, and strong solutions and the existence of weak solutions. For $\rho_0 \geq 0$, we obtain the existence of global strong solutions.

Key Words: Liquid crystal, compressible hydrodynamic flow, global solutions.

1 Introduction

In this paper, we consider the one dimensional initial-boundary value problem for $(\rho, u, n) : [0,1] \times [0,+\infty) \to \mathbf{R}_+ \times \mathbf{R} \times S^2$:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + (P(\rho))_x = \mu u_{xx} - \lambda(|n_x|^2)_x, \\ n_t + u n_x = \theta(n_{xx} + |n_x|^2 n), \end{cases} \tag{1.1}$$

for $(x,t) \in (0,1) \times (0,+\infty)$, with the initial condition:

$$(\rho,\ u,\ n)\big|_{t=0} = (\rho_0,\ u_0,\ n_0) \text{ in } [0,1], \tag{1.2}$$

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where $n_0 : [0,1] \to S^2$ and the boundary condition:

$$(u,\ n_x)\big|_{\partial I} = (0,\ 0),\ t > 0, \tag{1.3}$$

where $\rho \geq 0$ denotes the density function, u denotes the velocity field, n denotes the optical axis vector of the liquid crystal that is a unit vector (i.e., $|n| = 1$), $\mu > 0, \lambda > 0, \theta > 0$ are viscosity of the fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. $P = R\rho^\gamma$, for some constants $\gamma > 1$ and $R > 0$, is the pressure function.

The hydrodynamic flow of compressible (or incompressible) liquid crystals was first derived by Ericksen [2] and Leslie [3] in 1960's. However, its rigorous mathematical analysis was not taken place until 1990's, when Lin [4] and Lin-Liu [5, 6, 7] made some very important progress towards the existence of global weak solutions and partial regularity of the incompressible hydrodynamic flow equation of liquid crystals.

When the Ossen-Frank energy configuration functional reduces to the Dirichlet energy functional, the hydrodynamic flow equation of liquid crystals in $\Omega \subset \mathbf{R}^d$ can be written as follows (see Lin [4]):

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla(P(\rho)) = \mu\Delta u - \lambda \mathrm{div}(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} I_d), \\ n_t + u \cdot \nabla n = \theta(\Delta n + |\nabla n|^2 n), \end{cases} \tag{$\star$}$$

where $u \otimes u = (u^i u^j)_{1 \leq i,j \leq d}$, and $\nabla n \odot \nabla n = (n_{x_i} \cdot n_{x_j})_{1 \leq i,j \leq d}$.

Observe that for $d = 1$, the system ($\star$) reduces to (1.1). When the density function ρ is a positive constant, then ($\star$) becomes the hydrodynamic flow equation of incompressible liquid crystals (i.e., div $u = 0$). In a series of papers, Lin [4] and Lin-Liu [5, 6, 7] addressed the existence and partial regularity theory of suitable weak solution to the incompressible hydrodynamic flow of liquid crystals of variable length. More precisely, they considered the approximate equation of incompressible hydrodynamic flow of liquid crystals: (i.e., $\rho = 1$, and $|\nabla n|^2$ in $(\star)_3$ is replaced by $\dfrac{(1-|n|^2)n}{\epsilon^2}$), and proved [5], among many other results, the local existence of classical solutions and the global existence of weak solutions in dimension two and three. For any fixed $\epsilon > 0$, they also showed the existence and uniqueness of global classical solution either in dimension two or dimension three when the fluid viscosity μ is sufficiently large; in [7], Lin and Liu extended the classical theorem by Caffarelli-Kohn-Nirenberg [1] on the Navier-Stokes equation that asserts the one dimensional parabolic Hausdorff measure of the singular set of any *suitable* weak solution is zero. See also [9, 10, 18] for relevant results. For the incompressible case $\rho = 1$ and div $u = 0$, it remains to be an open problem that for $\epsilon \downarrow 0$ whether a sequence of solutions (u_ϵ, n_ϵ) to the approximate equation converges to a solution of the original equation $(\star)$. It is also a very interesting question to ask whether there exists a global weak solution to the incompressible hydrodynamic flow equation $(\star)$ similar to the Leray-Hopf type solutions in the context of Naiver-Stokes equation. We answer this question in [8] for $d = 2$.

When dealing with the compressible hydrodynamic flow equation $(\star)$, there seems very few results available. This motivates us to address the existence and uniqueness of global classical, strong solutions and the existence of weak solutions for $0 < c_0^{-1} \leq \rho_0 \leq c_0$ and the existence of strong solutions for $\rho_0 \geq 0$ when the dimension $d = 1$.

We remark that when the optical axis n is a constant unit vector, (1.1) becomes the Navier-Stokes equation for compressible isentropic flow with density-independent viscosity, which has been well studied recently. For example, the existence of global strong solutions to the compressible Navier-Stokes equation for $\rho_0 \geq 0$ was obtained by Choe-Kim [17] in one dimension and by Choe-Kim [16] in higher dimensions. Notice that $\rho = 0$ corresponds to the vacuum state, whose existence makes the analysis much more complicated. Okada [12] investigated the free boundary problem for one-dimensional Navier-Stokes equations with one boundary fixed and the other connected to vacuum and proved the existence of global weak solutions. Luo, Xin, and Yang [13] studied the free boundary value problem of the one-dimensional viscous gas which expands into the vacuum and established the regularity, behaviors of weak solutions near the interfaces (separating the gas and vacuum) and expanding rate of the interfaces. The reader can also refer to [14, 15] for related works.

Notations:

(1) $I = [0, 1]$, $\partial I = \{0, 1\}$, $Q_T = I \times [0, T]$ for $T > 0$.

(2) For $p \geq 1$, denote $L^p = L^p(I)$ as the L^p space with the norm $\|\cdot\|_{L^p}$. For $k \geq 1$ and $p \geq 1$, denote $W^{k,p} = W^{k,p}(I)$ for the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, $H^k = W^{k,2}(I)$, and

$$W^{k,p}(I, S^2) = \left\{ u \in W^{k,p}(I, \mathbf{R}^3) \mid |u(x)| = 1 \text{ a.e. } x \in I \right\}.$$

(3) For an even integer $k \geq 0$ and $0 < \alpha < 1$, let $C^{k+\alpha, \frac{k+\alpha}{2}}(Q_T)$ denote the Schauder space of functions on Q_T, whose derivatives up to kth order in x-variable and up to $\frac{k}{2}$th order in t-variable are Hölder continuous with exponents α and $\frac{\alpha}{2}$ respectively, with the norm $\|\cdot\|_{C^{k+\alpha, \frac{k+\alpha}{2}}}$.

Our first main theorem is concerned with the existence of global classical solutions.

Theorem 1.1 *For $0 < \alpha < 1$, assume that $\rho_0 \in C^{1,\alpha}$ satisfies $0 < c_0^{-1} \leq \rho_0 \leq c_0$ for some c_0, $u_0 \in C^{2,\alpha}$ and $n_0 \in C^{2,\alpha}(I, S^2)$. Then there exists a unique global classical solution $(\rho, u, n) : I \times \mathbf{R}_+ \to \mathbf{R}_+ \times \mathbf{R} \times S^2$ to the initial boundary value problem (1.1)-(1.3) satisfying that for any $T > 0$ there exists $c = c(c_0, T) > 0$ such that*

$$(\rho_x,\ \rho_t) \in C^{\alpha, \frac{\alpha}{2}}(Q_T),\ c^{-1} \leq \rho \leq c,\ (u,\ n) \in C^{2+\alpha, \frac{2+\alpha}{2}}(Q_T).$$

Our second main theorem is concerned with the existence of global strong solutions and weak solutions under the assumption that $\rho_0 \in H^1(I)$ satisfies $0 < c_0^{-1} \leq \rho_0 \leq c_0$. More precisely,

Theorem 1.2 *(i) If $\rho_0 \in H^1$, $0 < c_0^{-1} \leq \rho_0 \leq c_0$ for some c_0, $u_0 \in L^2(I)$, and $n_0 \in H^1(I, S^2)$, then there exists a global weak solution $(\rho, u, n) : I \times \mathbf{R}_+ \to \mathbf{R}_+ \times \mathbf{R} \times S^2$ to (1.1)-(1.3) such that for any $T > 0$ there exists $c = c(c_0, T) > 0$ such that*

$$\rho_x \in L^\infty(0, T; L^2),\ \rho_t \in L^2(0, T; L^2),\ 0 < c^{-1} \leq \rho \leq c,$$

$$u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1),$$

$$n \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2).$$

(ii) If $\rho_0 \in H^1$, $0 < c_0^{-1} \leq \rho_0 \leq c_0$ for some c_0, $u_0 \in H_0^1$, and $n_0 \in H^2(I, S^2)$, then there exists a unique global strong solution to the initial boundary value problem

(1.1)-(1.3) satisfying that for any $T > 0$ there exists $c = c(c_0, T) > 0$ such that

$$\rho \in L^\infty(0,T;H^1), \quad \rho_t \in L^\infty(0,T;L^2), \quad 0 < c^{-1} \le \rho \le c,$$
$$u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2), \quad u_t \in L^2(0,T;L^2),$$
$$n \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \quad n_t \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2).$$

When ρ_0 is only nonnegative, we establish the existence of global strong solutions to (1.1)-(1.3).

Theorem 1.3 *Assume $0 \le \rho_0 \in H^1$, $u_0 \in H_0^1$ and $n_0 \in H^2(I, S^2)$. Then there exists a global strong solution to the initial boundary value problem (1.1)-(1.3) such that for any $T > 0$,*

$$\rho \in L^\infty(0,T;H^1), \quad \rho_t \in L^\infty(0,T;L^2), \quad \rho \ge 0,$$
$$u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2), \quad \sqrt{\rho} u_t \in L^2(0,T;L^2),$$
$$n \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \quad n_t \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2).$$

Remark 1.1 (i) It is unknown whether the strong solution obtained in Theorem 1.3 is unique. If, in additions, $u_0 \in H^2$ satisfies the compatibility condition:

$$(\mu u_{0x})_x - (P(\rho_0))_x - \lambda(|n_{0x}|^2)_x = \rho_0^{\frac{1}{2}} g,$$

for some $g \in L^2$, then, by a method similar to [16] or [17], we can prove that the strong solution (ρ, u, n) obtained in Theorem 1.2 satisfies

$$u \in L^\infty(0,T;H^2), \ \sqrt{\rho} u_t \in L^\infty(0,T;L^2), u_t \in L^2(0,T;H^1)$$

and hence the uniqueness of solutions can be shown by the argument similar to Theorem 1.2 and that of [16] and [17].
(ii) We believe that there exists a global weak solution (ρ, u, n) to (1.1)-(1.3) under the assumption that $0 \le \rho_0 \in L^\gamma$, $u_0 \in L^2$, and $n_0 \in H^1(I, S^2)$. This will be discussed in a forthcoming paper.

Since the constant R and μ, λ, θ in (1.1) don't play any role in the analysis, we assume henceforth that

$$\mu = \lambda = \theta = R = 1.$$

The paper is organized as follows. In section 2, we prove the existence of the short time classical solutions of (1.1). In section 3, we derive some a priori estimates for classical solutions of (1.1), and prove the existence and uniqueness for both classical and strong solutions and the existence of weak solutions to (1.1)-(1.3) for $\rho_0 \geq c_0 > 0$. In section 4, we prove the existence of strong solution for $\rho_0 \geq 0$.

2 Existence of local classical solutions

In this section, we employ the contraction mapping theorem to prove that there exists a unique short time classical solution to (1.1)-(1.3) when ρ_0 has a positive lower bound.

The main result of this section can be stated as follows.

Theorem 2.1 *For $\alpha \in (0,1)$, assume that $\rho_0 \in C^{1,\alpha}$ satisfies $0 < c_0^{-1} \leq \rho_0 \leq c_0$, $u_0 \in C^{2,\alpha}$, and $n_0 \in C^{2,\alpha}(I, S^2)$. Then there exists $T > 0$ depending on ρ_0, u_0, n_0 such that the initial boundary value problem (1.1)-(1.3) has a unique classical solution $(\rho, u, n) : I \times [0,T) \to \mathbf{R}_+ \times \mathbf{R} \times S^2$ satisfying*

$$(\rho_x,\ \rho_t) \in C^{\alpha,\frac{\alpha}{2}}(Q_T),\ c^{-1} \leq \rho \leq c,\ (u,\ n) \in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T).$$

We may assume throughout this section that

$$\int_0^1 \rho_0(\xi) d\xi = 1. \tag{2.1}$$

To prove Theorem 2.1, we introduce for any $T > 0$ the Lagrangian coordinate (y, τ) on $I \times [0,T)$:

$$y(x,t) = \int_0^x \rho(\xi,t) d\xi,\ \tau(x,t) = t.$$

It is easy to check that $(x,t) \to (y,\tau)$ is a C^1-bijective map from $I \times [0,T) \to I \times [0,T)$, provide $\rho(x,t) \in C^1(I \times [0,T))$ is positive and $\int_0^1 \rho(\xi,t)\, d\xi = 1$ for all $t \in [0,T)$.

Direct calculations imply

$$\frac{\partial}{\partial t} = -\rho u \frac{\partial}{\partial y} + \frac{\partial}{\partial \tau},\ \ \frac{\partial}{\partial x} = \rho \frac{\partial}{\partial y},$$

and $(\rho, u, n)(x,t)$ solves (1.1)-(1.3) is equivalent to $(\rho, u, n)(y, \tau) := (\rho, u, n)(x,t)$ solves the following system:

$$\begin{cases} \rho_\tau + \rho^2 u_y = 0, \\ u_\tau + (P(\rho))_y = (\rho u_y)_y - (\rho^2 |n_y|^2)_y, \\ n_\tau = \rho(\rho n_y)_y + \rho^2 |n_y|^2 n, \end{cases} \tag{2.2}$$

and

$$(\rho, u, n)|_{\tau=0} = (\rho_0, u_0, n_0), \text{ in } I \tag{2.3}$$

$$(u, n_y)|_{\partial I} = 0, \ \tau > 0. \tag{2.4}$$

Now we use the contraction mapping theorem to establish the existence and uniqueness of local, classical solutions to (2.2)-(2.4).

Proof of Theorem 2.1:

Let $\alpha > 0$ and (ρ_0, u_0, n_0) be given by Theorem 2.1. For $K > 0$ and $T > 0$, to be determined later, define the space $X = X(T, K)$ by

$$\Big\{(v, m) : Q_T \to \mathbf{R} \times \mathbf{R^3} \mid (v, m) \in C^{2+\alpha, \frac{2+\alpha}{2}}, \ (v, m)\big|_{\tau=0} = (u_0, n_0), \ \|(v, m)\|_X \le K\Big\}$$

where

$$\|(v, m)\|_X := \|v\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(Q_T)} + \|m\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(Q_T)}.$$

It is evident that $(X, \|\cdot\|_X)$ is a Banach space. For any $(v, m) \in X$, we let $(\rho, u, n) : I \times [0, T) \to \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}^3$ solve the following system:

$$\begin{cases} \rho_\tau + \rho^2 v_y = 0, \\ u_\tau + (P(\rho))_y = (\rho u_y)_y - (\rho^2 |n_y|^2)_y, \\ n_\tau = \rho(\rho n_y)_y + \rho^2 |m_y|^2 n. \end{cases} \tag{2.2$'$}$$

along with the initial-boundary condition:

$$(\rho, u, n)\big|_{\tau=0} = (\rho_0, u_0, n_0); \ (\rho, u, n_x)\big|_{\partial I} = (\rho_0, 0, 0), \ \tau > 0. \tag{2.5}$$

The first equation of (2.2)$'$ yields that

$$\rho(y, \tau) = \frac{\rho_0(y)}{1 + \rho_0(y) \int_0^\tau v_y(y, s) ds}. \tag{2.6}$$

Since $(v,m)\in X$, we have $\|v\|_{C^1(Q_T)}\le K$ and hence (2.6) implies

$$\rho \le \frac{\rho_0}{1-|\rho_0\int_0^\tau v_y(y,s)ds|} \le \frac{c_0}{1-c_0KT} \le 2c_0, \tag{2.7}$$

$$\rho \ge \frac{\rho_0}{1+|\rho_0\int_0^\tau v_y(y,s)ds|} \ge \frac{c_0}{1+c_0KT} \ge \frac{c_0}{2}, \tag{2.8}$$

provided

$$T \le T_0 \equiv \frac{1}{2c_0K}.$$

Since $v\in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)$ and $\rho_0\in C^{1+\alpha}(I)$, (2.6) implies that $\rho,\rho_y\in C^{\alpha,\frac{\alpha}{2}}(Q_T)$. Since $m\in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)$, $\rho,\rho_y\in C^{\alpha,\frac{\alpha}{2}}(Q_T)$, it follows from (2.7), (2.8), and the Schauder theory that there is a unique solution (ρ,u,n), with $(\rho,u,n)\in C^{\alpha,\frac{\alpha}{2}}(Q_T)\times C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)\times C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)$ to $(2.2)'$ and (2.5). Define the solution map:

$$\mathbf{H}(v,m)=(u,n): X\to C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T,\mathbf{R}\times\mathbf{R}^3).$$

Claim. *There exist sufficiently large $K>0$ and sufficiently small $T>0$ such that $\mathbf{H}: X\to X$ is a contraction map.*

Proof of Claim. We will first prove that $\mathbf{H}$ maps X into X. Set $C_1=\|\rho_0\|_{C^{1,\alpha}}+\|u_0\|_{C^{2,\alpha}}+\|n_0\|_{C^{2,\alpha}}$.

Direct differentiation of (2.6) implies that

$$\rho_y=\frac{\rho_{0y}}{1+\rho_0\int_0^\tau v_y(y,s)ds}-\frac{\rho_0\rho_{0y}\int_0^\tau v_y(y,s)ds+\rho_0^2\int_0^\tau v_{yy}(y,s)ds}{(1+\rho_0\int_0^\tau v_y(y,s)ds)^2} \tag{2.9}$$

It follows from (2.6) and (2.9) that

$$\max\left\{\|\rho\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)},\ \|\rho_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\right\}\le c(C_1), \tag{2.10}$$

where $T\le T_1:=\min\{\frac{1}{2c_0K},(\frac{1}{K})^{\frac{2}{2-\alpha}}\}$.

Apply the Schauder theory to $(2.1)'_3$, we obtain that for any $T\le T_1$,

$$\|n\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}\le C(\|n_0\|_{C^{2+\alpha}(I)}+\|\rho^2|m_y|^2n\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}). \tag{2.11}$$

Direction calculations imply

$$\|n\|_{C^0(Q_T)}\le 1+KT,\ [n]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\le c(C_1)(1+KT^{\frac{\alpha}{2}}), \tag{2.12}$$

$$\|m_y\|_{C^0(Q_T)}\le\|m_y-m_{0y}\|_{C^0(Q_T)}+\|m_{0y}\|_{C^0(Q_T)}\le c(C_1)(1+KT^{\frac{\alpha}{2}}), \tag{2.13}$$

$$[m_y]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le [m_y - m_{0y}]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + [m_{0y}]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le c(C_1)(1 + KT^{\frac{\alpha}{2}}). \tag{2.14}$$

Hence if we choose $T \le T_2 = \min\{T_1, (\frac{1}{K})^{\frac{2}{\alpha}}\}$, then

$$\|\rho^2 |m_y|^2 n\|_{C^0(Q_T)} \le c(C_1), \tag{2.15}$$

and

$$\begin{aligned} & [\rho^2 |m_y|^2 n]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ \le \ & \|\rho\|^2_{C^0(Q_T)} \|m_y\|^2_{C^0(Q_T)} [n]_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|m_y\|^2_{C^0(Q_T)} \|n\|_{C^0(Q_T)} \|\rho\|^2_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ & + \|m_y\|^2_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \|\rho\|^2_{C^0(Q_T)} \|n\|_{C^0(Q_T)} \\ \le \ & c(C_1). \end{aligned} \tag{2.16}$$

It follows from (2.11), (2.15) and (2.16) that for $T \le T_2$,

$$\|n\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le c(C_1). \tag{2.17}$$

Now we need to estimate $\|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}$ as follows. It follows from $(2.1)'_2$ and the Schauder theory that

$$\begin{aligned} \|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le \ & C(\|u_0\|_{C^{2+\alpha}(I)} + \|(\rho^\gamma)_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|(\rho^2 |n_y|^2)_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ \le \ & c(C_1)[1 + \|\rho\|^{\gamma-1}_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \|\rho_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ & + \|\rho\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \|\rho_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \|n_y\|^2_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ & + \|\rho\|^2_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \|n_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \|n_{yy}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}] \\ \le \ & c(C_1), \end{aligned}$$

provide $T \le T_2$. Thus we conclude that if $K > 0$ is sufficiently large and $T \le T_2$, then $\mathbf{H}$ maps X into X.

Next we want to show $\mathbf{H} : X \to X$ is a contraction map. For $i = 1, 2$, let $(v_i, m_i) \in X$, and $(u_i, n_i) = \mathbf{H}(v_i, m_i)$. Set $\overline{\rho} = \rho_1 - \rho_2$, $\overline{v} = v_1 - v_2$, $\overline{m} = m_1 - m_2$, $\overline{u} = u_1 - u_2$ and $\overline{n} = n_1 - n_2$. Then it is easy to see

$$\left(\frac{\overline{\rho}}{\rho_1 \rho_2}\right)_\tau = -\overline{v}_y. \tag{2.18}$$

Since $\overline{\rho}|_{\tau=0} = 0$, integrating (2.18) with respect to τ yields

$$\overline{\rho} = -\rho_1 \rho_2 \int_0^\tau \overline{v}_y \, ds. \tag{2.19}$$

Since both ρ_1 and ρ_2 satisfy (2.10), we obtain

$$\max\{\|\overline{\rho}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}, \|\overline{\rho}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\} \le c(C_1)T^{1-\frac{\alpha}{2}}\|\overline{v}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}. \tag{2.20}$$

For $\bar{n}$, we have

$$\begin{aligned}\overline{n}_\tau &= \rho_1^2\overline{n}_{yy} + \overline{\rho}(\rho_1+\rho_2)n_{2yy} + \overline{\rho}\rho_{1y}n_{1y} + \rho_2\overline{\rho}_y n_{1y} + \rho_2\rho_{2y}\overline{n}_y \\ &\quad + \overline{\rho}(\rho_1+\rho_2)|m_{1y}|^2 n_1 + \rho_2^2\overline{m}_y(m_{1y}+m_{2y})n_1 + \rho_2^2|m_{2y}|^2\overline{n}. \end{aligned} \tag{2.21}$$

Since ρ_i, m_i, n_i $(i = 1, 2)$ satisfy (2.11)-(2.12), (2.13)-(2.14), we have by the Schauder theory that for $T \le T_2$,

$$\begin{aligned}\|\overline{n}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} &\le c(C_1)[\|\overline{\rho}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|\overline{\rho}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|\overline{n}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ &\quad + \|\overline{n}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|\overline{m}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}] \\ &\le c(C_1)[T^{1-\frac{\alpha}{2}}\|\overline{v}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + \|\overline{n}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ &\quad + \|\overline{n}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|\overline{m}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}]\end{aligned}$$

Since $\overline{m}|_{\tau=0} = \overline{n}|_{\tau=0} = 0$, we see that

$$\max\{\|\overline{n}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}, \|\overline{n}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\} \le CT^{\frac{\alpha}{2}}\|\overline{n}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)},$$

and

$$\|\overline{m}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le CT^{\frac{\alpha}{2}}\|\overline{m}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}.$$

Thus we obtain

$$\|\overline{n}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le C(C_1)T^{\frac{\alpha}{2}}[\|\overline{v}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + \|\overline{m}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}]. \tag{2.22}$$

Finally, we need to estimate $\|\overline{u}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}$ as follows. For $\overline{u}$, we have that

$$\overline{u}_\tau + (\rho_1^\gamma - \rho_2^\gamma)_y = (\overline{\rho}u_{1y})_y + (\rho_2\overline{u}_y)_y - [\overline{\rho}(\rho_1+\rho_2)|n_{1y}|^2]_y - [\rho_2^2(n_{1y}+n_{2y}).\overline{n}_y]_y. \tag{2.23}$$

Hence, by the Schauder theory, we have

$$\begin{aligned}\|\overline{u}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} &\le c(C_1)[\|\overline{\rho}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|\overline{\rho}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|(\rho_1^\gamma - \rho_2^\gamma)_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ &\quad + \|\overline{n}_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|\overline{n}_{yy}\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}] \\ &\le c(C_1)T^{\frac{\alpha}{2}}[\|\overline{v}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + \|\overline{m}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}],\end{aligned} \tag{2.24}$$

where we have used both (2.11) and (2.22) in the last step. It follows from (2.22) and (2.24) that if $T \le T_3 = \min\{T_2, (\frac{1}{4c(C_1)})^{\frac{2}{\alpha}}\}$, then

$$\|\overline{u}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + \|\overline{n}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le \frac{1}{2}(\|\overline{v}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + \|\overline{m}\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}).$$

Therefore $\mathbf{H}$ is a contraction map. By the fixed point theorem, there exists a unique $(u, n) \in X$ such that $\mathbf{H}(u, n) = (u, n)$. Furthermore, there is a unique ρ, with $\rho_y, \rho_\tau \in C^{\alpha,\frac{\alpha}{2}}(Q_{T_3})$, solving

$$\rho_\tau + \rho^2 u_y = 0$$

Hence (ρ, u, n) is the unique classical solution to problem (2.2)-(2.4) on $I \times [0, T_3]$. Multiplying $(2.2)_3$ by n and applying the Gronwall's inequality, we can obtain $|n| = 1$ for $(y, \tau) \in Q_{T_3}$. The proof of Theorem 2.1 is completed. □

3 Existence of global classical, strong, and weak solutions

In this section, we first prove that the short time classical solution, obtained in Theorem 2.1, can be extended to a global classical solution. This is based on several integral estimates for the short time classical solutions. The global weak, and strong solution are then achieved by both the approximation scheme and integral type a priori estimates for classical solutions.

For $0 < T < +\infty$, let $(\rho, u, n) : I \to [0, T) \to \mathbf{R}_+ \times \mathbf{R} \times S^2$ be the classical solutions obtained by Theorem 2.1. The first estimate we have is the energy inequality.

Lemma 3.1 *For any $0 \le t < T$, it holds*

$$\begin{aligned} &\int_I (\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + |n_x|^2)(t) + \int_0^t \int_I (u_x^2 + 2\left|n_{xx} + |n_x|^2 n\right|^2) \\ = &\int_I (\frac{\rho_0 u_0^2}{2} + \frac{\rho_0^\gamma}{\gamma - 1} + |(n_0)_x|^2). \end{aligned} \tag{3.1}$$

Proof. Multiplying $(1.1)_2$ by u and integrating the resulting equation over I, we get

$$\frac{d}{dt}\int_I \frac{\rho u^2}{2} - \int_I \rho^\gamma u_x + \int_I u_x^2 = \int_I |n_x|^2 u_x. \tag{3.2}$$

Now we claim that

$$-\int_I \rho^\gamma u_x = \frac{d}{dt}\int_I \frac{\rho^\gamma}{\gamma-1}. \tag{3.3}$$

In fact, by $(1.1)_1$, we have

$$\begin{aligned}
-\int_I \rho^\gamma u_x &= \int_I \rho^{\gamma-1}(\rho_t+\rho_x u) \\
&= \frac{d}{dt}\int_I \frac{\rho^\gamma}{\gamma} + \int_I (\frac{\rho^\gamma}{\gamma})_x u \\
&= \frac{d}{dt}\int_I \frac{\rho^\gamma}{\gamma} - \int_I \frac{\rho^\gamma}{\gamma} u_x.
\end{aligned}$$

This clearly yields (3.3).

Multiplying $(1.1)_3$ by $(n_{xx}+|n_x|^2 n)$ and integrating it over I, we obtain

$$\frac{d}{dt}\int_I |n_x|^2 + \int_I |n_x|^2 u_x + 2\int_I \left|n_{xx}+|n_x|^2 n\right|^2 = 0. \tag{3.4}$$

It follows from (3.2), (3.3), and (3.4) that

$$\frac{d}{dt}\int_I (\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma-1} + |n_x|^2) + \int_I u_x^2 + 2\int_I \left|n_{xx}+|n_x|^2 n\right|^2 = 0.$$

This clearly implies (3.1). The proof is complete. □

Lemma 3.2 *For any $0 \le t < T$, it holds*

$$\int_0^t \int_I |n_{xx}|^2 \le c(E_0)(1+t), \tag{3.5}$$

where

$$E_0 := \int_I (\frac{\rho_0 u_0^2}{2} + \frac{\rho_0^\gamma}{\gamma-1} + |(n_0)_x|^2)$$

denotes the total energy of the initial data.

Proof. By the Gagliardo-Nirenberg inequality, we have

$$\int_I |n_x|^4 \le c\|n_x\|^3_{L^2(I)}\|n_{xx}\|_{L^2(I)} \le \frac{1}{2}\int_I |n_{xx}|^2 + c(\int_I |n_x|^2)^3.$$

Since $|n| = 1$, this implies

$$\begin{aligned}
\int_I |n_{xx}|^2 &= \int_I |n_{xx}+|n_x|^2 n|^2 + \int_I |n_x|^4 \\
&\le \frac{1}{2}\int_I |n_{xx}|^2 + \int_I |n_{xx}+|n_x|^2 n|^2 + c(\int_I |n_x|^2)^3.
\end{aligned}$$

Hence

$$\int_I |n_{xx}|^2 \le 2\int_I |n_{xx} + |n_x|^2 n|^2 + c(\int_I |n_x|^2)^3.$$

This, combined with (3.1), yields (3.5). The proof of the Lemma is complete. □

Now we want to estimate $\|n_{xx}\|_{L^\infty([0,T],L^2(I))}$ in terms of both E_0 and $\|n_0\|_{H^2(I)}$ as follows.

Lemma 3.3 *For any $0 \le t < T$, it holds*

$$\int_I |n_{xx}|^2(t) + \int_0^t \int_I (|n_{xt}|^2 + |n_{xxx}|^2) \le c(E_0, \|n_0\|_{H^2(I)}) \exp\{c(E_0)t\}. \tag{3.6}$$

Proof. Differentiating $(1.1)_3$ with respect to x, multiplying the resulting equation by n_{xt}, and integrating it over I, we have

$$\begin{aligned}
&\int_I |n_{xt}|^2 + \frac{d}{dt}\int_I \frac{1}{2}|n_{xx}|^2 \\
&= \int_I [|n_x|^2 n_x \cdot n_{xt} + 2(n_x \cdot n_{xx})n \cdot n_{xt}] - \int_I u_x n_x \cdot n_{xt} - \int_I u n_{xx} \cdot n_{xt} \\
&\le \frac{1}{2}\int_I |n_{xt}|^2 + c\int_I (|n_x|^6 + |n_x|^2|n_{xx}|^2 + u_x^2|n_x|^2 + u^2|n_{xx}|^2).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_I |n_{xt}|^2\, dx + \frac{d}{dt}\int_I |n_{xx}|^2 \\
&\le c\int_I (|n_x|^6 + |n_x|^2|n_{xx}|^2 + u_x^2|n_x|^2 + u^2|n_{xx}|^2) \\
&\le c\left[\|n_x\|_{L^\infty}^4 \int_I |n_x|^2 + \|n_x\|_{L^\infty}^2 \int_I u_x^2 + (\|n_x\|_{L^\infty}^2 + \|u\|_{L^\infty}^2)\int_I |n_{xx}|^2\right] \\
&\le c(1+E_0)(\int_I |n_{xx}|^2)^2 + c\int_I u_x^2 \int_I |n_{xx}|^2,
\end{aligned}$$

where we have used the Sobolev embedding inequality:

$$\|n_x\|_{L^\infty(I)} \le c\|n_{xx}\|_{L^2(I)},\ \|u\|_{L^\infty(I)} \le c\|u_x\|_{L^2(I)}.$$

This, combined with Lemma 3.1, 3.2 and the Gronwall inequality, implies that any $t \in [0,T)$,

$$\int_I |n_{xx}|^2(t) + \int_0^t \int_I |n_{xt}|^2 \le c(E_0, \|n_0\|_{H^2(I)}) \exp\{c(E_0)t\}.$$

Since

$$n_{xxx} = n_{xt} + u_x n_x + u n_{xx} - 2(n_x \cdot n_{xx})n - |n_x|^2 n_x,$$

we also obtain

$$\int_0^t \int_I |n_{xxx}|^2 \le c(E_0, \|n_0\|_{H^2(I)}) \exp\{c(E_0)t\}.$$

The proof is now completed. □

Now we want to improve the estimation of both lower and upper bounds of ρ in terms of E_0, $\|\rho_0\|_{H^1}$, and the upper and lower bounds of ρ_0. This turns out to be the most difficult step.

Lemma 3.4 *There exist $c = c(c_0, \|\rho_0\|_{H^1}, E_0, \gamma) > 0$ and $C = C(c_0, \|\rho_0\|_{H^1}, E_0) > 0$ depending only on c_0, $\|\rho_0\|_{H^1}$, γ, and E_0 such that for any $0 \le t < T$, we have*

$$\int_I \rho |(\frac{1}{\rho})_x|^2(t) + \int_0^t \int_I \rho^{\gamma-3}\rho_x^2 \le C\exp(Ct), \tag{3.7}$$

and

$$\frac{1}{c\exp(Ct)} \le \rho(x,t) \le c\exp(Ct),\ \ x \in I. \tag{3.8}$$

Proof. Using $(1.1)_1$, we have

$$\begin{aligned}
\frac{d}{dt}\int_I \rho|(\frac{1}{\rho})_x|^2 &= \int_I \rho_t|(\frac{1}{\rho})_x|^2 + 2\int_I \rho(\frac{1}{\rho})_x(\frac{1}{\rho})_{xt} \\
&= -\int_I (\rho u)_x|(\frac{1}{\rho})_x|^2 + 2\int_I \rho(\frac{1}{\rho})_x(-\frac{\rho_t}{\rho^2})_x \\
&= -\int_I (\rho u)_x|(\frac{1}{\rho})_x|^2 + 2\int_I \rho(\frac{1}{\rho})_x(\frac{(\rho u)_x}{\rho^2})_x \\
&= -\int_I (\rho u)_x|(\frac{1}{\rho})_x|^2 + 2\int_I \rho(\frac{1}{\rho})_x[((-\frac{1}{\rho})_x u)_x + (\frac{u_x}{\rho})_x] \\
&= -\int_I (\rho u)_x|(\frac{1}{\rho})_x|^2 + 2\int_I [-\rho|(\frac{1}{\rho})_x|^2 u_x + \frac{1}{2}|(\frac{1}{\rho})_x|^2\rho_x u \\
&\quad +\frac{1}{2}|(\frac{1}{\rho})_x|^2\rho u_x] + 2\int_I \rho|(\frac{1}{\rho})_x|^2 u_x + 2\int_I (\frac{1}{\rho})_x u_{xx} \\
&= 2\int_I (\frac{1}{\rho})_x u_{xx}.
\end{aligned}$$

Thus we obtain

$$\frac{1}{2}\frac{d}{dt}\int_I \rho|(\frac{1}{\rho})_x|^2 = \int_I (\frac{1}{\rho})_x u_{xx}. \tag{3.9}$$

Multiplying $(1.1)_2$ by $(\frac{1}{\rho})_x$, integrating the resulting equation over I, and utilizing

(3.9), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho|(\frac{1}{\rho})_x|^2+\gamma\int_I \rho^{\gamma-3}\rho_x^2+\frac{d}{dt}\int_I \rho u(-\frac{1}{\rho})_x\\
&=\frac{1}{2}\frac{d}{dt}\int_I \rho|(\frac{1}{\rho})_x|^2-\int_I(\frac{1}{\rho})_x(\rho^\gamma)_x-\frac{d}{dt}\int_I \rho u(\frac{1}{\rho})_x\\
&=\int_I(\frac{1}{\rho})_x(|n_x|^2)_x-\int_I \rho u(\frac{1}{\rho})_{xt}+\int_I(\rho u^2)_x(\frac{1}{\rho})_x\\
&=2\int_I(\frac{1}{\rho})_x n_x\cdot n_{xx}-\int_I(\rho u)_x\frac{\rho_t}{\rho^2}-\int_I(\rho u^2)_x\frac{\rho_x}{\rho^2}\\
&\le\int_I\frac{1}{\rho}|n_{xx}|^2+\int_I\rho|(\frac{1}{\rho})_x|^2|n_x|^2+\int_I[\frac{1}{\rho^2}(|(\rho u)_x|^2-(\rho u^2)_x\rho_x)]\\
&\le\|\frac{1}{\rho}\|_{L^\infty}\int_I|n_{xx}|^2+\|n_x\|_{L^\infty}^2\int_I\rho|(\frac{1}{\rho})_x|^2+\int_I u_x^2\\
&\le[\|\frac{1}{\rho}\|_{L^\infty}+\int_I\rho|(\frac{1}{\rho})_x|^2]\int_I|n_{xx}|^2+\int_I u_x^2. \qquad (3.10)
\end{aligned}
$$

Since $(1.1)_1$ implies $\int_I\rho(\xi,t)=\int_I\rho_0(\xi)=1$, there exists $a(t)\in I$ such that $\rho(a(t),t)=\int_I\rho(\xi,t)d\xi=1$. Hence we have

$$
\begin{aligned}
\frac{1}{\rho(x,t)} &= \frac{1}{\rho(a(t),t)}+\int_{a(t)}^x(\frac{1}{\rho(\xi,t)})_\xi\\
&= 1+\int_{a(t)}^x\frac{-\rho_\xi}{\rho^2}\\
&\le 1+\|\frac{1}{\rho}\|_{L^\infty}^{\frac{1}{2}}(\int_I\frac{|\rho_x|^2}{\rho^3})^{\frac{1}{2}}\\
&\le 1+\frac{1}{2}\|\frac{1}{\rho}\|_{L^\infty}+\frac{1}{2}\int_I\rho|(\frac{1}{\rho})_x|^2.
\end{aligned}
$$

Taking the supremum over $x\in I$, this implies

$$
\|\frac{1}{\rho}\|_{L^\infty}\le 2+\int_I\rho|(\frac{1}{\rho})_x|^2. \qquad (3.11)
$$

Plugging (3.11) into (3.10), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I\rho|(\frac{1}{\rho})_x|^2+\gamma\int_I\rho^{\gamma-3}\rho_x^2+\frac{d}{dt}\int_I\rho u(-\frac{1}{\rho})_x\\
\le\ & c[\int_I|n_{xx}|^2+\int_I|n_{xx}|^2\int_I\rho|(\frac{1}{\rho})_x|^2]+\int_I u_x^2. \qquad (3.12)
\end{aligned}
$$

Integrating (3.12) over $(0,t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_I \rho|(\frac{1}{\rho})_x|^2+\gamma\int_0^t\int_I \rho^{\gamma-3}\rho_x^2\\
&\le \int_I \rho u(\frac{1}{\rho})_x+\frac{1}{2}\int_I \rho_0|(\frac{1}{\rho_0})_x|^2-\int_I \rho_0 u_0(\frac{1}{\rho_0})_x\\
&+c\int_0^t\int_I (u_x^2+|n_{xx}|^2)+c\int_0^t[\int_I |n_{xx}|^2\int_I \rho|(\frac{1}{\rho})_x|^2]\\
&\le \frac{1}{4}\int_I \rho|(\frac{1}{\rho})_x|^2+c\int_0^t[\int_I |n_{xx}|^2\int_I \rho|(\frac{1}{\rho})_x|^2]\\
&+c[\int_I \rho u^2+\int_0^t\int_I (u_x^2+|n_{xx}|^2)]+\frac{1}{2}\int_I \rho_0|(\frac{1}{\rho_0})_x|^2-\int_I \rho_0 u_0(\frac{1}{\rho_0})_x.
\end{aligned}$$

Since

$$\begin{aligned}
&c[\int_I \rho u^2+\int_0^t\int_I (u_x^2+|n_{xx}|^2)]+\frac{1}{2}\int_I \rho_0|(\frac{1}{\rho_0})_x|^2-\int_I \rho_0 u_0(\frac{1}{\rho_0})_x\\
&\le c(E_0)(1+t)+\frac{1}{2}\|\frac{1}{\rho_0}\|_{L^\infty}^3\|\rho_0\|_{H^1}^2+\|\frac{1}{\rho_0}\|_{L^\infty}^{\frac{3}{2}}\|\rho_0 u_0^2\|_{L^1}\|\rho_0\|_{H^1}\\
&\le C(E_0,c_0,\|\rho_0\|_{H^1})(1+t),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_I \rho|(\frac{1}{\rho})_x|^2+\gamma\int_0^t\int_I \rho^{\gamma-3}\rho_x^2\\
&\le C(E_0,c_0,\|\rho_0\|_{H^1})(1+t)+c\int_0^t(\int_I |n_{xx}|^2\int_I \rho|(\frac{1}{\rho})_x|^2).
\end{aligned}$$

Since Lemma 3.2 implies $\int_0^t\int_I |n_{xx}|^2 \le c(E_0)(1+t)$, we have, by the Gronwall's inequality, that

$$\begin{aligned}
\int_I \rho|(\frac{1}{\rho})_x|^2+\int_0^t\int_I \rho^{\gamma-3}\rho_x^2 &\le C(E_0,c_0,\|\rho_0\|_{H^1})(1+t^2)\exp(c\int_0^t\int_I |n_{xx}|^2)\\
&\le C(E_0,c_0,\|\rho_0\|_{H^1})\exp\{c(E_0)t\}.
\end{aligned}$$

This yields (3.7). It follows from (3.11) and (3.7) that

$$\rho \ge \frac{1}{c\exp\{c(E_0)t\}},\ \forall (x,t)\in I\times[0,T).$$

Since $\gamma>1$, we can write $\gamma=1+2\delta$ for some $\delta>0$. Hence we have

$$\begin{aligned}
\|\rho^\delta\|_{L^\infty} &\le \int_I \rho^\delta+\delta\int_I \rho^{\delta-1}|\rho_x|\\
&\le (\int_I \rho^\gamma)^{\frac{\delta}{\gamma}}+\delta(\int_I \rho^\gamma)^{\frac{1}{2}}(\int_I \rho|(\frac{1}{\rho})_x|^2)^{\frac{1}{2}}\\
&\le c(E_0,c_0,\|\rho_0\|_{H^1},\gamma)\exp\{c(E_0)t\},
\end{aligned}$$

where we have used (3.7) in the last step. This clearly yields (3.8). The proof is now complete. □

Lemma 3.5 *There exists* $C = C(\gamma, E_0, c_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^1}, \|n_0\|_{H^2}) > 0$ *such that for any* $0 \le t < T$,

$$\int_I u_x^2(t) + \int_0^t \int_I (u_t^2 + u_{xx}^2) \le C \exp\{C \exp(Ct)\}. \tag{3.13}$$

Proof. It follows from $(1.1)_1$ and $(1.1)_2$ that

$$\rho u_t + \rho u u_x + \gamma \rho^{\gamma-1} \rho_x = u_{xx} - 2 n_x \cdot n_{xx} \tag{3.14}$$

Multiplying (3.14) by u_t, integrating the resulting equation over I and employing integration by parts, we have

$$\begin{aligned}
&\int_I \rho u_t^2 + \frac{1}{2}\frac{d}{dt}\int_I u_x^2 \\
= &\ -\int_I \rho u u_x u_t - \int_I \gamma \rho^{\gamma-1} \rho_x u_t - 2\int_I n_x \cdot n_{xx} u_t \\
= &\ -2\int_I n_x \cdot n_{xx} u_t - \int_I \rho u u_x u_t - \gamma \int_I \rho^{\gamma-1} \rho_x u_t \\
\le &\ \frac{1}{2}\int_I \rho u_t^2 + c[\int_I \frac{1}{\rho}|n_x|^2 |n_{xx}|^2 + \int_I \rho u^2 u_x^2 + \int_I \rho^{2\gamma-3} \rho_x^2] \\
\le &\ \frac{1}{2}\int_I \rho u_t^2 + c[\|\frac{1}{\rho}\|_{L^\infty} (\int_I |n_{xx}|^2)^2 + \|\rho\|_{L^\infty} (\int_I u_x^2)^2] + c\|\rho\|_{L^\infty}^\gamma \int_I \rho^{\gamma-3} \rho_x^2.
\end{aligned}$$

This, combined with Lemma 3.3 and 3.4, implies

$$\int_I \rho u_t^2 + \frac{d}{dt}\int_I u_x^2 \le C \exp(Ct) + C\exp(Ct)(\int_I u_x^2)^2$$

for some $C = C(\gamma, E_0, c_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^1}, \|n_0\|_{H^2}) > 0$. Thus, by Lemma 3.1 and the Gronwall's inequality, we have

$$\int_0^t \int_I \rho u_t^2 + \int_I u_x^2 \le C \exp\{C \exp(Ct)\}.$$

Since $\rho \ge \dfrac{1}{C\exp(Ct)}$, we have

$$\int_I u_x^2(t) + \int_0^t \int_I u_t^2 \le C \exp\{C \exp(Ct)\}. \tag{3.15}$$

By Lemma 3.3, 3.4, (3.14) and (3.15), we also have

$$\int_0^t \int_I u_{xx}^2 \le C \exp\{C \exp(Ct)\}.$$

This completes the proof. $\square$

In order to prove the existence of global classical solutions, we also need the following estimate.

Lemma 3.6 *There exists* $C = C(\gamma, c_0, E_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^2}, \|n_0\|_{H^2}) > 0$ *such that for any* $0 \le t < T$,

$$\int_I (u_t^2 + u_{xx}^2)(t) + \int_0^t \int_I u_{xt}^2 \le C \exp\{C \exp[C \exp(Ct)]\}. \tag{3.16}$$

Proof. Differentiating (3.14) with respect to t, multiplying the resulting equation by u_t, integrating it over I, and using integration by parts, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_I \rho u_t^2 + \int_I u_{xt}^2 \\
= & \int_I (|n_x|^2)_t u_{xt} - \frac{1}{2}\int_I \rho_t u_t^2 - \int_I (\rho_t u u_x + \rho u_t u_x) u_t - \int_I \rho u u_{xt} u_t + \int_I (\rho^\gamma)_t u_{xt} \\
= & \ 2\int_I n_x \cdot n_{xt} u_{xt} + \int_I (\rho u)_x u_t^2 + \int_I (\rho u)_x u u_x u_t - \int_I \rho u_t^2 u_x - \gamma \int_I \rho^{\gamma-1}(\rho u)_x u_{xt} \\
= & \int_I (2n_x \cdot n_{xt} - 2\rho u u_t - \rho u^2 u_x - \gamma\rho^{\gamma-1}\rho_x u - \gamma\rho^\gamma u_x) u_{xt} \\
& - \int_I \rho u u_t u_x^2 - \int_I \rho u^2 u_{xx} u_t - \int_I \rho u_t^2 u_x \\
\le & \ \frac{1}{2}\int_I u_{xt}^2 + c\int_I (|n_x|^2|n_{xt}|^2 + \rho^2 u^2 u_t^2 + \rho^2 u^4 u_x^2) + c\gamma^2 \int_I (\rho^{2\gamma-2}\rho_x^2 u^2 + \rho^{2\gamma} u_x^2) \\
& + c\int_I (\rho u^2 u_x^4 + \rho u^4 u_{xx}^2) + c(1 + \|u_x\|_{L^\infty}) \int_I \rho u_t^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \frac{d}{dt}\int_I \rho u_t^2 + \int_I u_{xt}^2 \\
\le & \ c\|n_x\|_{L^\infty}^2 \int_I |n_{xt}|^2 + c(1 + \|u_x\|_{L^\infty} + \|\rho\|_{L^\infty}\|u\|_{L^\infty}^2) \int_I \rho u_t^2 \\
+ & \ c(\gamma^2\|\rho\|_{L^\infty}^{2\gamma} + \|\rho\|_{L^\infty}^2\|u\|_{L^\infty}^4 + \|\rho\|_{L^\infty}\|u\|_{L^\infty}^2\|u_x\|_{L^\infty}^2) \int_I u_x^2 \\
+ & \ c\|\rho\|_{L^\infty}\|u\|_{L^\infty}^4 \int_I u_{xx}^2 + c\gamma^2\|\rho\|_{L^\infty}^{\gamma+1}\|u\|_{L^\infty}^2 \int_I \rho^{\gamma-3}\rho_x^2.
\end{aligned}$$

It follows from Lemma 3.3 and 3.5 that

$$\max\{\|n_{xx}\|_{L^\infty([0,t],L^2(I))}, \|u_x\|_{L^\infty([0,t],L^2(I))}\} \le C\exp\{C\exp(Ct)\}.$$

Thus we have

$$\max\{\|n_x\|_{L^\infty},\ \|u\|_{L^\infty}\} \le C\exp\{C\exp(Ct)\}.$$

Also observe that

$$\|u_x\|_{L^\infty(I)}^2 \le \int_I |u_{xx}|^2 + \|u_x\|_{L^2(I)}^2.$$

Therefore, by the Gronwall's inequality, we have

$$\int_I \rho u_t^2(t) + \int_0^t \int_I u_{xt}^2 \le C(\gamma, c_0, E_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^2}, \|n_0\|_{H^2}) \exp\{C \exp[C \exp(Ct)]\}.$$

Since $\rho \ge \dfrac{1}{c\exp(Ct)}$, this together with (3.14) and Lemma 3.3-3.5 yields (3.16). The proof is complete. □

Lemma 3.7 ([19]) *Suppose that*

$$\sup_{x\in I} |u(x,t_1) - u(x,t_2)| \le \mu_1 |t_1 - t_2|^\alpha, \ \forall t_1, t_2 \in [0,T),$$

and

$$\sup_{t\in[0,T)} |u_x(x_1,t) - u_x(x_2,t)| \le \mu_2 |x_1 - x_2|^\beta, \ \forall x_1, x_2 \in I.$$

Then

$$\sup_{x\in I} |u_x(x,t_1) - u_x(x,t_2)| \le \mu |t_1 - t_2|^\delta, \ \forall t_1, t_2 \in [0,T),$$

where $\delta = \frac{\alpha\beta}{1+\beta}$, *and* μ *depends only on* $\alpha, \beta, \mu_1, \mu_2$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose it were false. Then there exists a maximal time interval $0 < T_* < +\infty$ such that there exists a unique classical solution $(\rho, u, n) : I \times [0, T_*) \to \mathbf{R}_+ \times \mathbf{R} \times S^2$ of (1.1)-(1.3), but at least one the following properties fails:

(i) $(\rho_x, \rho_t) \in C^{\alpha, \frac{\alpha}{2}}(Q_{T_*})$,

(ii) $0 < c^{-1} \le \rho \le c < +\infty, \ \forall (x,t) \in Q_{T_*}$,

(iii) $(u, n) \in C^{2+\alpha, \frac{2+\alpha}{2}}(Q_{T*})$.

Notice that (3.8) of Lemma 3.4 implies (ii) holds. Hence either (i) or (iii) fails. On the other hand, it follows from Lemma 3.3, Lemma 3.5, Lemma 3.6, and the Sobolev embedding Theorem that

$$\max\left\{\|n\|_{C^{1,\frac{1}{2}}(Q_{T^*})}, \|u\|_{C^{1,\frac{1}{2}}(Q_{T^*})}\right\} \le C(E_0, c_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^2}, \|n_0\|_{H^2}, T_*) < +\infty.$$

Hence Lemma 3.3 and $(1.1)_3$ imply that n and n_x is uniformly Hölder continuous in t and x respectively with exponent $\frac{1}{2}$. Lemma 3.7 then implies n_x is also uniformly Hölder continuous in t with exponent $\frac{1}{6}$. Thus

$$\max\{\|n_x\|_{C^{\frac{1}{3},\frac{1}{6}}(Q_{T^*})}, \|u_x\|_{C^{\frac{1}{3},\frac{1}{6}}(Q_{T^*})}\} \leq C(E_0, c_0, \|\rho_0\|_{H^1}, \|u_0\|_{H^2}, \|n_0\|_{H^2}, T_*) < +\infty.$$

It follows from $(1.1)_3$ and the Schauder theory that $\|n\|_{C^{2+\frac{1}{3},1+\frac{1}{6}}(Q_{T^*})} < +\infty$ and hence $\|n_x\|_{C^{1,\frac{1}{2}}(Q_{T^*})} < +\infty$. Hence, applying the Schauder theory to $(1.1)_3$ again, we have $\|n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T^*})} < +\infty$.

Write $F(x,t) = -(|n_x|^2)_x$. Then $\|F\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})} < +\infty$. In term of the Lagrangian coordinate, $(1.1)_1$ and $(1.1)_2$ become

$$\begin{cases} \rho_\tau + \rho^2 u_y = 0, \\ u_\tau + (\rho^\gamma)_y = (\rho u_y)_y + F, \end{cases} \tag{3.17}$$

Moreover, the estimates obtained by Lemma 3.4-3.6, in the Lagrangian coordinate, become

$$0 < c^{-1} \leq \rho \leq c < +\infty, \tag{3.18}$$

$$\int_I \rho_y^2 \leq c < +\infty, \tag{3.19}$$

$$\int_I u_y^2 + u_{yy}^2 \leq c < +\infty. \tag{3.20}$$

Combining $(3.17)_1$ with (3.18)-(3.20), we conclude $\|\rho\|_{C^{\frac{1}{2},\frac{1}{4}}(Q_{T^*})} < +\infty$. On the other hand, we have

$$\max\left\{\|F\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})}, \|u\|_{C^{\frac{1}{3},\frac{1}{6}}(Q_{T^*})}, \|u_y\|_{C^{1,\frac{1}{2}}(Q_{T^*})}\right\} < +\infty.$$

Now we claim that

$$\|\rho_y\|_{C^{\frac{1}{2},\frac{1}{4}}(Q_{T^*})} < +\infty. \tag{3.21}$$

In fact, $(3.17)_1$ and $(3.17)_2$ imply

$$(u + (\ln\rho)_y)_\tau = F - \gamma\rho^\gamma[u + (\ln\rho)_y] + \gamma\rho^\gamma u.$$

Hence we have

$$u + (\ln\rho)_y = (u_0 + (\ln\rho_0)_y)e^{-\gamma\int_0^\tau \rho^\gamma} + \int_0^\tau (F + \gamma\rho^\gamma u)e^{-\gamma\int_s^\tau \rho^\gamma}\, ds.$$

This clearly implies (3.21). Thus, applying the Schauder theory to $(3.17)_2$, we conclude that

$$\|u\|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(Q_{T^*})} < +\infty.$$

In particular,

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})} + \|u_y\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})} < +\infty.$$

Applying these estimates to $(3.17)_1$, we obtain that $\|\rho\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})} < +\infty$. Repeating the above argument once again, we see that

$$\max\left\{\|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_{T^*})}, \|\rho_y\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_{T^*})}, \|\rho_\tau\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_{T^*})}\right\} < +\infty.$$

This contradicts the choice of T^*. Hence $T^* = \infty$. The proof of Theorem 1.1 is now complete. □

Now we recall the following well-known Lemma.

Lemma 3.8 [10]. *Assume $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow E$. Then the following embedding are compact:*

$$(i) \quad \left\{\varphi : \varphi \in L^q(0,T;X), \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y)\right\} \hookrightarrow\hookrightarrow L^q(0,T;E), \quad \text{if} \quad 1 \le q \le \infty;$$

$$(ii) \quad \left\{\varphi : \varphi \in L^\infty(0,T;X), \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y)\right\} \hookrightarrow\hookrightarrow C([0,T];E), \quad \text{if} \quad 1 < r \le \infty.$$

Proof of Theorem 1.2.

Part (i): First, by the standard mollification, we may assume that for any $\alpha \in (0,1)$, there exist a sequence of initial data $(\rho_0^\epsilon, u_0^\epsilon, n_0^\epsilon) \in C^{1,\alpha}(I) \times C^{2,\alpha}(I) \times C^{2,\alpha}(I, S^2)$ such that

(i) $0 < c_0^{-1} \le \rho_0^\epsilon \le c_0 < +\infty$ on I,

(ii)

$$\lim_{\epsilon \downarrow 0} [\|\rho_0^\epsilon - \rho_0\|_{H^1} + \|u_0^\epsilon - u_0\|_{L^2} + \|n_0^\epsilon - n_0\|_{H^1}] = 0.$$

(To assure $|n_0^\epsilon| = 1$, we construct $n_0^\epsilon = \frac{\eta_\epsilon * n_0}{|\eta_\epsilon * n_0|}$)

Now let $(\rho^\epsilon, u^\epsilon, n^\epsilon)$ be the unique global classical solution of (1.1) along with the initial condition $(\rho_0^\epsilon, u_0^\epsilon, n_0^\epsilon)$ and the boundary condition $(u^\epsilon, \frac{\partial n^\epsilon}{\partial x})|_{\partial I} = (0,0)$ for $t > 0$. It follows from Lemma 3.1, Lemma 3.2, and Lemma 3.4 that for any

$0 < T < +\infty$, the following properties hold:

$$\frac{1}{c\exp(cT)} \leq \rho^\epsilon \leq c\exp(cT), \text{ in } I \times [0,T],$$
$$\|\rho^\epsilon\|_{L^\infty(0,T;H^1(I))} + \|\rho_t^\epsilon\|_{L^2(0,T;L^2(I))} \leq C(T),$$
$$\|u^\epsilon\|_{L^\infty(0,T;L^2(I))} + \|u^\epsilon\|_{L^2(0,T;H_0^1(I))} \leq C(T),$$
$$\|n^\epsilon\|_{L^\infty(0,T;H^1(I))} + \|n^\epsilon\|_{L^2(0,T;H^2(I))} + \|n_t^\epsilon\|_{L^2(0,T;L^2(I))} \leq C(T).$$

After taking possible subsequences, we may assume that as $\epsilon \to 0$,

$$(\rho^\epsilon, \rho_x^\epsilon) \rightharpoonup (\rho, \rho_x) \text{ weak}^* \ L^\infty(0,T;L^2(I)), \tag{3.22}$$

$$\rho_t^\epsilon \rightharpoonup \rho_t \ \textit{weakly in } L^2(0,T;L^2(I)), \tag{3.23}$$

$$\rho^\epsilon \to \rho \text{ strongly in } C(Q_T), \tag{3.24}$$

$$u^\epsilon \rightharpoonup u \text{ weak}^* \ L^\infty(0,T;L^2(I)) \text{ and weakly in } L^2(0,T;H_0^1(I)), \tag{3.25}$$

$$(n^\epsilon, n_x^\epsilon, n_{xx}^\epsilon) \rightharpoonup (n, n_x, n_{xx}) \text{ weakly in } L^2(0,T;L^2(I)), \tag{3.26}$$

$$(n^\epsilon, n_x^\epsilon) \rightharpoonup (n, n_x) \text{ weak}^* \ L^\infty(0,T;L^2(I)), \tag{3.27}$$

$$n_t^\epsilon \rightharpoonup n_t \text{ weakly in } L^2(0,T;L^2(I)), \tag{3.28}$$

$$n^\epsilon \to n \text{ strongly in } C(Q_T) \cap L^2(0,T;C^1(I)), \tag{3.29}$$

where we have used Lemma 3.8.

It is easy to see that (3.24) and (3.25) imply

$$\rho_t^\epsilon + (\rho^\epsilon u^\epsilon)_x \to \rho_t + (\rho u)_x \text{ in } \mathcal{D}'(Q_T)$$

so that $\rho_t + (\rho u)_x = 0$.

Since

$$(\rho^\epsilon u^\epsilon)_t = u_{xx}^\epsilon - (|n_x^\epsilon|^2)_x - (\rho^\epsilon (u^\epsilon)^2)_x - ((\rho^\epsilon)^\gamma)_x \in L^2(0,T;H^{-1}(I))$$

and $\rho^\epsilon u^\epsilon \rightharpoonup \rho u$ weak* $L^\infty(0,T;L^2(I))$, we have from Lemma 3.8 that

$$\rho^\epsilon u^\epsilon \to \rho u \text{ strongly in } C(0,T;H^{-1}(I)).$$

This, combined with (3.25), implies that $\rho^\epsilon(u^\epsilon)^2 \to \rho u^2$ in $\mathcal{D}'(Q_T)$. Since

$$u^\epsilon_{xx} - (|n^\epsilon_x|^2)_x - ((\rho^\epsilon)^\gamma)_x \to u_{xx} - (|n_x|^2)_x - (\rho^\gamma)_x \text{ in } \mathcal{D}'(Q_T),$$

we see that

$$(\rho u)_t + (\rho u^2)_x + (\rho^\gamma)_x = u_{xx} - (|n_x|^2)_x \text{ in } \mathcal{D}'(Q_T).$$

Since

$$u^\epsilon n^\epsilon_x \to u n_x,\ |n^\epsilon_x|^2 n^\epsilon \to |n_x|^2 n \text{ in } \mathcal{D}'(Q_T),$$

we also have that

$$n_t + u n_x = n_{xx} + |n_x|^2 n \text{ in } \mathcal{D}'(Q_T).$$

This completes the proof of (i).

To prove part (ii). Observe that since $u_0 \in H^1_0$ and $n_0 \in H^2$, we have

$$\lim_{\epsilon\to 0}[\|u^\epsilon_0 - u_0\|_{H^1(I)} + \|n^\epsilon_0 - n_0\|_{H^2(I)}] = 0.$$

Thus, Lemma 3.3, and Lemma 3.5 imply

$$\sup_{0\le t\le T}\int_I (|u^\epsilon_x|^2 + |n^\epsilon_t|^2 + |n^\epsilon_{xx}|^2) + \int_{Q_T}(|u^\epsilon_{xx}|^2 + |u^\epsilon_t|^2 + |n^\epsilon_{xt}|^2 + |n^\epsilon_{xxx}|^2) \\ \le C(T). \tag{3.30}$$

It follows from (3.30) that $u \in L^\infty(0,T;H^1_0(I)) \cap L^2(0,T;H^2)$, $u_t \in L^2(0,T;L^2(I))$, $n \in L^\infty(0,T;H^2(I)) \cap L^2(0,T;H^3(I))$, and $n_t \in L^2(0,T;H^1(I)) \cap L^\infty(0,T;L^2(I))$. This proves the existence.

To prove the uniqueness, let (ρ_i, u_i, n_i) be two solutions to (1.1)-(1.3) obtained as above. Denote $\widetilde{\rho} = \rho_1 - \rho_2, \widetilde{u} = u_1 - u_2, \widetilde{n} = n_1 - n_2$, Then

$$\begin{cases} \widetilde{\rho}_t + (\widetilde{\rho} u_1)_x + (\rho_2 \widetilde{u})_x = 0, \\ \rho_1 \widetilde{u}_t - \widetilde{u}_{xx} = -\widetilde{\rho} u_{2t} - \widetilde{\rho} u_2 u_{2x} - \rho_1 \widetilde{u} u_{2x} - \rho_1 u_1 \widetilde{u}_x - (\rho_1^\gamma - \rho_2^\gamma)_x \\ \qquad\qquad -2n_{1x}\cdot \widetilde{n}_{xx} - 2\widetilde{n}_x \cdot n_{2xx}, \\ \widetilde{n}_t + u_1 \widetilde{n}_x + \widetilde{u} n_{2x} = \widetilde{n}_{xx} + |n_{1x}|^2 \widetilde{n} + [\widetilde{n}_x \cdot (n_{1x} + n_{2x})] n_2, \end{cases} \tag{3.31}$$

for $(x,t) \in (0,1) \times (0,+\infty)$, with the initial and boundary condition:

$$(\widetilde{\rho},\ \widetilde{u},\ \widetilde{n})\big|_{t=0} = 0 \text{ in } [0,1],\ (\widetilde{u},\ \widetilde{n}_x)\big|_{\partial I} = 0 \quad t > 0.$$

Multiplying $(3.31)_1$ by $\widetilde{\rho}$, integrating the resulting equation over I, and using the integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_I|\widetilde{\rho}|^2 &= \int_I\widetilde{\rho}u_1\widetilde{\rho}_x-\int_I(\rho_{2x}\widetilde{u}+\rho_2\widetilde{u}_x)\widetilde{\rho}\\ &= -\frac{1}{2}\int_I|\widetilde{\rho}|^2u_{1x}-\int_I(\rho_{2x}\widetilde{u}+\rho_2\widetilde{u}_x)\widetilde{\rho}\\ &\le \frac{1}{2}\|u_{1x}\|_{L^\infty}\int_I|\widetilde{\rho}|^2+\|\widetilde{u}\|_{L^\infty}\|\rho_{2x}\|_{L^2}\|\widetilde{\rho}\|_{L^2}+\|\rho_2\|_{L^\infty}\|\widetilde{u}_x\|_{L^2}\|\widetilde{\rho}\|_{L^2}.\end{aligned}$$

Since $\widetilde{u}(0,t)=0$, we have $\widetilde{u}(y,t)=\int_0^y\widetilde{u}_x(x,t)dx$ for $(y,t)\in Q_T$ and hence

$$\|\widetilde{u}\|_{L^\infty}\le\|\widetilde{u}_x\|_{L^2},\quad t\in[0,T]. \tag{3.32}$$

It follows from (3.32), the regularities of (ρ_i,u_i), Hölder inequality, and Sobolev inequality that

$$\begin{aligned}\frac{d}{dt}\int_I|\widetilde{\rho}|^2 &\le c\|u_1\|_{H^2}\int_I|\widetilde{\rho}|^2+c\|\widetilde{u}_x\|_{L^2}\|\widetilde{\rho}\|_{L^2}\\ &\le c(\|u_1\|_{H^2}+1)\int_I|\widetilde{\rho}|^2+\int_I|\widetilde{u}_x|^2.\end{aligned} \tag{3.33}$$

Multiplying $(3.31)_2$ by $\widetilde{u}$, integrating the resulting equation over I, and using the integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I\rho_1|\widetilde{u}|^2+\int_I|\widetilde{u}_x|^2\\ =\ &\frac{1}{2}\int_I\rho_{1t}|\widetilde{u}|^2-\int_I\rho_1u_1\widetilde{u}_x\widetilde{u}-\int_I\widetilde{\rho}\widetilde{u}u_{2t}-\int_I\widetilde{\rho}\widetilde{u}u_2u_{2x}-\int_I\rho_1|\widetilde{u}|^2u_{2x}\\ +\ &\int_I(\rho_1^\gamma-\rho_2^\gamma)\widetilde{u}_x+2\int_I(n_{1x}\cdot\widetilde{n}_x)\widetilde{u}_x+2\int_I(n_{1xx}\cdot\widetilde{n}_x)\widetilde{u}-2\int_I(\widetilde{n}_x\cdot n_{2xx})\widetilde{u}.\end{aligned}$$

Since $\rho_{1t}+(\rho_1u_1)_x=0$, we have

$$\frac{1}{2}\int_I\rho_{1t}|\widetilde{u}|^2-\int_I\rho_1u_1\widetilde{u}_x\widetilde{u}=\frac{1}{2}\int_I\rho_{1t}|\widetilde{u}|^2+\frac{1}{2}\int_I|\widetilde{u}|^2(\rho_1u_1)_x=0.$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I\rho_1|\widetilde{u}|^2+\int_I|\widetilde{u}_x|^2\\ \le\ &\|\widetilde{u}\|_{L^\infty}\|\widetilde{\rho}\|_{L^2}\|u_{2t}\|_{L^2}+\|\widetilde{\rho}\|_{L^2}\|\widetilde{u}\|_{L^\infty}\|u_2\|_{L^\infty}\|u_{2x}\|_{L^2}+\|u_{2x}\|_{L^\infty}\int_I\rho_1|\widetilde{u}|^2\\ +\ &c[\|\widetilde{\rho}\|_{L^2}\|\widetilde{u}_x\|_{L^2}+\|\widetilde{u}_x\|_{L^2}\|\widetilde{n}_x\|_{L^2}\|n_{1x}\|_{L^\infty}+\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}\|\widetilde{n}_x\|_{L^2}\|\frac{n_{1xx}+n_{2xx}}{\sqrt{\rho_1}}\|_{L^\infty}].\end{aligned}$$

It follows from (3.32), Hölder inequality, Sobolev's inequality, and the regularities of (ρ_i, u_i) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho_1|\widetilde{u}|^2 + \int_I |\widetilde{u}_x|^2 \\
\leq\ & c\|\widetilde{u}_x\|_{L^2}(\|\widetilde{\rho}\|_{L^2}\|u_{2t}\|_{L^2} + \|\widetilde{\rho}\|_{L^2} + \|\widetilde{n}_x\|_{L^2}) + c\|u_2\|_{H^2}\int_I \rho_1|\widetilde{u}|^2 \\
& + c(\|n_1\|_{H^3} + \|n_2\|_{H^3})\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}\|\widetilde{n}_x\|_{L^2} \\
\leq\ & \frac{1}{2}\|\widetilde{u}_x\|_{L^2}^2 + c\|\widetilde{\rho}\|_{L^2}^2(1 + \|u_{2t}\|_{L^2}^2) + c(\|u_2\|_{H^2} + \|n_1\|_{H^3}^2 + \|n_2\|_{H^3}^2)\int_I \rho_1|\widetilde{u}|^2 \\
& + c\|\widetilde{n}_x\|_{L^2}^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{d}{dt}\int_I \rho_1|\widetilde{u}|^2 + \int_I |\widetilde{u}_x|^2 \ &\leq\ c(1 + \|u_{2t}\|_{L^2}^2)\int_I |\widetilde{\rho}|^2 + c\int_I |\widetilde{n}_x|^2 \\
&+\ c(\|u_2\|_{H^2} + \|n_1\|_{H^3}^2 + \|n_2\|_{H^3}^2)\int_I \rho_1|\widetilde{u}|^2. \quad (3.34)
\end{aligned}$$

Multiplying $(3.31)_3$ by $\widetilde{n}$, integrating the resulting equation over I, and using the integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I |\widetilde{n}|^2 + \int_I |\widetilde{n}_x|^2 \\
=\ & -\int_I u_1\widetilde{n}_x \cdot \widetilde{n} - \int_I \widetilde{u}n_{2x} \cdot \widetilde{n} + \int_I |n_{1x}|^2|\widetilde{n}|^2 + \int_I [\widetilde{n}_x \cdot (n_{1x} + n_{2x})]n_2 \cdot \widetilde{n} \\
\leq\ & \|u_1\|_{L^\infty}\|\widetilde{n}_x\|_{L^2}\|\widetilde{n}\|_{L^2} + \|n_{2x}\|_{L^\infty}\|\rho_1^{-\frac{1}{2}}\|_{L^\infty}\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}\|\widetilde{n}\|_{L^2} + \|n_{1x}\|_{L^\infty}^2\int_I |\widetilde{n}|^2 \\
& + \|n_2\|_{L^\infty}\|n_{1x} + n_{2x}\|_{L^\infty}\|\widetilde{n}\|_{L^2}\|\widetilde{n}_x\|_{L^2}.
\end{aligned}$$

By Hölder inequality and Sobolev's inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I |\widetilde{n}|^2 + \int_I |\widetilde{n}_x|^2 \\
\leq\ & c\|\widetilde{n}_x\|_{L^2}\|\widetilde{n}\|_{L^2} + c\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}\|\widetilde{n}\|_{L^2} + c\int_I |\widetilde{n}|^2 \\
\leq\ & \frac{1}{2}\|\widetilde{n}_x\|_{L^2}^2 + c\|\widetilde{n}\|_{L^2}^2 + c\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}^2.
\end{aligned}$$

Hence

$$\frac{d}{dt}\int_I |\widetilde{n}|^2 + \int_I |\widetilde{n}_x|^2 \leq c\|\widetilde{n}\|_{L^2}^2 + c\|\sqrt{\rho_1}\widetilde{u}\|_{L^2}^2. \tag{3.35}$$

Adding together (3.33), (3.34), and (3.35), we obtain

$$\frac{d}{dt}\int_I (|\widetilde{\rho}|^2 + \rho_1|\widetilde{u}|^2 + c|\widetilde{n}|^2) \leq A(t)\int_I (|\widetilde{\rho}|^2 + \rho_1|\widetilde{u}|^2 + c|\widetilde{n}|^2), \tag{3.36}$$

where

$$A(t) = c[2+c+\int_I |u_{2t}|^2(t) + \|u_1(\cdot,t)\|_{H^2} + \|u_2(\cdot,t)\|_{H^2} + \|n_1(\cdot,t)\|_{H^3}^2 + \|n_2(\cdot,t)\|_{H^3}^2].$$

It follows from the regularities of u_i and n_i that $\int_0^T A(t)dt < +\infty$. (3.36) and the Gronwall's inequality imply

$$\begin{aligned}&\int_I (|\widetilde{\rho}|^2 + \rho_1|\widetilde{u}|^2 + c|\widetilde{n}|^2)(x,t)dx \\ &\le \int_I (|\widetilde{\rho}|^2 + \rho_1|\widetilde{u}|^2 + c|\widetilde{n}|^2)(x,0)dx \exp\{\int_0^T A(s)ds\} = 0.\end{aligned}$$

Since $\rho_1 > 0$ and $c > 0$, we have $(\widetilde{\rho}, \widetilde{u}, \widetilde{n}) = 0$. This proves the uniqueness. Hence the proof of Theorem 1.2 (ii) is complete. □

4 Global strong solutions for $\rho_0 \ge 0$.

In this section, we establish the existence of global strong solutions for $\rho_0 \ge 0$. The proof of Theorem 1.3 is based on several estimates of the approximate solutions without the hypothesis that the initial density function has a positive lower bound.

For a small $\epsilon > 0$, let $\rho_0^\epsilon = \eta_\epsilon \star \rho_0 + \epsilon$, $u_0^\epsilon = \eta_\epsilon \star u_0$, $n_0^\epsilon = \frac{\eta_\epsilon * n_0}{|\eta_\epsilon * n_0|}$. Then $\rho_0^\epsilon \ge \epsilon$ and

$$\lim_{\epsilon \downarrow 0} [\|\rho_0^\epsilon - \rho_0\|_{H^1} + \|u_0^\epsilon - u_0\|_{H^1} + \|n_0^\epsilon - n_0\|_{H^2}] = 0.$$

Let $(\rho^\epsilon, u^\epsilon, n^\epsilon)$ be the unique global classical solution of (1.1) along with the initial condition $(\rho_0^\epsilon, u_0^\epsilon, n_0^\epsilon)$ and the boundary condition $(u^\epsilon, \frac{\partial n^\epsilon}{\partial x})|_{\partial I} = (0,0)$. Now we outline several integral estimates for $(\rho^\epsilon, u^\epsilon, n^\epsilon)$. For simplicity, we write $(\rho, u, n) = (\rho^\epsilon, u^\epsilon, n^\epsilon)$. The first Lemma follows from the global energy inequality, Nirenberg inequality, and the second order estimate of $(1.1)_3$.

Lemma 4.1 *For any $T > 0$, it holds*

$$\sup_{0\le t\le T} \int_I (\rho u^2 + \rho^\gamma + |n_x|^2 + |n_{xx}|^2) + \int_0^T \int_I (u_x^2 + |n_{xt}|^2 + |n_{xxx}|^2) \le C(E_0, T). \quad (4.1)$$

Proof. It is exactly as same as that of Lemma 3.1, Lemma 3.2, and Lemma 3.3. □

The next Lemma is concerned with the upper bound estimate of ρ.

Lemma 4.2 *For any $T > 0$, there exists $C > 0$ independent of ϵ such that*

$$\|\rho\|_{L^\infty(I\times(0,T))} \le C. \tag{4.2}$$

Proof. Set

$$w(x,t) = \int_0^t (u_x - |n_x|^2 - \rho u^2 - \rho^\gamma) + \int_0^x (\rho_0 u_0)(\xi).$$

Then we have

$$w_t = u_x - |n_x|^2 - \rho u^2 - \rho^\gamma, \quad w_x = \rho u.$$

It follows from Lemma 4.1 that

$$\int_I (|w + |w_x|) \le C$$

and hence

$$\|w\|_{L^\infty(I)} \le C \int_I (|w| + |w_x|) \le C.$$

Since $\rho \ge 0$, it suffices to prove $\rho(y,s) \le C$ for any $(y,s) \in I \times (0,T)$. Let $x(y,t)$ solve

$$\begin{cases} \dfrac{dx(y,t)}{dt} = u(x(y,t),t), \ 0 \le t < s; \\ x(y,s) = y, \quad 0 \le y \le 1. \end{cases}.$$

Denote $f = \exp w$. Then we have

$$\begin{aligned} \frac{d}{dt}((\rho f)(x(y,t),t)) &= (\rho_t + \rho_x u) f + \rho f (w_t + u w_x) \\ &= [\rho_t + \rho_x u + \rho u_x - \rho |n_x|^2 - \rho^2 u^2 - \rho^{\gamma+1} + (\rho u)^2] f \\ &= (-\rho |n_x|^2 - \rho^{\gamma+1}) f \le 0. \end{aligned}$$

Thus

$$\rho(y,s) f(y,s) = \rho(x(y,s),s) f(x(y,s),s) \le \rho_0(x(y,0)) f(x(y,0),0) \le C.$$

Therefore

$$\rho(y,s) \le C \exp(-w(y,s)) \le C \exp(\|w\|_{L^\infty(I\times(0,T))}) \le C.$$

The proof is completed. □

Next we want to estimate $\|u_x\|_{L^\infty(0,T;L^2(I))}$ and $\|\rho u_t^2\|_{L^1(I\times[0,T])}$.

Lemma 4.3 *For any $T > 0$, there exists $C > 0$ independent of ϵ such that*

$$\sup_{0\le t\le T}\int_I u_x^2 + \int_0^T\int_I \rho u_t^2 \le C \tag{4.3}$$

Proof. It is similar to Lemma 3.5. Multiplying (3.14) by u_t, integrating the resulting equation over I, and employing integration by parts, we have

$$\begin{aligned}\int_I \rho u_t^2 + \frac{1}{2}\frac{d}{dt}\int_I u_x^2 &= \int_I |n_x|^2 u_{xt} + \int_I \rho^\gamma u_{xt} - \int_I \rho u u_x u_t \\ &\le \frac{1}{2}\int_I \rho u_t^2 dx + \|\rho\|_{L^\infty}\|u\|_{L^\infty}^2 \int_I u_x^2 + \int_I |n_x|^2 u_{xt} + \int_I \rho^\gamma u_{xt}.\end{aligned}$$

Thus, using Lemma 4.2 and $\|u\|_{L^\infty}^2 \le \int_I |u_x|^2$, we obtain

$$\begin{aligned}\int_I \rho u_t^2 + \frac{d}{dt}\int_I u_x^2 &\le 2[(\int_I u_x^2)^2 + \int_I \rho^\gamma u_{xt} + \int_I |n_x|^2 u_{xt}] \\ &= I + II + III. \end{aligned}\tag{4.4}$$

$$\begin{aligned} II &= \frac{d}{dt}\int_I \rho^\gamma u_x - \int_I \gamma\rho^{\gamma-1}\rho_t u_x \\ &= \frac{d}{dt}\int_I \rho^\gamma u_x + \int_I (\rho^\gamma)_x u u_x + \gamma\int_I \rho^\gamma u_x^2 \\ &= \frac{d}{dt}\int_I \rho^\gamma u_x + (\gamma-1)\int_I \rho^\gamma u_x^2 - \int_I \rho^\gamma u u_{xx} \\ &= \frac{d}{dt}\int_I \rho^\gamma u_x + (\gamma-1)\int_I \rho^\gamma u_x^2 - \int_I \rho^\gamma u(\rho u_t + \rho u u_x + (\rho^\gamma)_x + (|n_x|^2)_x) \\ &\le \frac{d}{dt}\int_I \rho^\gamma u_x + (\gamma-1)\int_I \rho^\gamma u_x^2 + \frac{1}{4}\int_I \rho u_t^2 + C(1 + \int_I |u_x|^2)\int_I u_x^2 \\ &- \int_I \rho^\gamma(\rho^\gamma)_x u - 2\int_I \rho^\gamma u n_x \cdot n_{xx} \\ &\le \frac{d}{dt}\int_I \rho^\gamma u_x + C(1 + \int_I u_x^2)\int_I u_x^2 + \frac{1}{4}\int_I \rho u_t^2 + \frac{1}{2}\int_I \rho^{2\gamma} u_x \\ &+ C\int_I u^2|n_x|^2 + C\int_I |n_{xx}|^2 \\ &\le \frac{d}{dt}\int_I \rho^\gamma u_x + C(1 + \int_I u_x^2)\int_I u_x^2 + \frac{1}{4}\int_I \rho u_t^2 + C(\int_I |n_x|^2\int_I u_x^2 + \int_I |n_{xx}|^2) \\ &\le \frac{d}{dt}\int_I \rho^\gamma u_x + C[1 + (\int_I u_x^2)^2] + \frac{1}{4}\int_I \rho u_t^2. \end{aligned}$$

$$
\begin{aligned}
III &= \frac{d}{dt}\int_I |n_x|^2 u_x - 2\int_I n_x \cdot n_{xt} u_x \\
&\le \frac{d}{dt}\int_I |n_x|^2 u_x + \|n_x\|_{L^\infty}^2 \int_I u_x^2 + \int_I |n_{xt}|^2 \\
&\le \frac{d}{dt}\int_I |n_x|^2 u_x + \|n_{xx}\|_{L^2}^2 \int_I u_x^2 + \int_I |n_{xt}|^2 \\
&\le \frac{d}{dt}\int_I |n_x|^2 u_x + \int_I u_x^2 + \int_I |n_{xt}|^2.
\end{aligned}
$$

Combining (4.4) with the estimates of II and III, we obtain

$$
\begin{aligned}
&\int_I \rho u_t^2 + \frac{d}{dt}\int_I u_x^2 \\
\le\ & C[1 + (\int_I u_x^2)^2] + \frac{d}{dt}\int_I \rho^\gamma u_x + \frac{d}{dt}\int_I |n_x|^2 u_x + C\int_I |n_{xt}|^2. \qquad (4.5)
\end{aligned}
$$

Integrating (4.5) over $[0, T]$ yields

$$
\begin{aligned}
&\int_0^T \int_I \rho u_t^2 + \int_I u_x^2 \\
\le\ & \int_I u_{0x}^2 + C(T + \int_0^T (\int_I u_x^2)^2) + \int_I (\rho^\gamma u_x - \rho_0^\gamma u_{0x}) + \int_I (|n_x|^2 u_x - |n_{0x}|^2 u_{0x}) \\
\le\ & C(1+T) + C\int_0^T (\int_0^1 u_x^2)^2 + \frac{1}{8}\int_I u_x^2 + \int_I \rho^{2\gamma} + \frac{1}{8}\int_I u_x^2 + C\int_I |n_x|^4 \\
\le\ & C(1+T) + C\int_0^T (\int_I u_x^2)^2 + \frac{1}{4}\int_I u_x^2 + C\int_I |n_x|^2 \int_I |n_{xx}|^2.
\end{aligned}
$$

Thus we have

$$
\int_0^T \int_I \rho u_t^2 + \int_I u_x^2 \le C(1+T) + C\int_0^T (\int_I u_x^2)^2.
$$

This, combined with the inequality $\int_0^T \int_I u_x^2 \le C$ and the Gronwall inequality, implies (4.3). This completes the proof. □

We also need to estimate $\|\rho\|_{L^\infty(0,T;L^2(I))}$ as follows.

Lemma 4.4 *For any $T > 0$, there exists $C > 0$ independent of ϵ such that*

$$
\sup_{0 \le t \le T} \int_I \rho_x^2 + \int_0^T \|u_x\|_{L^\infty}^2 \le C. \qquad (4.6)
$$

Proof. By (3.14), we have

$$\begin{aligned}
\|u_x\|_{L^\infty}^2 &\leq 2\|u_x-\rho^\gamma-|n_x|^2\|_{L^\infty}^2+2\|\rho^\gamma+|n_x|^2\|_{L^\infty}^2\\
&\leq C[\|u_x-\rho^\gamma-|n_x|^2\|_{L^2}^2+\|u_{xx}-(\rho^\gamma)_x-(|n_x|^2)_x\|_{L^2}^2]\\
&+ C[\|\rho^\gamma\|_{L^\infty}^2+(\int_I|n_{xx}|^2)^2]\\
&\leq C[1+\|n_x\|_{L^\infty}^2\int_I|n_x|^2]+C\|\rho u_t+\rho u u_x\|_{L^2}^2\\
&\leq C[1+(\int_I u_x^2)^2+\int_I|n_x|^2\int_I|n_{xx}|^2]+C\int_0^1\rho u_t^2.
\end{aligned}$$

Thus we obtain

$$\int_0^T\|u_x\|_{L^\infty}^2\leq C.$$

To estimate $\int_I\rho_x^2$, take the derivative of $(1.1)_1$ with respect to x, multiply the resulting equation by ρ_x, and integrate it over I and employ integration by parts. Then we have

$$\begin{aligned}
\frac{d}{dt}\int_I\rho_x^2 &= 2\int_I(\rho u)_x\rho_{xx}-(\rho u)_x\rho_x\Big|_0^1\\
&= \int_I(\rho_x u\rho_{xx}+\rho u_x\rho_{xx})-\rho\rho_x u_x\Big|_0^1\\
&= -\frac{1}{2}\int_I\rho_x^2u_x+\rho u_x\rho_x\Big|_0^1-\int_I\rho_x^2u_x-\int_I\rho\rho_x u_{xx}-\rho\rho_x u_x\Big|_0^1\\
&= -\frac{3}{2}\int_I\rho_x^2u_x-\int_I\rho\rho_x u_{xx}\\
&\leq C\|u_x\|_{L^\infty}\int_I\rho_x^2-\int_I\rho\rho_x[\rho u_t+\rho u u_x+(\rho^\gamma)_x+(|n_x|^2)_x]dx\\
&\leq C(1+\|u_x\|_{L^\infty})\int_I\rho_x^2+C\int_I\rho u_t^2+C(\int_I u_x^2)^2+C(\int_I|n_{xx}|^2)^2\\
&\leq C+C(1+\|u_x\|_{L^\infty})\int_I\rho_x^2+C\int_I\rho u_t^2.
\end{aligned}$$

(4.6) follows from the Gronwall inequality. The proof is now completed. □

Finally we need to estimate $\|u_{xx}\|_{L^2}$.

Lemma 4.5 *For any $T>0$, there exists $C>0$ independent of ϵ such that*

$$\int_0^T\int_I u_{xx}^2\leq C.$$

Proof. It follows from (3.14) that

$$u_{xx}=\rho u_t+\rho u u_x+(\rho^\gamma)_x+(|n_x|^2)_x.$$

Hence

$$\begin{aligned}\int_{Q_T} u_{xx}^2 &\le C\int_{Q_T}\rho u_t^2 + C\int_0^T \|u\|_{L^\infty}^2\int_I u_x^2 + C\int_{Q_T}\rho_x^2\\ &+ C\int_0^T \|n_x\|_{L^\infty}^2\int_I |n_{xx}|^2\\ &\le C[1+\int_0^T(\int_I u_x^2)^2+\int_0^T(\int_I |n_{xx}|^2)^2]\le C.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3. It is similar to that of Theorem 1.2, we sketch it here. It follows from Lemma 4.1-4.5 that

$$\begin{aligned}&\sup_{0\le t\le T}(\|\rho^\epsilon\|_{H^1}+\|\rho_t^\epsilon\|_{L^2}+\|u^\epsilon\|_{H^1}+\|n^\epsilon\|_{H^2}+\|n_t^\epsilon\|_{L^2})\\ &+\int_0^T(\|(\rho^\epsilon u^\epsilon)_t\|_{L^2}^2+\|u_{xx}^\epsilon\|_{L^2}^2+\|n_{xxx}^\epsilon\|_{L^2}^2+\|n_t^\epsilon\|_{H^1}^2)dt\le C.\end{aligned}$$

After taking possible subsequences, we may assume

$$(\rho^\epsilon,\rho_x^\epsilon,\rho_t^\epsilon)\rightharpoonup(\rho,\rho_x,\rho_t)\text{ weak}^*\ \ L^\infty\left(0,T;L^2(I)\right),$$

$$(u^\epsilon,u_x^\epsilon)\rightharpoonup(u,u_x)\text{ weak}^*\ L^\infty\left(0,T;L^2(I)\right),$$

$$u_{xx}^\epsilon\rightharpoonup u_{xx}\text{ weakly in }\ L^2\left(0,T;L^2(I)\right),$$

$$(\rho^\epsilon u^\epsilon)_t\rightharpoonup v\text{ weakly in }L^2\left(0,T;L^2(I)\right),$$

$$(n^\epsilon,n_x^\epsilon,n_{xx}^\epsilon,n_t^\epsilon)\rightharpoonup(n,n_x,n_{xx},n_t)\text{ weak}^*L^\infty\left(0,T;L^2(I)\right),$$

$$(n_{xt}^\epsilon,n_{xxx}^\epsilon)\rightharpoonup(n_{xt},n_{xxx})\text{ weakly in }L^2\left(0,T;L^2(I)\right).$$

Since ρ^ϵ is bounded in $L^\infty\left(0,T;H^1(I)\right)$, and ρ_t^ϵ bounded in $L^\infty\left(0,T;L^2(I)\right)$, Lemma 3.8 implies that as $\epsilon\to 0$

$$\rho^\epsilon\to\rho\text{ strongly in }\ C(Q_T).$$

Hence, as $\epsilon\to 0$, we have

$$\rho^\epsilon u^\epsilon\rightharpoonup\rho u\text{ weak}^*\ L^\infty(0,T;L^2(I)).$$

Thus $v=(\rho u)_t$. In fact, Lemma 3.8 yields

$$\rho^\epsilon u^\epsilon\to\rho u\text{ strongly in }C(Q_T),$$

as ρu and $(\rho u)_t$ are bounded in $L^\infty(0,T;H^1(I))$ and $L^2(0,T;L^2(I))$ respectively. Combining these convergence together, we have

$$\rho^\epsilon(u^\epsilon)^2 \rightharpoonup \rho u^2, \quad \text{weak}^* \quad L^\infty(0,T;L^2(I)).$$

Since $(\rho^\epsilon(u^\epsilon)^2)_x$ is bounded in $L^\infty(0,T;L^2(I))$, it follows

$$(\rho^\epsilon(u^\epsilon)^2)_x \rightharpoonup (\rho u^2)_x, \ \text{weak}^* L^\infty(0,T;L^2(I)).$$

Lemma 3.8 also implies

$$(n^\epsilon)_x \to n_x \text{ strongly in } C(Q_T).$$

Since $(|n_x^\epsilon|^2)_x$ is bounded in $L^\infty(0,T;L^2(I))$, we have

$$(|n_x^\epsilon|^2)_x \rightharpoonup (|n_x|^2)_x, \ \text{weak}^* L^\infty(0,T;L^2(I)).$$

Similarly, we can get

$$(\rho^\epsilon u^\epsilon)_x \rightharpoonup (\rho u)_x, \ \text{weak}^* \ L^\infty(0,T;L^2(I)),$$

$$P(\rho^\epsilon)_x \rightharpoonup P(\rho)_x, \ \text{weak}^* L^\infty(0,T;L^2(I)),$$

$$n^\epsilon \to n \text{ strongly in } \ C(Q_T),$$

$$|n_x^\epsilon|^2 n^\epsilon \to |n_x|^2 n \text{ strongly in } C(Q_T),$$

$$u^\epsilon n_x^\epsilon \rightharpoonup u n_x \ \text{weak}^* L^\infty(0,T;L^2(I)).$$

Based on these convergence, we can conclude that (ρ,u,n) is a strong solution of the system (1.1) along with the initial and boundary conditions. The proof of Theorem 1.3 is completed. □

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