MA 213 — Calculus III Fall 2013
Exam II November 13, 2013

This is a closed book exam. There are give (5) problems on six (6) pages (including this cover page). Check and be sure that you have a complete exam.

No books or notes may be used during the exam. You can not use a graphing calculator. In addition, *a*ny device capable of electronic communication (cell phone, pager, etc.) must be turned off and out of sight during the exam.

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. If you need more space then use the backs of the exam pages.

Show your work. Answers without justification will receive no credit. Partial credit for a problem will be given only when there is coherent written evidence that you have solved part of the problem. In particular, answers that are obtained simply as the output of calculator routines will receive no credit. Finally, be aware that it is not the responsibility of the grader to determine which part of your response is to be graded. Be sure to erase or mark out any work that you do not want graded.

Name: ________________________

Section: __________

Last four digits of student identification number: __________

Problem	Score	Total
1		20
2		20
3		20
4		20
5		20
		100

1 (a) (10 pts). Calculate the directional derivative in the direction $\mathbf{v}$ at the given point. Remember to normalize the direction vector first.

$$f(x, y, z) = z^2 - xy^2, \ \mathbf{v} = \langle -1, 2, 2 \rangle, \ P = (2, 1, 3).$$

Solution:

$$D_u f(p) = \frac{5}{3}.$$

(b) (10 pts). Find an equation of the tangent plane to the surface at the given point.

$$xz + 2x^2y + y^2z^3 = 11, \ P = (2, 1, 1).$$

Solution: An equation of the tangent plane is:

$$9(x-2) + 10(y-1) + 5(z-1) = 0.$$

2 (20 points). Find all critical points of the function and analyze them using the Second Derivative Test to determine whether they are local maximal, minimal, or saddle points.

$$f(x,y) = x^4 - 4xy + 2y^2.$$

Solution:

$$f_x = 4(x^3 - y);\ \ f_y = 4(y - x).$$

There are 3 critical points: $(0,0)$, $(1,1)$, and $(-1,-1)$.

The second derivatives are:

$$f_{xx} = 12x^2,\ \ f_{xy} = -4,\ \ f_{yy} = 4,$$

so that the determinant is

$$D = f_{xx}f_{yy} - f_{xy}^2 = 48x^2 - 16 = 16(3x^2 - 1).$$

1) $D(0,0) = -16 \Rightarrow (0,0)$ is a saddle point.
2) $D(1,1)) = 32 > 0,\ \ f_{xx}(1,1) = 12 > 0 \Rightarrow (1,1)$ is a minimum point.
3) $D(-1,-1)) = 32 > 0,\ \ f_{xx}(-1,-1) = 12 > 0 \Rightarrow (-1,-1)$ is a minimum point.

3 (20 pts). Use the polar coordinate to calculate the double integral

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{9+x^2+y^2}\, dx\, dy.$$

Solution:

$$D = \left\{(x,y):\ 0 \le y \le \sqrt{2},\ y \le x \le \sqrt{4-y^2}\right\} = \left\{(r,\theta):\ 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{4}\right\}.$$

Hence the integral equals to

$$\begin{aligned} &= \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \frac{r}{9+r^2}\, dr \\ &= \frac{\pi}{8} \ln(9+r^2)\Big|_{r=0}^{r=\sqrt{2}} = \frac{\pi}{8} \ln(\frac{11}{9}). \end{aligned}$$

4 (20 pts). Use the Fubini's theorem (or equivalently, the iterated integration) to evaluate the triple integral

$$\int\int\int_E e^x \, dV,$$

where

$$E = \Big\{(x, y, z) \in \mathbb{R}^3 \mid 0 \le y \le 1,\ 0 \le x \le y,\ 0 \le z \le e^x + y\Big\}.$$

Solution: The integral equals to

$$\begin{aligned}
&= \int_0^1 \int_0^y \int_0^{e^x+y} e^x \, dz,\, dx\, dy \\
&= \int_0^1 \int_0^y e^x (e^x + y) \, dy\, dx \\
&= \int_0^1 (\frac{1}{2}e^{2y} + ye^y - \frac{1}{2} - y) \, dy \\
&= -(\frac{y}{2} + \frac{y^2}{2})|_0^1 + \frac{1}{4}e^{2y}|_0^1 + (ye^y - e^y)|_0^1 \\
&= \frac{1}{2}(e^2 - 1).
\end{aligned}$$

5 (20 pts). Find the total mass of the centroid of the region $\mathcal{W}$ lying above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$, where the density function ρ is given by $\rho(x, y, z) = z$.

Solution: The integration region W is given by

$$W = \Big\{(x, y, z) : \ \sqrt{6 - x^2 - y^2} \le z \le 4 - (x^2 + y^2)\Big\}.$$

Note that

$$\sqrt{6 - x^2 - y^2} = 4 - (x^2 + y^2) \Leftrightarrow x^2 + y^2 = 2.$$

Hence W can be represented by

$$W = \Big\{(x, y, z) : \ 0 \le x^2 + y^2 \le 2, \ \ \sqrt{6 - x^2 - y^2} \le z \le 4 - (x^2 + y^2)\Big\}.$$

In the cylinerical coordinates, W can be written as

$$W = \Big\{(r, \theta, z) : \ 0 \le r \le \sqrt{2}, \ 0 \le \theta \le 2\pi, \ \sqrt{6 - r^2} \le 2 \le 4 - r^2\Big\}.$$

Now the mass is given by

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\sqrt{6-r^2}}^{4-r^2} zr \, dz dr d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{1}{2}[(4 - r^2)^2 - (6 - r^2)]r \, dr d\theta = \frac{8\pi}{3}. \end{aligned}$$