Solution to the final review problems

December 11, 2013

1.

$$\begin{aligned}\int_C x\sin y\,dx + xyz\,dz &= \int_0^1 (t\sin t^2 + t^6(3t^2))\,dt \\ &= \frac{1}{2}\int_0^1 \sin u\,du + 3\int_0^1 t^8\,dt \\ &= \frac{1}{2}(1-\cos 1) + \frac{1}{3}.\end{aligned}$$

□

2. The circle can ba parametrized as: $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$. Hence

$$\begin{aligned}\int_C x^3 y\,dx - x\,dy &= \int_0^{2\pi} (\cos^3 t\sin t(-\sin t) - \cos^2 t)\,dt \\ &= -\int_0^{2\pi} (1-\sin^2 t)\sin^2 t\,d(\sin t) - \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt \\ &= -\frac{\sin^3 t}{3} + \frac{\sin^5 t}{5}\Big|_{t=0}^{2\pi} - \pi - \frac{\sin 2t}{4}\Big|_{t=0}^{2\pi} \\ &= -\pi.\end{aligned}$$

□

3.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \int_0^1 (-t^7, e^{-t^3})\cdot(d(t^2), d(-t^3)) \\ &= \int_0^1 (-2t^8 - 3t^2 e^{-t^3})\,dt \\ &= -\frac{2}{9} + e^{-t^3}\Big|_{t=0}^{1} \\ &= -\frac{2}{9} + (e^{-1} - 1).\end{aligned}$$

□

4.

$$\begin{aligned}\int_C x^2 y\,dx - xy^2\,dy &= \int_{\{x^2+y^2\le 1\}} ((-xy^2)_x - (x^2y)_y)\,dxdy \\ &= -\int_{\{x^2+y^2\le 1\}} (x^2+y^2)\,dxdy \\ &= -\int_0^{2\pi}\int_0^1 r^3\,drd\theta \\ &= -\frac{\pi}{2}.\end{aligned}$$

$\square$

5.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \int_{\{(x-2)^2+(y-3)^2\le 1\}} [2(\tan^{-1}(\frac{y}{x}))_x - (y-\ln(x^2+y^2))_y]\,dxdy \\ &= \int_{\{(x-2)^2+(y-3)^2\le 1\}} [2\frac{\frac{-y}{x^2}}{1+\frac{y^2}{x^2}} - 1 + \frac{2y}{x^2+y^2}]\,dxdy \\ &= -\int_{\{(x-2)^2+(y-3)^2\le 1\}} 1\,dxdy = -2.\end{aligned}$$

$\square$

6.

$$\begin{aligned}\text{curl}(\mathbf{F}) &= \langle (\frac{z}{\sqrt{x^2+y^2+z^2}})_y - (\frac{y}{\sqrt{x^2+y^2+z^2}})_z, (\frac{z}{\sqrt{x^2+y^2+z^2}})_x - (\frac{x}{\sqrt{x^2+y^2+z^2}})_z, \\ &\quad (\frac{y}{\sqrt{x^2+y^2+z^2}})_x - (\frac{x}{\sqrt{x^2+y^2+z^2}})_y\rangle \\ &= \langle 0,0,0\rangle.\end{aligned}$$

Direct calculation also implies

$$\text{curl}(\mathbf{G}) = 0.$$

$\square$

7. For the first vector field F, we have

$$\begin{aligned}\text{curl}(\mathbf{F}) &= \langle (-\sin z)_y - (x\cos(xy))_z,\ (y\cos(xy))_z + (\sin z)_x,\ (x\cos(xy))_x - (y\cos(xy))_y\rangle \\ &= \langle 0, 0, (\cos(xy) - xy\sin(xy)) - (\cos(xy) - xy\sin(xy))\rangle = \mathbf{0}.\end{aligned}$$

Hence it is conservative. To find f such that $F = \nabla f$, we need to solve

$$f_x = y\cos(xy),\ f_y = x\cos(xy),\ f_z = -\sin z.$$

Hence

$$f(x,y,z) = \cos z + g(x,y),$$

and

$$g_x = y\cos(xy),\ g_y = x\cos(xy).$$

Solving g we have

$$g(x,y) = \sin(xy) + c.$$

Putting all together implies $f(x,y,z) = \sin(xy) + \cos z + c$.

For the second vector field F, we have

$$\begin{aligned}\text{curl}(\mathbf{F}) &= \langle (y^2)_y - (x^2+2yz)_z,\ (2xy)_z - (y^2)_x,\ (x^2+2yz)_x - (2xy)_y\rangle \\ &= \langle 2y-2y, 0-0, 2x-2x)\rangle = \mathbf{0}.\end{aligned}$$

Hence it is also conservative. To find f such that $F = \nabla f$, we need to solve

$$f_x = 2xy,\ f_y = x^2 + 2yz,\ f_z = y^2.$$

hence

$$f(x,y,z) = x^2y + g(y,z)$$

where g satisfies

$$g_y + x^2 = x^2 + 2yz,$$

so that $g_y = 2yz$ and hence $g(y,z) = y^2z + h(z)$ and $f(x,y,z) = x^2y + y^2z + h(z)$.

Note that

$$f_z = y^2 + h'(z) = y^2.$$

Hence $h(z) = c$. Putting together we obtain

$$f(x,y,z) = x^2y + y^2z + c.$$

□

8.

$$\begin{aligned} A &= \int_0^1 \int_0^1 \sqrt{1 + z_x^2 + z_y^2}\, dxdy \\ &= \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dxdy \\ &= \frac{2}{3} \int_0^1 (1 + x + y)^{\frac{3}{2}} \Big|_{y=0}^1 dx \\ &= \frac{2}{3} \int_0^1 [(2 + x)^{\frac{3}{2}} - (1 + x)^{\frac{3}{2}}]\, dx \\ &= \frac{4}{15} [(2 + x)^{\frac{5}{2}} - (1 + x)^{\frac{5}{2}}] \Big|_0^1 \\ &= \frac{4}{15} (3^{\frac{5}{2}} - 2^{\frac{7}{2}} + 1). \end{aligned}$$

□

9. First we need to calculate the area element in the parametric form: $G(u,v) = \langle u^2, uv, \frac{1}{2}v \rangle$.

$$G_u = \langle 2u, v, 0 \rangle,\ G_v = \langle 0, u, v \rangle,$$

and

$$\mathbf{n}(u,v) = G_u \times G_v = \langle v^2, -2uv, 2u^2 \rangle.$$

$$\begin{aligned} A &= \int_{\{0 \le u \le 1,\ 0 \le v \le 2\}} |\mathbf{n}(u,v)|\, dudv \\ &= \int_{\{0 \le u \le 1,\ 0 \le v \le 2\}} \sqrt{v^4 + 2u^2v^2 + 4u^4}\, dudv \\ &= \int_{\{0 \le u \le 1,\ 0 \le v \le 2\}} (2u^2 + v^2)\, dudv \\ &= (\frac{2}{3} u^3 |_0^1)(\int_0^2 dv) + (\frac{1}{3} v^3 |_0^2)(\int_0^1 du) \\ &= \frac{4}{3} + \frac{8}{3} = 4. \end{aligned}$$

□

10. First we calculate the normal vector in the graphic form: $z(x,y) = 4 - x^2 - y^2$.

$$\mathbf{n}(x,y) = \langle -z_x, -z_y, 1\rangle = \langle 2x, 2y, 1\rangle.$$

Hence

$$\begin{aligned}\int\int_S \mathbf{F}\cdot d\mathbf{S} &= \int\int_{\{0\le x\le 1,\ 0\le y\le 1\}} \langle xy, y(4-x^2-y^2), (4-x^2-y^2)x\rangle \cdot \langle 2x, 2y, 1\rangle\, dxdy \\ &= \int_0^1\int_0^1 (4x + 8y^2 - x^3 - xy^2 - 2y^4 - 2x^2y^2 + 2x^2y)\, dxdy \\ &= 2 + \frac{8}{3} - \frac{1}{4} - \frac{2}{5} - \frac{1}{2}\cdot\frac{1}{3} + 2\frac{1}{3}\cdot\frac{1}{2} - 2(\frac{1}{3})^2 = 3\frac{173}{180}.\end{aligned}$$

□

11. First we want to find the equation of the surface (plane) P that passes through $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. It turns out that the equation is:

$$P:\ z = 1 - x - y,\ 0 \le x, y,\ x + y \le 1.$$

Hence the normal vector pointed upward is

$$\mathbf{n} = \langle 1, 1, 1\rangle.$$

Before applying Stokes' theorem, we need to compute the curl of F:

$$\begin{aligned}\mathrm{curl}(\mathbf{F}) &= \langle F_z^2 - F_y^3,\ F_x^3 - F_z^1,\ F_y^1 - F_x^2\rangle \\ &= \langle (y+z^2)_z - (z+x^2)_y, (z+x^2)_x - (x+y^2)_z,\ (x+y^2)_y - (y+z^2)_x\rangle \\ &= \langle 2z, 2x, 2y\rangle\end{aligned}$$

By Stokes' theorem we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \int_P \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} \\ &= \int_{\{0\le x\le 1,\ 0\le y\le 1-x\}} (2x + 2y + 2(1-x-y))\, dxdy \\ &= 2\int_0^1\int_0^{1-x} dy\, dx = 1.\end{aligned}$$

□

12. By Stokes' theorem, we have

$$\int\int_S \mathrm{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{s},$$

where $C = \{(x,y,z):\ x^2 + y^2 = 9,\ z = 0\}$ is the unit circle in the z-plane that is oriented counterclockwise.

Recall that

$$F(x,y,0) = \langle 2y\cos 0, e^x \sin 0, xe^y\rangle = \langle 2y, 0, xe^y\rangle,$$

and

$$d\mathbf{s} = \langle dx, dy, 0\rangle.$$

Hence

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C 2y\,dx.$$

Parametrize C by

$$x = 3\cos t,\ \ y = 3\sin t,\ \ 0 \le t \le 2\pi.$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C 2y\,dx = 2\int_0^{2\pi} 3\sin t d(3\cos t) = -18\int_0^{2\pi} \sin^2 t\,dt = -18\pi.$$

This implies

$$\text{curl}(\mathbf{F}) = -18\pi.$$

□