MA 266—066 & 067 Fall 2016
Exam 2 November 9, 2016

Exam Scores

Question	Score	Total
1		15
2		15
3		15
4		15
5		15
6		15
7		10
Total		100

Name: ______________________________

Section: ____________________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- Additional blank sheets for scratch work are available upon request.
- **All Question Are Free Response Questions:**
  Show all your work on the page of the problem. Show all your work. Clearly indicate your answer and the reasoning used to arrive at that answer.

Unsupported answers for the questions may not receive credit!

1. Find the solution of the initial value problem of the following equation:

$$\begin{cases} y'' - 2y' + 5y = 0, \\ y(\frac{\pi}{2}) = -2, \\ y'(\frac{\pi}{2}) = 2. \end{cases}$$

$$\lambda^2 - 2\lambda + 5 = 0 \Rightarrow (\lambda - 1)^2 + 2^2 = 0 \Rightarrow \lambda = 1 \pm 2i$$

$$y(t) = e^t(c_1 \cos 2t + c_2 \sin 2t)$$

$$y'(t) = e^t(c_1 \cos 2t + c_2 \sin 2t) + e^t(-2c_1 \sin 2t + 2c_2 \cos 2t)$$

$$y(\tfrac{\pi}{2}) = e^{\pi/2}(-c_1) = -2 \Rightarrow c_1 = 2e^{-\pi/2}$$

$$y'(\tfrac{\pi}{2}) = \underline{e^{\pi/2}(-c_1)} + e^{\pi/2}(-2c_2) = 2 \Rightarrow \underline{c_2 = -2e^{-\pi/2}}$$

$$y(t) = e^t(2e^{-\pi/2}\cos 2t - 2e^{-\pi/2}\sin 2t)$$

$$= \boxed{2e^{t-\pi/2}(\cos 2t - \sin 2t)}$$

2. Find the general solution of the following 4th order differential equation:

$$y^{(4)} - 5y^{(2)} - 36y = 0.$$

$\lambda^4 - 5\lambda^2 - 36 \Rightarrow (\lambda^2 - 9)(\lambda^2 + 4) = 0$

$\Rightarrow \lambda_1 = 3,\ \lambda_2 = -3,\ \lambda_{3,4} = \pm 2i$

$\Rightarrow y(t) = \boxed{c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos 2t + c_4 \sin 2t}$

3. Find the suitable form of a particular solution of $Y(t)$ of the equation:

$$y^{(3)} + 3y^{(2)} + 3y' + y = \sin(2t) + te^{-t}.$$

(Hint: $r^3 + 3r^2 + 3r + 1 = (r+1)^3$)

- $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \Rightarrow (\lambda+1)^3 = 0$

  $\lambda = -1 \ (m=3)$

- $y_c(t) = (c_1 + c_2 t + c_3 t^2)e^{-t}$.

- $Y(t) = (A\cos 2t + B\sin 2t) + \underline{t}^3(Ct + D)e^{-t}$

4. (**a**) (5 points) First verify that the given functions $y_1 = t$ and $y_2 = te^t$ satisfy the corresponding homogeneous equation.

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \ t > 0.$$

(**b**) (10 points) Then use the variation of parameters to find a particular solution of the given nonhomogeneous equation:

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \ t > 0; \ \ y_1(t) = t, \ y_2(t) = te^t.$$

a) $t^2y_1'' - t(t+2)y_1' + (t+2)y_1$

$= t^2(t)'' - t(t+2)\,t' + (t+2)t$

$= 0 - t(t+2) + (t+2)t = 0$

$t^2(te^t)'' - t(t+2)(te^t)' + (t+2)\,te^t$

$= t^2\left[(1+t)e^t + e^t\right] - t(t+2)\left[e^t + te^t\right] + (t+2)te^t$

$= e^t\{t^2(2+t) - t(t+2)(1+t) + (t+2)t] = 0.$

b) $y_p(t) = u_1\, t + u_2\, te^t$

$$\begin{bmatrix} t & te^t \\ 1 & (1+t)e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2t \end{bmatrix}$$

$u_1' = \dfrac{-2t^2e^t}{t^2e^t} = -2 \qquad u_1 = -2t$

$u_2' = \dfrac{2t^2}{t^2e^t} = 2e^{-t} \qquad u_2 = -2e^{-t}$

$y_p(t) = -2t\cdot t - 2e^{-t}\,te^t = -2t - 2t^2.$

5. A mass weighing 4 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb-s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/s. Find its position u at any time t.

$$k = \frac{4}{\frac{1}{4}} = 16, \quad \gamma = 2, \quad m = \frac{4}{32} = \frac{1}{8}$$

$$u(0) = 0, \quad u'(0) = \frac{1}{4}$$

$$\left.\begin{array}{l} \frac{1}{8}u'' + 2u' + 16u = 0 \\ \qquad u(0) = 0, \ u'(0) = \frac{1}{4} \end{array}\right\}$$

$$\left.\begin{array}{l} u'' + 16u' + 128u = 0 \\ \qquad u(0) = 0, \ u'(0) = 1/4 \end{array}\right\}$$

$$\lambda^2 + 16\lambda + 128 = 0 \qquad (\lambda + 8)^2 + 8^2 = 0$$

$$\Rightarrow \lambda = -8 \pm 8i$$

$$u(t) = e^{-t}(c_1 \cos 8t + c_2 \sin 8t)$$

$$u'(t) = -e^{-t}(c_1 \cos 8t + c_2 \sin 8t) + e^{-t}(-8c_1 \sin 8t + 8c_2 \cos 8t)$$

$$u(0) = c_1 = 0, \quad u'(0) = -c_1 + 8c_2 = \frac{1}{4}$$

$$c_1 = 0, \quad c_2 = \frac{1}{32} \qquad \Rightarrow \boxed{u(t) = \frac{1}{32} e^{-8t} \sin 8t}$$

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6. Use the characteristics method to solve the initial value problem

$$y''' - y'' + y' - y = 0; \quad y(0) = 2,\ y'(0) = -1,\ y''(0) = -2.$$

• $\lambda^3 - \lambda^2 + \lambda - 1 = 0$ $\qquad \lambda^2(\lambda-1) + (\lambda-1) = 0$

$\lambda_1 = 1,\ \lambda_2 = \pm i \qquad \Leftarrow (\lambda-1)(\lambda^2+1) = 0$

• $y(t) = c_1 e^t + c_2 \cos t + c_3 \sin t$

$y'(t) = c_1 e^t - c_2 \sin t + c_3 \cos t$

$y''(t) = c_1 e^t - c_2 \cos t - c_3 \sin t$

$$\begin{cases} c_1 + c_2 = 2 \\ c_1 + c_3 = -1 \\ c_1 - c_2 = -2 \end{cases}$$

$\Rightarrow c_1 = 0,\ c_2 = 2,\ c_3 = -1$

$$\boxed{y(t) = 2\cos t - \sin t}$$

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7. Use Laplace transform to solve the initial value problem

$$y'' + 2y' + y = 5e^{-t}; \quad y(0) = 2,\ y'(0) = -1.$$

$Y(s) = \mathcal{L}\{y\}(s).$

$$(s^2Y - s\cdot 2 + 1) + 2(sY - 2) + Y = \frac{5}{s+1}$$

$$\Rightarrow (s^2+2s+1)\,Y(s) = 2s+3 + \frac{5}{s+1}$$

$$\Rightarrow Y(s) = \frac{2s+3}{s^2+2s+1} + \frac{5}{(s+1)(s^2+2s+1)}$$

$$= \frac{2(s+1)+1}{(s+1)^2} + \frac{5}{(s+1)^3}$$

$$= \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{5}{(s+1)^3}$$

$$\Rightarrow \boxed{y(t) = 2e^{-t} + te^{-t} + \frac{5}{2}t^2e^{-t}.}$$