Solutions to Review Problems for Midterm II (MA 266, Fall 2016)

The second midterm exam will be taken place at

8:00-9:00 pm, Wednesday, November 9, 2016

Sections to be covered

The sections to be covered in this example contain

1. Chapter 3.3: Complex roots
2. Chapter 3.4: Repeated real roots
3. Chapter 3.5: Method of undetermined coefficients
4. Chapter 3.6: Variation of parameters
5. Chapter 3.7: Mechanical vibrations
6. Chapter 3.8: Forced vibrations
7. Chapter 4.1-Chapter 4.3: Higher order constant coefficient ODEs (Methods of characteristics and undetermined coefficients)
8. Chapter 6.1: Laplace transform
9. Chapter 6.2: Initial value problem by Laplace transform

Some Review Problems

1. Solve the initial value problem

$$y'' + 2y' + 2y = 0, \ y(\frac{\pi}{4}) = 2, \ y'(\frac{\pi}{4}) = -2.$$

Solution: The char. eqn. $r^2 + 2r + 2 = 0$ has two complex roots $r = -1 \pm i$. Hence the general solution is

$$y(t) = e^{-t}(c_1 \cos t + c_2 \sin t).$$

Direct calculation implies

$$y'(t) = e^{-t}[(c_2 - c_1)\cos t - (c_1 + c_2)\sin t].$$

The initial condition gives

$$e^{-\frac{\pi}{4}}\frac{\sqrt{2}}{2}(c_1 + c_2) = 2; \ e^{-\frac{\pi}{4}}\frac{\sqrt{2}}{2}[(c_2 - c_1) - (c_1 + c_2)] = -2.$$

Hence $c_1 = c_2 = \sqrt{2}e^{\frac{\pi}{4}}$, and $y(t) = \sqrt{2}e^{\frac{\pi}{4}-t}(\cos t + \sin t)$. □

2. Consider the initial value problem

$$y'' + 2ay' + (a^2 + 1)y = 0, \ y(0) = 1, \ y'(0) = 0.$$

(a) Find the solution $y(t)$ of this problem, and
(b) for $a = 1$ find the smallest T such that $|y(t)| < 0.1$ for $t > T$.

Solution: The char. eqn. $r^2 + 2ar + (a^2 + 1) = 0$ has two complex roots $r = -a \pm i$. A general solution is given by

$$y(t) = e^{-at}(c_1 \cos t + c_2 \sin t).$$

So that

$$y'(t) = e^{-at}[(-ac_1 + c_2)\cos t - (ac_2 - c_1)\sin t].$$

The initial condition gives

$$c_1 = 1, -ac_1 + c_2 = 0.$$

Thus $c_1 = 1, \ c_2 = a$, and $y(t) = e^{-at}(\cos t + a \sin t)$.

If $a = 1$, then $y(t) = e^{-t}\sqrt{2}\cos(t - \frac{\pi}{4})$. $|y(t)| < 0.1$ if $e^{-t}\sqrt{2} < 0.1$. Hence $T = \ln(10\sqrt{2})$. □

3. Solve the initial value problem

$$9y'' + 6y' + 82y = 0, \ y(0) = -1, \ y'(0) = 2.$$

Solution: The char. eqn. $9r^2 + 6r + 82 = 0$ has two roots $r = \frac{-1 \pm 9i}{3}$. Hence

$$y(t) = e^{-t/3}(c_1 \cos 3t + c_2 \sin 3t).$$

Applying the initial condition, we obtain $c_1 = -1, c_2 = 5/9$ so that

$$y(t) = e^{-t/3}(-\cos 3t + 5/9 \sin 3t).$$

□

4. Consider the initial value problem

$$9y'' + 12y' + 4y = 0,\ y(0) = a,\ y'(0) = -1.$$

(a) Solve the initial value problem, and
(b) find the critical value of a that separates solutions that become negative from those that are always positive.
Solution: The char. eqn. $9r^2 + 12r + 4 = 0$ has two equal roots $r = -2/3$ so that

$$y(t) = (c_1 + c_2 t)e^{-2/3t}.$$

The initial condition implies

$$c_1 = a,\ c_2 = \frac{2}{3}a - 1.$$

Hence $y(t) = [(a + (\frac{2}{3}a - 1)t]e^{-2/3t}$. The critical value of $a = 3/2$. □

5. Use the variation of parameters to find the second solution of

$$(x-1)y'' - xy' + y = 0,\ x > 1;\ y_1(x) = e^x.$$

Solution Set $y_2(x) = u(x)e^x$ so that

$$y_2'(x) = (u' + u)e^x,\ y''(x) = (u'' + 2u' + u)e^x.$$

Plugging them into the equation gives

$$(x-1)u'' + (x-2)u' = 0.$$

Hence $u'(x) = (x-1)e^{-x}$ and $u(x) = -xe^{-x}$. Hence $y_2(x) = -x$. □

6. Solve the initial value problem by the method of undetermined coefficients

$$y'' + 2y' + 5y = 4e^{-t}\cos 2t,\ y(0) = 1,\ y'(0) = 0.$$

Solution The homogeneous equation has general solutions given by

$$y_c(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t).$$

Try the particular solution as

$$y_p(t) = te^{-t}(A\cos 2t + B\sin 2t).$$

Plugging it into the equation, we obtain: $A = 0$ and $B = 1$. Thus general solution of the inhomogeneous equation is:

$$y(t) = e^{-t}[c_1 \cos 2t + (c_2 + t)\sin 2t].$$

Now applying the initial condition, we have $c_1 = 1$ and $c_2 = 1/2$. Thus

$$y(t) = e^{-t}[\cos 2t + (1/2 + t)\sin 2t].$$

$\square$

7. Find the correct form of a particular solution of

$$y'' + 4y = t^2 \sin 2t + (6t + 7) \cos 2t.$$

Solution The homogeneous equation has general solutions given by

$$y_c(t) = c_1 \cos 2t + c_2 \sin 2t.$$

The correct form of a particular solution is given by

$$y_p(t) = (At^3 + Bt^2 + Ct) \cos 2t + (Dt^3 + Et^2 + Ft) \sin 2t.$$

$\square$

8. Use the variation of parameters to find the general solution of

$$x^2 y'' - 3xy' + 4y = x^2 \ln x,\ x > 0;\ y_1(x) = x^2,\ y_2(x) = x^2 \ln x.$$

Solution $y_p(x) = u_1(x)x^2 + u_2(x)x^2 \ln x$, where

$$x^2 u_1' + x^2 \ln x u_2' = 0;\ 2xu_1' + x(1 + 2 \ln x)u_2' = \ln x.$$

Hence

$$u_1' = -\frac{(\ln x)^2}{x},\ u_2' = \frac{\ln x}{x}.$$

Thus

$$u_1 = -\frac{(\ln x)^3}{3},\ u_2 = \frac{(\ln x)^2}{2}.$$

$$y_p(x) = \frac{1}{6} x^2 (\ln x)^3.$$

$\square$

9. A spring is stretched 10 cm by a force of 3 N. A mass of 2kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is $5m/s$. If the mass is pulled down 5cm below its equilibrium position and given an initial downward velocity of $10cm/s$, determine its position u at any time t.

Solution $m = 2$kg, $\gamma = 3/5$N.s/m $= 3/49$ kg.s/m, and $k = 3N/0.1m = 30/9.8$ kg. The equation is:

$$2u'' + 3/49u' + 30/9.8u = 0;$$

along with the initial condition:

$$u(0) = 0.02m;\ u'(0) = 0.1m/s.$$

The rest is left over for you guys to check. $\square$

10. Find the general solutions of

$$y^{(4)} + y = 0,$$

$$y^{(4)} - 5y^{(2)} + 4y = 0.$$

Solution The first equation has general solution by

$$y(t) = e^{\frac{\sqrt{2}}{2}t}(c_1 \cos(\frac{\sqrt{2}}{2}t) + c_2 \sin(\frac{\sqrt{2}}{2}t)) + e^{-\frac{\sqrt{2}}{2}t}(c_3 \cos(\frac{\sqrt{2}}{2}t) + c_4 \sin(\frac{\sqrt{2}}{2}t)).$$

The second equation has

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}.$$

□

11. Find the general solution of

$$y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0.$$

Solution The char. eqn. is

$$r^4 + 6r^3 + 17r^2 + 22r + 14 = 0.$$

This equation can be factored as

$$(r^2 + 2r + 2)(r^2 + 4r + 7) = 0,$$

which has roots

$$r_1 = -1 \pm i,\ r_3 = -2 \pm \sqrt{3}i,$$

so that the general solution is given by

$$y_c(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t + c_3 e^{-2t} \cos(\sqrt{3}t) + c_4 e^{-2t} \sin(\sqrt{3}t).$$

□

12. Solve the initial value problem of

$$y''' - 3y'' + 2y' = t + e^t,\ y(0) = 1,\ y'(0) = -\frac{1}{4},\ y''(0) = -\frac{3}{2}.$$

Solution The general solution to the homogeneous equation is:

$$y_c(t) = c_1 + c_2 e^t + c_3 e^{2t}.$$

A particular solution to the inhomogeneous equation is given by

$$y_p(t) = At^2 + Bt + Cte^t.$$

Plugging it into the equation, we can solve

$$A = 1/4,\ B = 3/4,\ C = -1.$$

Hence a general solution is given by

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} + (1/4t^2 + 3/4t) - te^t.$$

Calculating

$$y' = c_2e^t + 2c_3e^{2t} + (1/2t + 3/4) - (1+t)e^t;$$
$$y'' = c_2e^t + 4c_3e^{2t} + 1/2 - (2+t)e^t.$$

Applying the initial condition, we get

$$\begin{cases} c_1 + c_2 + c_3 = 1, \\ c_2 + 2c_3 = 0 \\ c_2 + 4c_3 = 0. \end{cases}$$

Hence $c_1 = 1, c_2 = c_3 = 0$. The unique solution is then given by

$$y(t) = 1 + 3/4t + 1/4t^2 - te^t.$$

□

13. Find a suitable form of particular solution of

$$y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t.$$

Solution The general solution of the homogeneous equation is:

$$y_c(t) = c_1 + c_2t + c_3e^{-t}\cos t + c_4e^{-t}\sin t.$$

Hence the correct form for a particular solution is given by

$$y_p(t) = Ae^t + (Bt + C)e^{-t} + te^{-t}(D\cos t + E\sin t).$$

14. Use the definition to find the Laplace transform of the following functions:
(a) $f(t) = t^2$, $t > 0$.
(b)

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < +\infty. \end{cases}$$

Solution
(a)

$$\begin{aligned} \mathcal{L}(t^2)(s) &= \int_0^\infty e^{-st}t^2\,dt = \int_0^\infty t^2\,d(e^{-st}/-s) \\ &= t^2e^{-st}/-s\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}2t\,dt \\ &= \frac{2}{s}\int_0^\infty t\,d(e^{-st}/-s) = \frac{2}{s^2}\int_0^\infty e^{-st}\,dt \\ &= \frac{2}{s^3},\ s > 0. \end{aligned}$$

(b)

$$\mathcal{L}(f) \;\; = \;\; \int_0^1 te^{-st}\,dt + \int_1^\infty e^{-st}\,dt = \frac{1-e^{-s}}{s^2}$$

15. Use Laplace transform to solve the initial value problem:

$$y'' + \omega^2 y = \cos 2t,\;\; \omega^2 \neq 4,\;\; y(0) = 1,\;\; y'(0) = 0.$$

Solution: We have

$$(s^2+\omega^2)Y(s) = \frac{s}{s^2+4} + s.$$

Hence

$$Y(s) = \frac{s}{(s^2+4)(s^2+\omega^2)} + \frac{s}{s^2+\omega^2} = \frac{1}{\omega^2-4}[\frac{s}{s^2+4} - \frac{s}{s^2+\omega^2}] + \frac{s}{s^2+\omega^2}.$$

Hence

$$y(t) = \frac{1}{\omega^2-4}(\cos 2t - \cos\omega t) + \cos\omega t.$$

$$y^{(4)} - 4y''' + 6y'' - 4y' + y = 0;\quad y(0) = 0,\;\; y'(0) = 1,\;\; y''(0) = 0,\;\; y'''(0) = 1.$$

Solution:

$$s^4Y - s^2 - 1 - 4(s^3Y - s) + 6(s^2Y - 1) - 4sY + Y = 0.$$

Hence

$$Y(s) = \frac{s^2-4s+7}{s^4-4s^3+6s^2-4s+1} = \frac{s^2-4s+7}{(s-1)^4} = \frac{1}{(s-1)^2} - 2\frac{1}{(s-1)^3} + \frac{4}{(s-1)^4}.$$

Hence

$$y(t) = e^t t - e^t t^2 + 4e^4t^3 3! = e^t t - e^t t^2 + \frac{2}{3}e^t t^3.$$