Solution to HW#1, MA523

1) By the formula of integration by parts, one has

$$\begin{cases} \int_\Omega f \Delta g = -\int_\Omega \nabla f \nabla g + \int_{\partial\Omega} f \frac{\partial g}{\partial \nu}, & (1) \\ \int_\Omega g \Delta f = -\int_\Omega \nabla g \cdot \nabla f + \int_{\partial\Omega} g \frac{\partial f}{\partial \nu}. & (2) \end{cases}$$

Subtracting (1) from (2) yield the 2nd Green's identity.

2) For any $(x_0, t_0) \in \mathbb{R}^n \times (0, +\infty)$, define the function z that is the restriction of a solution u on its characteristic line:

$$z(s) = u(x_0 + bs, t_0 + s), \quad s \in \mathbb{R}.$$

Then we have

$$\begin{aligned} \frac{dz}{ds} &= (u_t + b \cdot Du)(x_0 + bs, t_0 + s) \\ &= -cu(x_0 + bs, t_0 + s) \qquad \text{(by eq'n)} \\ &= -cz(s) \end{aligned}$$

$$\Rightarrow \quad Z(s) = Z(0)\, e^{-cs} \Rightarrow Z(0) = Z(s)\, e^{cs}$$

Note $u(x_0, t_0) = Z(0) = Z(s)\, e^{cs} = u(x_0 + bs, t_0 + s)\, e^{cs}$

Set $s = -t_0$, we obtain

$$u(x_0, t_0) = u(x_0 - bt_0, 0)\, e^{-ct_0}$$

$$= g(x_0 - bt_0)\, e^{-ct_0} \quad \text{(use the initial condition).}$$

Hence $\boxed{u(x,t) = g(x - bt)\, e^{-ct}}$

3). Set $Z(s) = u(x + as, y + bs)$. Then

$$\frac{dZ}{ds} = (a u_x + b u_y)(x + as, y + bs)$$

$$= -cu(x + as, y + bs) = -cZ(s)$$

$$\Rightarrow \quad Z(s) = Z(0)\, e^{-cs}$$

$$\Rightarrow \quad u(x, y) = Z(0) = Z(s)\, e^{cs} = u(x + as, y + bs)\, e^{cs}$$

Choose $s = -\frac{y}{b}$. so that $u(x, y) = u(x - \frac{a}{b} y, 0)\, e^{-\frac{a}{b} y}$

$$= f(x-\tfrac{a}{b}y)e^{-\frac{c}{b}y}$$

Hence the solution is given by

$$\boxed{u(x,y) = f(x-\tfrac{a}{b}y)e^{-\frac{c}{b}y}}$$

4) set $z(s) = u(x+s, y-s)$. Then

$$\frac{dz}{ds} = (u_x - u_y)(x+s, y-s)$$

$$= e^{3(x+s)-4(y-s)} - 5u(x+s, y-s)$$

$$= e^{3x-4y+7s} - 5z(s)$$

$\Rightarrow$ (By integral of factor method)

$$\frac{d}{ds}\left(e^{5s}z(s)\right) = e^{3x-4y+12s}$$

$$\Rightarrow e^{5s}z(s) - e^{5\cdot 0}z(0) = \int_0^s e^{3x-4y+12\tau}\,d\tau$$

$$= e^{3x-4y}\frac{e^{12s}-1}{12}$$

$$\Rightarrow \quad z(0) = e^{5s} z(s) - \frac{e^{12s} - 1}{12} e^{3x-4y}$$

choose $s = y$, we obtain

$$u(x,y) = e^{5y}\, u(x+y, 0) - \frac{e^{12y} - 1}{12} e^{3x-4y}$$

$$= e^{5y} \cdot 0 + \frac{1 - e^{12y}}{12} e^{3x-4y}$$

$$= \frac{e^{3x-4y} - e^{3x+8y}}{12} \quad \leftarrow \text{solution}$$