

Heat Flow of Biharmonic Maps in Dimensions Four and Its Application

Changyou Wang

Abstract. *Let (M, g) be a four dimensional compact Riemannian manifold without boundary, $(N, h) \subset \mathbf{R}^k$ be a compact Riemannian submanifold without boundary. We establish the existence of a global weak solution to the heat flow of extrinsic biharmonic maps from M to N , which is smooth away from finitely many singular times. As a consequence, we prove that if $\Pi_4(N) = \{0\}$, then any free homotopy class $\alpha \in [M, N]$ contains at least one minimizing biharmonic map.*

§1. Introduction

Let (M, g) , (N, h) be smooth compact Riemannian manifolds without boundaries. Assume that (N, h) is isometrically embedded into an Euclidean space \mathbf{R}^k . For a non-negative integer l , and $1 \leq p < +\infty$, the Sobolev space $W^{l,p}(M, N)$ is defined by

$$W^{l,p}(M, N) = \{u \in W^{l,p}(M, \mathbf{R}^k) \mid u(x) \in N \text{ for a.e. } x \in M\}.$$

On $W^{2,2}(M, N)$, there are two natural, second order energy functionals:

$$H(u) = \int_M |\Delta u|^2 dv_g, \quad T(u) = \int_M |\tau(u)|^2 dv_g,$$

where Δ is the Laplace-Beltrami operator of (M, g) , dv_g is the volume element of (M, g) , $\tau(u) = (\Delta u)^T := \Delta u + A(u)(\nabla u, \nabla u)$ is the tension field of u , and $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of (N, h) in \mathbf{R}^k .

Recall that a map $u \in W^{2,2}(M, N)$ is called an extrinsic (or intrinsic, resp.) biharmonic map if u is a critical point of $H(\cdot)$ (or $T(\cdot)$, resp.). For a sufficiently small $\delta > 0$, let $\Pi : N_\delta \rightarrow N$ be the smooth nearest point projection map,

$P(y) = \nabla\Pi(y) : \mathbf{R}^k \rightarrow T_yN$ be the orthogonal projection to the tangent space at $y \in N$. Note that

$$A(y)(X, Y) = \nabla_X P(y)(Y), \quad \forall y \in N, X, Y \in T_yN.$$

It is readily seen (cf. Wang [W1, 2]) that the Euler-Lagrange equation for extrinsic biharmonic maps is

$$\Delta^2 u = \Delta(A(u)(\nabla u, \nabla u)) + \langle \Delta u, \Delta(P(u)) \rangle + 2\langle \nabla \Delta u, \nabla(P(u)) \rangle. \tag{1.1}$$

(1.1) is equivalent to the geometric form:

$$\Delta^2 u \perp T_u N, \tag{1.2}$$

in the sense of distributions.

Regularity issues for biharmonic maps have first been studied by Chang-Wang-Yang [CWY] for spheres $N = \mathbf{S}^{k-1}$, and later by Wang [W1,2] for general manifolds N (see also [W3] and Strzelecki [P]).

Motivated by the heat flow of harmonic maps from surfaces (see Struwe [S] and Sacks-Uhlenbeck [SaU]), and the problem by Eells-Lemaire [EL] *that is to find extrinsic (or intrinsic) biharmonic maps among any free homotopy class $\alpha \in [M, N]$ for $\dim(M) = 4$* , people are interested in the study of the heat flow of biharmonic maps $u : M \times \mathbf{R}_+ \rightarrow N$:

$$u_t + \Delta^2 u = \Delta(A(u)(\nabla u, \nabla u)) + \langle \Delta u, \Delta(P(u)) \rangle + 2\langle \nabla \Delta u, \nabla(P(u)) \rangle, \quad M \times \mathbf{R}_+, \tag{1.3}$$

$$u(x, 0) = \phi(x), \quad x \in M, \tag{1.4}$$

where $\phi \in W^{2,2}(M, N)$ is a given map.

For a smooth map $\phi \in C^\infty(M, N)$, the short time existence of smooth solutions to (1.3)-(1.4) is well-known, since (1.3) is a fourth order strongly parabolic system (see Lamm [L1] for more details). Moreover, if $\dim M \leq 3$, then such a short time smooth solution can be extended to be a globally smooth solution. For $\dim M \geq 4$, the short time smooth solution may develop a singularity at finite time. For $\dim M = 4$, Lamm [L2] proved that (1.3)-(1.4) has a globally smooth solution $u \in C^\infty(M \times \mathbf{R}_+, N)$, provided that $\phi \in C^\infty(M, N)$ has small Hessian energy $H(\phi)$. Without the smallness assumption, we establish a partially smooth, weak solution to (1.3)-(1.4). More precisely, we have

Theorem A. *For $\dim M = 4$ and any map $\phi \in W^{2,2}(M, N)$, there exists a global weak solution $u : M \times \mathbf{R}_+ \rightarrow N$ of (1.3)-(1.4) satisfying:*

(1) *For any $0 < T < +\infty$,*

$$\int_0^T \int_M |u_t|^2 dv_g dt + \int_M |\Delta u|^2(\cdot, T) dv_g \leq \int_M |\Delta \phi|^2 dv_g, \tag{1.5}$$

and $H(u(\cdot, t))$ is monotonically nonincreasing with respect to $t \geq 0$.

(2) There exist an $\epsilon_0 > 0$, a positive integer L depending only on ϕ, M, N , and $0 < t_1 < \dots < t_L < +\infty$ such that $u \in C^\infty(M \times (\mathbf{R}_+ \setminus \cup\{t_1, \dots, t_L\}), N)$, and

$$\mathcal{E}(u, t_i) := \lim_{r \downarrow 0} \limsup_{t \uparrow t_i} \sup_{x \in M} \int_{B_r(x)} |\Delta u|^2 dv_g \geq \epsilon_0^2, \quad 1 \leq i \leq L. \quad (1.6)$$

(3) The quantity ϵ_0 can be characterized by

$$\epsilon_0^2 = \inf \left\{ \int_{\mathbf{R}^4} |\Delta \omega|^2 dx \mid \omega \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N) \text{ nonconstant biharmonic maps} \right\}. \quad (1.7)$$

(4) For $1 \leq i \leq L$, there exist a nonconstant biharmonic map $\omega_i \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N)$, $t_{j_i} \uparrow t_i$, $\{x_{j_i}\} \subset M$ with $x_{j_i} \rightarrow x_i$, $r_{j_i} \rightarrow 0$, such that

$$u_{j_i}(\cdot) = u(x_{j_i} + r_{j_i} \cdot, t_{j_i}) \rightarrow \omega_i, \quad \text{in } C_{loc}^k(\mathbf{R}^4) \quad \forall k \geq 1. \quad (1.8)$$

Remark 1.1. After the submission of the paper, we have learned that Theorem A has also been independently proved by Gastel [G] recently.

Remark 1.2. It is an open question *whether the weak solution as in theorem A has at most finitely many singularities (i.e. at most finitely many singular points in each singular time).*

Remark 1.3. Motivated by the existence theory of the heat flow of harmonic maps in high dimensions by Chen-Struwe [CS], it is a very interesting problem to study the heat flow of biharmonic maps in dimensions at least five. The main difficulty is the lack of suitable monotonicity formulas of parabolic types. However, in a forthcoming article [W4], we are able to establish the existence of smooth solutions to (1.3)-(1.4), under the assumption that $\dim M \leq 8$ and $H(\phi)$ is sufficiently small.

Remark 1.4. It is also a very natural question to study the heat flow of intrinsic biharmonic maps. In [L3], Lamm proved the existence of smooth solutions to the heat flow of intrinsic biharmonic maps provided that $\dim M = 4$ and N has non-positive sectional curvature. However, without this condition on curvature it seems difficult to obtain a global, weak solution due to the lack of coercivity property of $T(\cdot)$.

Remark 1.5. The fourth order heat flow has been recently employed by Kuwert-Schätzle [KS1,2] in the study of Willmore functionals. Some earlier existence results on Willmore surfaces were established by Simon [S1]. The interested readers may find that there are similar analytic techniques between the Willmore flow and the heat flow of biharmonic maps studied here. In particular, the integral estimate and interpolation inequalities are common themes.

As a consequence of theorem A, we prove

Theorem B. *If $\dim M = 4$ and $\Pi_4(N) = \{0\}$, then any free homotopy class $\alpha \in [M, N]$ contains at least one biharmonic map $u \in C^\infty(M, N)$ that minimizes the Hessian energy, i.e. $H(u) = \min\{H(v) \mid v \in C^\infty(M, N), [v] = \alpha\}$.*

The paper is written as follows. In §2, we recall density of $C^\infty(M, N)$ in $W^{2,2}(M, N)$ (see, Brezis-Nirenberg [BN]) and prove the quantization effect (1.7). In §3, we review some integral estimates for smooth solutions of (1.3) by Lamm [L2], the characterization of the first singular time, and prove Theorem A. In §4, we give a proof of Theorem B.

Acknowledgement. The author thanks the referee for his careful readings and many helpful suggestions. This work is partially supported by NSF grant 0400718.

§2. Density of smooth maps and quantization effect (1.7)

In this section, we establish a Bochner type inequality and an ϵ -gradient estimate for smooth biharmonic maps and prove the quantization fact (1.7). We also recall a well-known fact (see, [BN]) on the density of smooth maps in $W^{2,2}(M, N)$ whenever $\dim M = 4$. First we have

Proposition 2.1. *There are $C_1, C_2 > 0$ depending only on N such that if $u \in C^\infty(\mathbf{R}^n, N)$ is an (extrinsic) biharmonic map, then $e(u) \equiv |\Delta u|^2 + C_1|\nabla u|^4$ satisfies*

$$\Delta e(u) + C_2 e(u)(1 + e(u)) \geq 0, \text{ in } \mathbf{R}^n. \tag{2.1}$$

Proof. Note $(\Delta u)^T = P(u)(\Delta u)$ and $\Delta u = (\Delta u)^T + A(u)(\nabla u, \nabla u)$. It follows from (1.2) that we have $\langle \Delta^2 u, (\Delta u)^T \rangle = 0$. Hence

$$\Delta(|\Delta u|^2) = 2\langle \Delta^2 u, \Delta u \rangle + 2|\nabla \Delta u|^2 = 2\langle \Delta^2 u, A(u)(\nabla u, \nabla u) \rangle + 2|\nabla \Delta u|^2. \tag{2.2}$$

This and (1.1) imply

$$\begin{aligned} \Delta(|\Delta u|^2) &= 2|\nabla \Delta u|^2 + 2\langle \Delta(A(u)(\nabla u, \nabla u)), A(u)(\nabla u, \nabla u) \rangle \\ &\quad + 2\langle \langle \nabla \Delta u, \nabla(P(u)) \rangle, A(u)(\nabla u, \nabla u) \rangle \\ &\quad + \langle \langle \Delta(P(u)), \Delta u \rangle, A(u)(\nabla u, \nabla u) \rangle \\ &= 2|\nabla \Delta u|^2 + II + III + IV. \end{aligned} \tag{2.3}$$

It is easy to see that there exist C_3, C_4 depending only on N such that

$$|III| \leq 2\|\nabla P\|_{L^\infty(N)}\|A\|_{L^\infty(N)}|\nabla \Delta u||\nabla u|^3 \leq \frac{1}{2}|\nabla \Delta u|^2 + C_3(|\nabla u|^4 + |\nabla u|^8), \tag{2.4}$$

and

$$\begin{aligned}
|IV| &\leq \|A\|_{L^\infty(N)} |\Delta(P(u))| |\Delta u| |\nabla u|^2 \\
&\leq \|A\|_{L^\infty(N)} |\Delta u| |\nabla u|^2 (\|\nabla P\|_{L^\infty(N)} |\Delta u| + \|\nabla^2 P\|_{L^\infty(N)} |\nabla u|^2) \\
&\leq C_4 (|\Delta u|^2 |\nabla u|^2 + |\Delta u| |\nabla u|^4) \\
&\leq C_4 (|\Delta u|^2 + |\Delta u|^4 + |\nabla u|^4 + |\nabla u|^8). \tag{2.5}
\end{aligned}$$

To estimate II , we first calculate $\Delta(A(u)(\nabla u, \nabla u))$ as follows.

$$\begin{aligned}
\Delta(A(u)(\nabla u, \nabla u)) &= \nabla^2 A(u)(\nabla u, \nabla u)(\nabla u, \nabla u) + \nabla A(u)(\nabla u, \nabla u)(\Delta u) \\
&\quad + 4\nabla A(u)(\nabla^2 u, \nabla u)(\nabla u) + 2A(u)(\nabla^2 u, \nabla^2 u) \\
&\quad + 2A(u)(\nabla \Delta u, \nabla u).
\end{aligned}$$

This implies that there exists a C_5 depending only on N such that

$$|\Delta(A(u)(\nabla u, \nabla u))| \leq C_5 (|\nabla u|^4 + |\nabla^2 u|^2 + |\nabla \Delta u| |\nabla u| + |\nabla^2 u| |\nabla u|^2). \tag{2.6}$$

Therefore we have

$$\begin{aligned}
|II| &\leq |A(u)(\nabla u, \nabla u)| |\Delta(A(u)(\nabla u, \nabla u))| \\
&\leq C_5 \|A\|_{L^\infty(N)} (|\nabla \Delta u| |\nabla u|^3 + |\nabla^2 u| |\nabla u|^4 \\
&\quad + |\nabla u|^6 + |\nabla^2 u|^2 |\nabla u|^2).
\end{aligned}$$

By the Hölder inequality, we have, for some C_6 depending only on N ,

$$|\nabla u|^6 \leq |\nabla u|^4 + |\nabla u|^8,$$

$$\|A\|_{L^\infty(N)} |\nabla \Delta u| |\nabla u|^3 \leq \frac{1}{2C_5} |\nabla \Delta u|^2 + C_6 (|\nabla u|^4 + |\nabla u|^8),$$

and

$$|\nabla^2 u| |\nabla u|^4 \leq |\nabla^2 u|^2 |\nabla u|^2 + |\nabla u|^6 \leq |\nabla^2 u|^2 |\nabla u|^2 + |\nabla u|^4 + |\nabla u|^8.$$

Putting these inequalities together, we obtain

$$|II| \leq \frac{1}{2} |\nabla \Delta u|^2 + C_6 [(|\nabla u|^4 + |\nabla u|^8) + |\nabla^2 u|^2 |\nabla u|^2]. \tag{2.7}$$

Putting (2.4)-(2.7) into (2.3), we have, for some C_7 depending only on N ,

$$\Delta(|\Delta u|^2) \geq |\nabla \Delta u|^2 - C_7 (|\Delta u|^2 + |\nabla u|^4) (1 + (|\Delta u|^2 + |\nabla u|^4)) - C_7 |\nabla^2 u|^2 |\nabla u|^2. \tag{2.8}$$

On the other hand, direct calculations imply

$$\begin{aligned}
\Delta(|\nabla u|^4) &= 8|\langle \nabla u, \nabla^2 u \rangle|^2 + 4|\nabla u|^2 |\nabla^2 u|^2 + 4|\nabla u|^2 \langle \nabla u, \nabla \Delta u \rangle \\
&\geq 4|\nabla^2 u|^2 |\nabla u|^2 - C_7^{-1} |\nabla \Delta u|^2 - C_7 (|\nabla u|^4 + |\nabla u|^8). \tag{2.9}
\end{aligned}$$

Therefore, if we choose $C_1 = C_7$, then there exists $C_2 > 0$ depending only on N such that

$$\Delta(|\Delta u|^2 + C_1 |\nabla u|^4) \geq -C_2 (|\Delta u|^2 + C_1 |\nabla u|^4) [1 + (|\Delta u|^2 + C_1 |\nabla u|^4)].$$

This yields (2.1). ■

Now we prove an ϵ_0 -gradient estimate for biharmonic maps.

Theorem 2.2. *For $n \geq 4$, there exists an $\epsilon_0 > 0$ depending only on N such that if $u \in C^\infty(\mathbf{R}^n, N)$ is an extrinsic biharmonic map and satisfies, for some $B_R \subset \mathbf{R}^n$,*

$$R^{4-n} \int_{B_R} (|\nabla^2 u|^2 + |\nabla u|^4) \leq \epsilon_0^2 \tag{2.10}$$

then, for any $k \geq 1$,

$$r^k \max_{y \in B_r(x)} |\nabla^k u|(y) \leq C(k, \epsilon_0), \quad \forall x \in B_{\frac{R}{4}}, \quad r \leq \frac{R}{4}. \tag{2.11}$$

Proof. We remark that the argument to prove partial regularity for stationary biharmonic maps by [CWY] and [W2] also yields a proof of theorem 2.2. Here we give a direct proof that is based on Proposition 2.1, that is similar to that by Schoen [Sr] on smooth harmonic maps.

For simplicity, we assume $x = 0$. Since u is a C^∞ -biharmonic map, it is a stationary biharmonic map (see [CWY] and [W1,2] for the definition). Hence it follows [W2] Lemma 5.2 and Lemma 5.3 that there exists a $\theta_0 \in (0, 1)$ such that

$$s^{4-n} \int_{B_s(x)} (|\nabla^2 u|^2 + |\nabla u|^4) \leq C\epsilon_0^2, \quad \forall B_s(x) \subset B_{2\theta_0 R}. \tag{2.12}$$

Denote $R_0 = \theta_0 R$ and $e(u) = |\Delta u|^2 + C_1 |\nabla u|^4$ with C_1 the same constant as in Proposition 2.1. Then there exists $r_0 \in [0, R_0)$ such that

$$(R_0 - r_0)^4 \max_{B_{r_0}} e(u) = \max_{0 \leq s \leq R_0} (R_0 - s)^4 \max_{B_s} e(u). \tag{2.13}$$

Moreover, there exists a $x_0 \in B_{r_0}$ such that

$$e_0 = e(u)(x_0) = \max_{B_{r_0}} e(u). \tag{2.14}$$

Set $\rho_0 = \frac{1}{2}(R_0 - r_0)$. Then we have

$$\max_{B_{\rho_0}(x_0)} e(u) \leq \max_{B_{r_0+\rho_0}} e(u) \leq \frac{(R_0 - r_0)^4 e_0}{(R_0 - (r_0 + \rho_0))^4} = 16e_0. \tag{2.15}$$

Set $\delta_0 = e_0^{\frac{1}{4}} \rho_0$. We need to show $\delta_0 \leq 1$. For otherwise, we have $\delta_0 > 0$ or $e_0^{-\frac{1}{4}} < \rho_0$. Let $v \in C^\infty(B_{\delta_0}, N)$ be defined by

$$v(y) = u(x_0 + \frac{y}{e_0^{\frac{1}{4}}}), \quad y \in B_{\delta_0}.$$

Then v is an extrinsic biharmonic map, and (2.15) implies

$$\max_{B_{\delta_0}} e(v) \leq 16, \quad e(v)(0) = 1. \tag{2.16}$$

Therefore Proposition 2.1 implies

$$\Delta e(v) + Ce(v) \geq 0, \quad \text{in } B_{\delta_0}. \quad (2.17)$$

Hence, by the Harnack inequality (see [GT] Theorem 8.17, page 184), we conclude

$$1 = e(v)(0) \leq C \int_{B_1} e(v)(y) dy. \quad (2.18)$$

On the other hand, by rescalings, we have

$$\int_{B_1} e(v)(y) dy = (e_0^{-\frac{1}{4}})^{4-n} \int_{B_{e_0^{-\frac{1}{4}}}(x_0)} e(u)(x) dx \leq C\epsilon_0^2, \quad (2.19)$$

where it used that

$e_0^{-\frac{1}{4}} \leq \rho_0 = \frac{1}{2}(R_0 - r_0) \leq R_0$ and (2.12). (2.19) contradicts with (2.18), provided that ϵ_0 is chosen to be sufficiently small. Hence $\delta_0 = e_0^{\frac{1}{4}}\rho_0 \leq 1$ and (2.13) implies

$$\left(\frac{\theta_0 R}{2}\right)^4 \max_{B_{\frac{\theta_0 R}{2}}} e(u) \leq 16.$$

This, combined with simple covering arguments, yields (2.11) for $k = 2$. (2.11) for $k \geq 3$ can be deduced by the standard theory of 4th order linear elliptic equations. \blacksquare

As a direct consequence, we can prove the following quantization result, which was previously proved in [L2] (Theorem 1.1) by a different method.

Corollary 2.3. *There exists $\epsilon_0 > 0$ depending only on N such that*

$$\epsilon_0^2 := \inf \left\{ \int_{\mathbf{R}^4} |\Delta \omega|^2, \omega \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N) \text{ nonconstant biharmonic maps} \right\} > 0. \quad (2.20)$$

Proof. Suppose (2.20) were false. Then there exist a sequence of nonconstant extrinsic biharmonic maps $\{\omega_k\} \subset C^\infty \cap W^{2,2}(\mathbf{R}^4, N)$ such that

$$\int_{\mathbf{R}^4} |\Delta \omega_k|^2 \leq k^{-1}. \quad (2.21)$$

Since $\omega_k \in W^{2,2}(\mathbf{R}^4, N)$, we have, by integration of parts,

$$\int_{\mathbf{R}^4} |\nabla^2 \omega_k|^2 = \int_{\mathbf{R}^4} |\Delta \omega_k|^2 \leq k^{-1}.$$

This, combined with the Sobolev embedding inequality, implies

$$\int_{\mathbf{R}^4} |\nabla \omega_k|^4 \leq C \left(\int_{\mathbf{R}^4} |\nabla^2 \omega_k|^2 \right)^2 \leq Ck^{-2}.$$

Therefore we have

$$\int_{\mathbf{R}^4} |\Delta \omega_k|^2 + |\nabla \omega_k|^4 \leq Ck^{-1}. \quad (2.22)$$

Hence, Theorem 2.2 implies that for k sufficiently large, we have

$$R^4 \max_{B_R} (|\Delta\omega_k|^2 + |\nabla\omega_k|^4) \leq Ck^{-1}, \forall R > 0. \tag{2.23}$$

Letting R tend to infinity, we have that ω_k is constant, for k sufficiently large. This contradicts with the choice of ω_k . Therefore (2.20) is true. ■

Related to the lower bound estimate of ϵ_0 , we have the following conjecture.

Conjecture 2.4. *For $N \subset \mathbf{R}^k$ of dimension 4, we have*

$$\begin{aligned} \epsilon_0^2 &= \inf \{ \int_{\mathbf{R}^4} |\Delta\omega|^2 : \omega \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N) \text{ nonconstant biharmonic} \} \\ &\geq 4S(4)^2 \sqrt{|S^4|}, \end{aligned} \tag{2.24}$$

where $S(4)$ is the best Sobolev constant of $W^{1,2}(\mathbf{R}^4) \subset L^4(\mathbf{R}^4)$.

Our motivation for (2.24) is follows. Since $\omega \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N)$ is biharmonic map, it follows from removability of isolated singularities (see [W1,2]) that $\lim_{|x| \rightarrow \infty} \omega(x)$ exists and $\omega : \mathbf{R}^4 \rightarrow N$ has a well-defined topological degree. Moreover, by the Sobolev embedding inequality, the geometry-mean inequality, and the integral representation formula of degree, we have

$$\begin{aligned} \int_{\mathbf{R}^4} |\Delta\omega|^2 &= \int_{\mathbf{R}^4} |\nabla^2\omega|^2 \geq S(4)^2 \left(\int_{\mathbf{R}^4} |\nabla\omega|^4 \right)^{\frac{1}{2}} \\ &\geq 4S(4)^2 \sqrt{ \left| \int_{\mathbf{R}^4} \det(\nabla\omega) \right| } \\ &= 4S(4)^2 \sqrt{ |\deg(\omega)| |S^4| } \end{aligned}$$

where $|S^4|$ denotes the volume of the unit sphere S^4 . Hence, if we can prove that any nonconstant extrinsic biharmonic map $\omega \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N)$ has nonzero degree, then (2.24) holds. However, this is unknown. ■

We end this section with a well-known result on the density theorem (see, for example, Brezis-Nirenberg [BN]).

Theorem 2.5. *If the dimension of M is 4, then $C^\infty(M, N)$ is dense in $W^{2,2}(M, N)$ with respect to $W^{2,2}$ -norm.*

§3. Lamm’s integral estimates and proof of Theorem A

In this section, we first recall some integral estimates for smooth solutions of (1.3)-(1.4), which are essentially due to Lamm [L2]. Then, by combining Theorem 2.2 and 2.4, we give a proof of Theorem A.

Throughout this section, we assume $\dim M = 4$ and denote by $\text{inj}(M)$ the injectivity radius of M . For $0 < r < \text{inj}(M)$, denote by $B_r(x) \subset M$ the geodesic ball of radius r and center x . We start with

Theorem 3.1. *There exists a $\epsilon_1 > 0$, depending only on M, N , such that for $0 < T < \infty$, if $u \in C^\infty(M \times (0, T), N)$ is a solution of (1.3)-(1.4) satisfying*

$$\sup_{0 < t < T} \sup_{x \in M} \int_{B_{r_0}(x)} |\nabla u|^4 \leq \epsilon_1^2, \quad (3.1)$$

for some $0 < r_0 < \text{inj}(M)$, then we have

$$\max_{\frac{T}{2} \leq t \leq T} \|u\|_{C^k(M)} \leq C(k, \epsilon_1, r_0^{-1}, T, \|\nabla^2 \phi\|_{L^2(M)}), \quad \forall k \geq 1. \quad (3.2)$$

For the convenience of the readers, we outline some key Lemmas needed in the proof of Theorem 3.1. The reader can consult [L2] for more details.

Lemma 3.2. *For $T > 0$, if $u \in C^\infty(M \times (0, T), N)$ solves (1.3)-(1.4), then we have*

$$\int_0^T \int_M |u_t|^2 + \int_M |\Delta u|^2(\cdot, T) \leq \int_M |\Delta \phi|^2. \quad (3.3)$$

Moreover, for any $\eta \in C_0^\infty(M)$,

$$\begin{aligned} \int_0^T \int_M \eta^4 |u_t|^2 + \int_M \eta^4 |\Delta u|^2(x, T) &\leq C \int_0^T \int_M (|\Delta \eta|^2 + |\nabla \eta|^4) |\Delta u|^2 \\ &+ \int_M \eta^4 |\Delta \phi|^2 + C \int_0^T \int_M |\nabla \eta|^2 |\nabla \Delta u|^2. \end{aligned} \quad (3.4)$$

Proof. Multiplying (1.3) by u_t , integrating the resulting equation over M , using $u_t + \Delta^2 u \perp T_u N$, and then integrating it over $[0, T]$, one can obtain (3.3). For (3.4), multiplying (1.3) by $\eta^4 u_t$, integrating over M , using integration by parts and the Hölder inequality, one has

$$\begin{aligned} \int_M \eta^4 |u_t|^2 + \frac{1}{2} \frac{d}{dt} \int_M \eta^4 |\Delta u|^2 &= \int_M \Delta \eta^4 \langle \Delta u, u_t \rangle + 2 \int_M \langle \nabla \Delta u, u_t \rangle \cdot \nabla \eta^4 \\ &= 4 \int_M \eta^2 (\eta \Delta \eta + 3 |\nabla \eta|^2) \langle \Delta u, u_t \rangle + 8 \int_M \eta^3 \langle \nabla \Delta u, u_t \rangle \cdot \nabla \eta \\ &\leq \frac{1}{4} \int_M \eta^4 |u_t|^2 + C \int_M (|\Delta \eta|^2 + |\nabla \eta|^4) |\Delta u|^2 + |\nabla \eta|^2 |\nabla \Delta u|^2. \end{aligned}$$

Integrating this over $t \in (0, T)$, we get (3.4). ■

Now we need to establish $W^{4,2}$ -estimate for u . More precisely,

Lemma 3.3. *There exists an $\epsilon_1 > 0$ depending only on M, N , such that for $T > 0$, if $u \in C^\infty(M \times (0, T), N)$ is a solution of (1.3)-(1.4) satisfying*

$$\sup_{0 < t < T} \sup_{x \in M} \int_{B_{r_0}(x)} |\nabla u|^4 \leq \epsilon_1^2$$

for some $0 < r_0 < \text{inj}(M)$, then we have

$$\int_M |\nabla^4 u|^2 \leq C \left(\int_M |u_t|^2 + r_0^{-4} \int_M |\nabla^2 \phi|^2 \right). \quad (3.5)$$

Proof. For any $B_{r_0}(x) \subset M$, let $\eta \in C_0^\infty(B_{r_0}(x))$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{\frac{r_0}{2}}(x)$, $|\nabla\eta| \leq 4r_0^{-1}$, and $|\nabla^2\eta| \leq Cr_0^{-2}$. Multiplying (1.3) by $\eta^4\Delta^2u$, integrating over M , and applying Hölder inequality, we have

$$\int_M \eta^4 |\Delta^2 u|^2 \leq C \left[\int_M \eta^4 |u_t|^2 + \int_M \eta^4 (|\Delta u|^4 + |\Delta u|^2 |\nabla u|^4 + |\nabla \Delta u|^2 |\nabla u|^2 + |\nabla u|^8) \right]. \quad (3.6)$$

Applying the interpolation inequalities (see [L2] Lemma 2.4), we have

$$\begin{aligned} \int_M \eta^4 |\Delta u|^4 &\leq C \left(\int_{B_{r_0}(x)} |\nabla u|^4 \right)^{\frac{1}{2}} \left(\int_M \eta^4 |\nabla^4 u|^2 + |\nabla^2 \eta|^2 |\nabla^2 u|^2 \right) \\ &\leq C \epsilon_1 \left(\int_M \eta^4 |\nabla^4 u|^2 + |\nabla^2 \eta|^2 |\nabla^2 u|^2 \right). \end{aligned} \quad (3.7)$$

Similarly, one has

$$\int_M \eta^4 |\nabla u|^2 |\nabla u|^4 \leq C \epsilon_1 \left(\int_M \eta^4 |\nabla^4 u|^2 + |\nabla^2 \eta|^2 |\nabla^2 u|^2 \right), \quad (3.8)$$

$$\int_M \eta^4 |\nabla \Delta u|^2 |\nabla u|^2 \leq C \epsilon_1 \left(\int_M \eta^4 |\nabla^4 u|^2 + |\nabla^2 \eta|^2 |\nabla^2 u|^2 \right), \quad (3.9)$$

$$\int_M \eta^4 |\nabla u|^8 \leq C \epsilon_1 \left(\int_M \eta^4 |\nabla^4 u|^2 + |\nabla^2 \eta|^2 |\nabla^2 u|^2 \right). \quad (3.10)$$

Putting (3.7)-(3.10) into (3.6) and choosing ϵ_1 to be sufficiently small, we have

$$\int_M \eta^4 |\nabla^4 u|^2 \leq C \left(\int_M \eta^4 |u_t|^2 + (|\nabla \eta|^4 + |\nabla^2 \eta|^2) |\nabla^2 u|^2 \right). \quad (3.11)$$

This implies

$$\int_{B_{\frac{r_0}{2}}(x)} |\nabla^4 u|^2 \leq C \left(\int_{B_{r_0}(x)} |u_t|^2 + r_0^{-4} \int_{B_{r_0}(x)} |\nabla^2 u|^2 \right). \quad (3.12)$$

Now we can apply the Vitali's covering Lemma to get

$$\int_M |\nabla u|^4 \leq C \left(\int_M |u_t|^2 + r_0^{-4} \int_M |\nabla^2 u|^2 \right). \quad (3.13)$$

This, combined with (3.3), implies (3.5). \blacksquare

Now we need to have the uniform control of $\int_M |u_t|^2$. For this, we recall the following Lemma, whose proof can be found in [L2] Lemma 3.6.

Lemma 3.4. *There exists an $\epsilon_1 > 0$ depending only on M, N such that if $u \in C^\infty(M \times (0, T), N)$ solves (1.3)-(1.4) and satisfies*

$$\sup_{0 < t < T} \sup_{x \in M} \int_{B_{r_0}(x)} |\nabla u|^4 \leq \epsilon_1^2$$

for some $0 < r_0 < \text{inj}(M)$. Then there exist $\beta \in (0, \frac{1}{2})$ and $0 < \delta < \min\{T, \beta r_0^4\}$ such that for any $0 < s < t < T$, with $|t - s| \leq \delta$, we have

$$\int_M |u_t|^2(\cdot, s) \leq C(1 + \int_M |u_t|^2(\cdot, t)), \quad (3.14)$$

where $C = C(\int_M |\nabla^2 \phi|^2, M, N) > 0$.

Proof of Theorem 3.1.

It follows from Lemma 3.3 and 3.4 that $\nabla^4 u \in L^\infty([\frac{T}{4}, T], L^2(M))$. Hence, by the Sobolev embedding theorem, we conclude that $u_t + \Delta^2 u \in L^p(M \times [\frac{T}{4}, T])$ for any $1 < p < \infty$. Therefore, by the parabolic L^p theory and Schauder estimate, we can achieve the desired estimate (3.2). ■

To prove Theorem A, we also need the following estimate on the lower bound of the time interval in which smooth solutions of (1.3)-(1.4) exist.

Lemma 3.5. *There exist $0 < \epsilon_2 \ll \epsilon_1$ and $\beta_0 \in (0, \frac{1}{4})$ such that if $\phi \in C^\infty(M, N)$ satisfies*

$$\sup_{x \in M} \int_{B_{2r_0}(x)} |\nabla^2 \phi|^2 \leq \epsilon_2^2, \quad (3.15)$$

for some $0 < r_0 < \frac{\text{inj}(M)}{2}$. Then there exist $T_0 \geq \beta_0 r_0^4$ and $u \in C^\infty(M \times [0, T_0], N)$ solving (1.3)-(1.4).

Proof. Let $T_0 > 0$ be the maximum such that there exists a smooth solution $u \in C^\infty(M \times [0, T_0], N)$ of (1.3)-(1.4). Let $0 \leq t_0 \leq T_0$ be the maximum such that

$$\sup_{0 < s < t_0} \sup_{x \in M} \int_{B_{\frac{3r_0}{2}}(x)} |\nabla u|^4 \leq 2\epsilon_2^2. \quad (3.16)$$

Note that (3.15) implies $t_0 > 0$. It follows from (3.4), Lemma 3.3, Lemma 3.4 that for any $x \in M$,

$$2\epsilon_2^2 = \int_{B_{\frac{3r_0}{2}}(x)} |\Delta u|^2(x, t_0) \leq \int_{B_{2r_0}(x)} |\Delta \phi|^2 + C \frac{t_0}{r_0^4} \int_M |\nabla^2 \phi|^2 \leq \epsilon_2^2 + C \frac{t_0}{r_0^4}.$$

This implies $t_0 \geq \frac{\epsilon_2^2}{Cr_0^4}$. This, combined with the Sobolev inequality and (3.16), implies

$$\sup_{0 < t < \frac{\epsilon_2^2}{Cr_0^4}} \sup_{x \in M} \int_{B_{r_0}(x)} |\nabla u|^4 \leq C\epsilon_2^2 \leq \epsilon_1^2.$$

Hence Theorem 3.1 implies the conclusion of Lemma 3.5. ■

Proof of Theorem A.

Note that (3) of Theorem A follows from Theorem 2.2. For $\phi \in W^{2,2}(M, N)$, it follows from Theorem 2.5 that there exist $\phi_n \in C^\infty(M, N)$ such that $\lim_{n \rightarrow \infty} \|\phi_n - \phi\|_{W^{2,2}(M)} = 0$. Hence there exists a $r_0 \in (0, \frac{\text{inj}(M)}{2})$ such that

$$\sup_n \sup_{x \in M} \int_{B_{2r_0}(x)} |\Delta \phi_n|^2 \leq \epsilon_2^2$$

where $\epsilon_2 > 0$ is the same constant as in Lemma 3.5. Let $u_n \in C^\infty(M \times [0, T_n], N)$ be smooth solutions to (1.3) under the initial condition $u_n(x, 0) = \phi_n(x)$. Then Lemma 3.5 implies $T_n \geq \beta_0 r_0^4$. Moreover, Theorem 3.1 implies that we have uniform C^k -estimates of u_n in $M \times [0, \beta_0 r_0^4]$. Hence, after taking possible subsequence, we can assume that $u_n \rightarrow u$ weakly in $W^{2,2}(M, N)$, strongly in $W^{1,2}(M, N)$, and in $C^k(M \times [\delta, \beta_0 r_0^4])$ for any $\delta > 0$. It is clear that $u \in C^\infty(M \times (0, \beta_0 r_0^4), N)$ is a solution of (1.3) and satisfies $u(x, 0) = \phi(x)$ in the sense of trace. Now we assume that $T_0 \geq \beta_0 r_0^4$ is the maximum time such that $u \in C^\infty(M \times (0, T_0), N)$ solves (1.3)-(1.4). It follows from Theorem 3.1 that T_0 can be characterized by

$$\lim_{r \rightarrow 0} \limsup_{t \uparrow T_0} \sup_{x \in M} \int_{B_r(x)} |\nabla u|^4(\cdot, t) \geq \epsilon_1^2. \tag{3.17}$$

Now we claim that there exists an $\hat{\epsilon}_0 \geq \epsilon_0$ such that

$$\lim_{r \rightarrow 0} \limsup_{t \uparrow T_0} \sup_{x \in M} \int_{B_r(x)} |\Delta u|^2(\cdot, t) \geq \hat{\epsilon}_0^2. \tag{3.18}$$

The proof of (3.18) is given at the end of the proof. For the moment, assume that (3.18) is true, we want to show that the Hessian energy drops at least $\hat{\epsilon}_0^2$ at T_0 . In fact, it follows from (3.3) that there is a well-defined trace $u(x, T_0) = \lim_{t \uparrow T} u(x, T)$ weakly in $W^{2,2}(M, N)$. In particular, $u(\cdot, T_0) \in W^{2,2}(M, N)$. Moreover, (3.18) implies that there exists $\{x_i\} \subset M$, with $x_i \rightarrow x_0 \in M$, $r_i \downarrow 0$, and $t_i \uparrow T_0$ such that

$$\lim_{i \rightarrow \infty} \int_{B_{r_i}(x_i)} |\Delta u|^2(\cdot, t_i) \geq \hat{\epsilon}_0^2. \tag{3.19}$$

Now we claim

$$\int_M |\Delta u|^2(\cdot, T_0) \leq \int_M |\Delta \phi|^2 - \hat{\epsilon}_0^2. \tag{3.20}$$

In fact, by the lower semicontinuity and (3.3), we have, for any $r > 0$,

$$\begin{aligned} \int_{M \setminus B_r(x_0)} |\Delta u|^2(\cdot, T_0) &\leq \liminf_{i \rightarrow \infty} \int_{M \setminus B_r(x_0)} |\Delta u|^2(\cdot, t_i) \\ &= \liminf_{i \rightarrow \infty} \left(\int_M |\Delta u|^2(\cdot, t_i) - \int_{B_r(x_0)} |\Delta u|^2(\cdot, t_i) \right) \\ &\leq \liminf_{i \rightarrow \infty} \int_M |\Delta u|^2(\cdot, t_i) - \lim_{i \rightarrow \infty} \int_{B_{r_i}(x_i)} |\Delta u|^2(\cdot, t_i) \end{aligned}$$

$$\leq \int_M |\Delta\phi|^2 - \hat{\epsilon}_0^2. \quad (3.21)$$

Taking r into zero, (3.21) yields (3.20).

Now we use $u(\cdot, T_0)$ as the initial data to extend the above solution beyond T_0 to obtain a weak solution $u : M \times (0, T_1) \rightarrow N$ of (1.3)-(1.4) for some $T_1 > T_0$. Since $H(u(t))$ drops at least $\hat{\epsilon}_0^2$ at each singular time, we have that after at most $\lceil \frac{\int_M |\Delta\phi|^2}{\hat{\epsilon}_0^2} \rceil$ -times, the solution can be extended smoothly to be a global weak solution that satisfies (1.5) and (1.6) with ϵ_0 replaced by $\hat{\epsilon}_0$.

To prove (4), we proceed as follows (see also [L2] Theorem 1.1). For $1 \leq i \leq L-1$, $t \in (t_i, t_{i+1})$, and $0 < r < \text{inj}(M)$, define the concentration function

$$Q(t, r) = \sup \left\{ \int_{B_r(x)} (|\Delta u|^2 + |\nabla u|^4)(z, \tau) dz \mid B_r(x) \subset M, t_i \leq \tau \leq t \right\}$$

Then we have (i) $Q(t, r)$ is monotonically nondecreasing and continuous w.r.t. $r > 0$, (ii) $Q(t, 0) = 0$, and (iii) $Q(t, r)$ is monotonically nondecreasing w.r.t. $t < t_{i+1}$, and

$$\lim_{t \uparrow t_{i+1}} Q(t, r) \geq \epsilon_1^2,$$

where ϵ_1 is given by (3.17). Therefore, we can conclude that there exist $t_n \uparrow t_{i+1}$, $r_n \downarrow 0$, and $x_n \rightarrow x_0 \in M$ such that

$$Q(t_n, r_n) = \frac{\epsilon_1^2}{C_0} = \int_{B_{r_n}(x_n)} (|\Delta u|^2 + |\nabla u|^4)(\cdot, t_n) \quad (3.22)$$

where $C_0 > 0$ is to be determined. (3.22) implies, for some $\delta_0 > 0$,

$$\int_{B_{r_n}(x)} (|\Delta u|^2 + |\nabla u|^4)(\cdot, t_n) \leq \frac{\epsilon_1^2}{C_0}, \quad \forall B_{r_n}(x) \subset B_{\delta_0}(x_0). \quad (3.23)$$

This and Lemma 3.5 imply

$$\int_{B_{r_n}(x)} (|\nabla^2 u|^2 + |\nabla u|^4)(\cdot, t) \leq \frac{C\epsilon_1^2}{C_0} \leq \epsilon_1^2, \quad \forall x \in B_{\frac{\delta_0}{2}}(x_0), \quad t_n \leq t \leq t_n + \beta_0 r_n^4 \quad (3.24)$$

provided that C_0 is sufficiently large. Define $u_n(x, t) = u(x_n + r_n x, t_n + r_n^4 t) \in C^\infty(B_{r_n^{-1}\delta_0} \times [r_n^{-4}(t_i - t_n), \beta_0], N)$. Then u_n are smooth solutions of (1.3). Applying Theorem 3.1 to u_n , we conclude that, after taking possible subsequences, $u_n \rightarrow \omega$ in $C_{\text{loc}}^k(\mathbf{R}^4 \times (-\infty, \beta_0))$ for any $k \geq 4$. Moreover, $\|\frac{\partial u_n}{\partial t}\|_{L^2} \rightarrow 0$ implies $\omega \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N)$ is a nontrivial biharmonic map. This proves (4).

Now we return to the proof of (3.18). For any $\epsilon > 0$, let $R > 0$ such that

$$\int_{B_R} |\Delta\omega|^2 \geq \int_{\mathbf{R}^4} |\Delta\omega|^2 - \epsilon.$$

Then we have, for any $r > 0$,

$$\begin{aligned}
 \epsilon_0^2 - \epsilon &\leq \int_{\mathbf{R}^4} |\Delta\omega|^2 - \epsilon \leq \int_{B_R} |\Delta\omega|^2 \\
 &= \lim_{n \rightarrow \infty} \int_{B_R} |\Delta u_n|^2(\cdot, 0) \\
 &= \lim_{n \rightarrow \infty} \int_{B_{Rr_n}(x_n)} |\Delta u|^2(\cdot, t_n) \\
 &\leq \limsup_{t \uparrow T_0} \sup_{x \in M} \int_{B_r(x)} |\Delta u|^2(\cdot, t).
 \end{aligned} \tag{3.25}$$

Since ϵ is arbitrary, this clearly implies (3.18). Hence the proof of Theorem A is complete. ■

§4 Proof of Theorem B

This section is devoted to the proof of Theorem B.

Proof of Theorem B.

The aim is to prove that the global weak solution obtained by Theorem A is smooth, provided that the initial data is properly chosen. Our idea is similar to that of Struwe [Sm3] on the heat flow of harmonic maps from surfaces.

Let $\epsilon_0 > 0$ be given by (1.7). For any $\epsilon < \frac{\epsilon_0}{2}$ and $\alpha \in [M, N]$, let $u_0 \in \alpha \cap C^\infty(M, N)$ be such that

$$\int_M |\Delta u_0|^2 - \epsilon^2 \leq C_\alpha \equiv \inf_{v \in \alpha \cap C^2(M, N)} \int_M |\Delta v|^2, \tag{4.1}$$

and $u : M \times \mathbf{R}_+ \rightarrow N$ be the global weak solution of (1.3), with $u(x, 0) = u_0(x)$ for $x \in M$, given by Theorem A. We want to show that $u \in C^\infty(M \times [0, +\infty), N)$. For otherwise, let $0 < T_0 < +\infty$ be the first singular time of u . By (4) of Theorem A, we know that there exist $x_n \rightarrow x_0 \in M$, $t_n \uparrow T_0$, and $\lambda_n \downarrow 0$, and a bubble $\omega \in C^\infty \cap W^{2,2}(\mathbf{R}^4, N)$ such that

$$u_n(\cdot) \equiv u(x_n + \lambda_n \cdot, t_n) \rightarrow \omega, \text{ in } C_{\text{loc}}^4(\mathbf{R}^4, N). \tag{4.2}$$

For any $C_0 > 0$, let $R = R(C_0) > 0$ be so large that

$$\int_{\mathbf{R}^4 \setminus B_R} |\Delta\omega|^4 + |\nabla\omega|^4 \leq \frac{\epsilon_0^2}{C_0}. \tag{4.3}$$

Define $\omega_R : B_R \rightarrow N$ by $\omega_R(x) = \omega(\frac{R^2}{|x|^2}x)$, $x \in B_R$. It follows by removability of isolated singularities (see [W1,2]) that $\hat{\omega} = \omega \circ \Phi \in C^\infty(S^4, N)$, where $\Phi : S^4 \rightarrow \mathbf{R}^4$ is the stereographic projection, and hence $\omega_R \in C^\infty(B_R, N)$.

Let $\eta \in C_0^\infty(B_R)$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 0$ in $B_{\frac{R}{4}}$, $\eta \equiv 1$ on $B_R \setminus B_{\frac{R}{2}}$, $|\nabla\eta| \leq 8R^{-1}$, and $|\nabla^2\eta| \leq CR^{-2}$. Define $v_n(x) = (1 - \eta(x))\omega_R(x) + \eta u_n(x)$, $x \in$

B_R . Then we have

$$\begin{aligned} \max_{B_R} \text{dist}(v_n(x), N) &\leq \max_{B_R \setminus B_{\frac{R}{4}}} |(1 - \eta(x))\omega_R(x) + \eta u_n(x) - \omega_R(x)| \\ &\leq \max_{B_R \setminus B_{\frac{R}{4}}} |\omega_R(x) - u_n(x)| \\ &\leq \max_{B_R \setminus B_{\frac{R}{4}}} (|\omega_R(x) - \omega(x)| + |u_n(x) - \omega(x)|) \\ &\leq \text{osc}_{B_{4R} \setminus B_{\frac{R}{4}}} \omega + o(1) = o(1, R^{-1}) \end{aligned}$$

where $\lim_{n, R \rightarrow \infty} o(1, R^{-1}) = 0$. Therefore for any $\delta > 0$, $v_n(B_R) \subset N_\delta$ for sufficiently large R and n . For $\delta > 0$ sufficiently small, let $\Pi : N_\delta \rightarrow N$ be the smooth nearest point projection map, and define $w_n : B_R \rightarrow N$ by $w_n(x) = \Pi(v_n(x))$, $x \in B_R$, and $\bar{w}_n : M \rightarrow N$ by

$$\begin{aligned} \bar{w}_n(x) &= w_n\left(\frac{x - x_n}{\lambda_n}\right), x \in B_{\lambda_n R}(x_n) \\ &= u(x, t_n), x \in M \setminus B_{\lambda_n R}(x_n). \end{aligned}$$

It is clear that $\bar{w}_n \in C^\infty(M, N)$. Moreover, since $\Pi_4(N) = \{0\}$, it follows that $\bar{w}_n \in \alpha$. Therefore we have

$$\begin{aligned} C_\alpha &\leq \int_M |\Delta \bar{w}_n|^2 \\ &= \int_{M \setminus B_{\lambda_n R}(x_n)} |\Delta u|^2(\cdot, t_n) + \int_{B_{\lambda_n R}(x_n)} |\Delta \bar{w}_n|^2 \\ &= \int_M |\Delta u|^2(\cdot, t_n) - \int_{B_{\lambda_n R}(x_n)} |\Delta u|^2(\cdot, t_n) + \int_{B_{\lambda_n R}(x_n)} |\Delta \bar{w}_n|^2. \end{aligned} \quad (4.4)$$

Note that, by (4.2) and change of variables, we have, for sufficiently large n, R ,

$$\int_{B_{\lambda_n R}(x_n)} |\Delta u|^2(\cdot, t_n) = \int_{B_R} |\Delta_{g_n} u_n|^2 dv_{g_n} \geq \frac{3}{4} \int_{B_R} |\Delta \omega|^2 \geq \frac{\epsilon_0^2}{2}, \quad (4.5)$$

where $g_n(x) = g(x_n + \lambda_n x)$, $x \in B_R$ and Δ_{g_n} is the Laplace operator w.r.t. g_n . On the other hand, direct calculations imply

$$\begin{aligned} \int_{B_{\lambda_n R}(x_n)} |\Delta \bar{w}_n|^2 &= \int_{B_R} |\Delta_{g_n} w_n|^2 dv_{g_n} \\ &\leq C \int_{B_R} |\Delta_{g_n} ((1 - \eta)\omega_R + \eta u_n)|^2 dv_{g_n} \\ &= C \int_{B_{\frac{R}{4}}} |\Delta_{g_n} \omega_R|^2 dv_{g_n} \\ &\quad + C \int_{B_R \setminus B_{\frac{R}{4}}} |\Delta_{g_n} ((1 - \eta)\omega_R + \eta u_n)|^2 dv_{g_n} \\ &\leq C \int_{R^4 \setminus B_R} (|\Delta \omega|^2 + |\nabla \omega|^4) \\ &\quad + C \int_{B_R \setminus B_{\frac{R}{4}}} (|\Delta \omega_R|^2 + |\Delta u_n|^2 + R^{-4} |u_n - \omega_R|^2 + R^{-2} |\nabla(u_n - \omega_R)|^2) \\ &\leq \frac{C \epsilon_0^2}{C_0} + o(1, R^{-1}) \leq \frac{\epsilon_0^2}{4} + o(1, R^{-1}) \leq \frac{\epsilon_0^2}{2} \end{aligned} \quad (4.6)$$

provided that C_0, n, R are sufficiently large. Therefore, by (1.5) and (4.4)-(4.6), we have

$$C_\alpha \leq \int_M |\Delta u|^2(\cdot, t_n) - \frac{\epsilon_0^2}{2} \leq \int_M |\Delta u_0|^2 - \frac{\epsilon_0^2}{2}.$$

This contradicts with (4.1). Hence $u \in C^\infty(M \times [0, +\infty), N)$.

Next we want to prove that there exists $t_n \rightarrow \infty$ and a smooth biharmonic map $u_\infty \in C^\infty(M, N)$ such that $u(\cdot, t_n) \rightarrow u_\infty$ in $C^2(M, N)$.

To prove this, we first observe that (1.5) implies there exist $t_n \uparrow +\infty$ such that $\|u_t(\cdot, t_n)\|_{L^2(M)} \rightarrow 0$ and there exists $u_\infty \in W^{2,2}(M, N)$ such that $u(\cdot, t_n) \rightarrow u_\infty$ weakly in $W^{2,2}(M, N)$. Moreover, $u(\cdot, t_n) \rightarrow u_\infty$ in $C^k(M \setminus \Sigma, N)$ for any $k \geq 1$, where $\Sigma \subset M$ is given by

$$\Sigma = \cap_{r>0} \{x \in M \mid \liminf_{n \rightarrow \infty} \int_{B_r(x)} |\nabla u|^4(\cdot, t_n) \geq \epsilon_1^2\}$$

where ϵ_1 is the same constant as in Theorem 3.1. It is easy to see that a simple covering argument implies $\Sigma \subset M$ is a finite set. Theorem 3.1 and Lemma 3.5 imply the smooth convergence of $u(\cdot, t_n)$ to u_∞ away from Σ . This implies that $u_\infty \in W^{2,2}(M, N) \cap C^\infty(M \setminus \Sigma, N)$ is a biharmonic map. Hence, by the removability of isolated singularities ([W1,2]), we conclude that $u_\infty \in C^\infty(M, N)$ is a biharmonic map. We may assume that for any given $x_0 \in \Sigma$, there is a $r_0 > 0$ such that $u(\cdot, t_n) \rightarrow u_\infty$ in $C_{\text{loc}}^k(B_{r_0}(x_0) \setminus \{x_0\}, N)$.

Now we do a surgery of $u(\cdot, t_n)$ near x_0 as follows. Let $\eta \in C_0^\infty(B_{r_0}(x_0))$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 0$ in $B_{\frac{r_0}{4}}(x_0)$, $\eta \equiv 1$ in $B_{r_0}(x_0) \setminus B_{\frac{r_0}{2}}(x_0)$, $|\nabla \eta| \leq 8r_0^{-1}$, and $|\nabla^2 \eta| \leq Cr_0^{-2}$. Define $v_n : B_{r_0}(x_0) \rightarrow N$ by

$$v_n(x) = (1 - \eta(x))u_\infty(x) + \eta(x)u(x, t_n), \quad x \in B_{r_0}(x_0).$$

Then for any $\delta_0 > 0$, there exists $n_0 \geq 1$ such that for $n \geq n_0$, we have

$$\begin{aligned} \max_{x \in B_{r_0}(x_0)} \text{dist}(v_n(x), N) &\leq \max_{x \in B_{r_0}(x_0)} |v_n(x) - u_\infty(x)| \\ &\leq \max_{B_{r_0}(x_0) \setminus B_{\frac{r_0}{4}}(x_0)} |u_\infty(x) - u(x, t_n)| \leq \delta_0. \end{aligned}$$

Therefore, we can project v_n to N to get $w_n(x) = \Pi(v_n(x))$ for $x \in B_{r_0}(x_0)$. This gives $\bar{u}_n \in C^\infty(M, N)$ by

$$\begin{aligned} \bar{u}_n(x) &= w_n(x), \quad \forall x \in B_{r_0}(x_0) \\ &= u(x, t_n), \quad \forall x \in M \setminus B_{r_0}(x_0). \end{aligned}$$

Since $\Pi_4(N) = \{0\}$, we also have $\bar{u}_n \in \alpha$ and hence

$$\begin{aligned} C_\alpha &\leq \int_M |\Delta \bar{u}_n|^2 = \int_M |\Delta u|^2(\cdot, t_n) - \int_{B_{r_0}(x_0)} |\Delta u|^2(\cdot, t_n) + \int_{B_{r_0}(x_0)} |\Delta w_n|^2 \\ &\leq \int_M |\Delta u|^2(\cdot, t_n) - \epsilon_0^2 + \int_{B_{r_0}(x_0)} |\Delta w_n|^2 \end{aligned}$$

where we have used the fact that $x_0 \in \Sigma$ and

$$\int_{B_{r_0}(x_0)} |\Delta u|^2(\cdot, t_n) \geq \epsilon_0^2. \quad (4.7)$$

The proof of (4.7) is same as that of (3.18). On the other hand, we have

$$\begin{aligned} \int_{B_{r_0}(x_0)} |\Delta w_n|^2 &\leq C \int_{B_{r_0}} |\Delta v_n|^2 + |\nabla v_n|^4 \\ &\leq C(r_0) \left\{ \int_{B_{r_0}(x_0) \setminus B_{\frac{r_0}{4}}(x_0)} (|\Delta u_\infty|^2 + |\nabla u_\infty|^4) \right. \\ &\quad + \int_{B_{r_0}(x_0) \setminus B_{\frac{r_0}{4}}(x_0)} (|\Delta(u_\infty - u(\cdot, t_n))|^2 \\ &\quad + |u_\infty - u(\cdot, t_n)|^2 + |\nabla(u_\infty - u(\cdot, t_n))|^2 \\ &\quad \left. + |\nabla(u_\infty - u(\cdot, t_n))|^4) \right\} \\ &\leq \frac{\epsilon_0^2}{2} \end{aligned}$$

for sufficiently large t_n . We conclude that $C_\alpha \leq C_\alpha - \epsilon_0^2 + \frac{\epsilon_0^2}{2}$. This is impossible. Therefore $u(\cdot, t_n) \rightarrow u_\infty$ in $C^k(M, N)$. In particular, for any small $\epsilon > 0$, there is a biharmonic map $u_\epsilon \in C^\infty(M, N) \cap \alpha$ such that

$$\int_M |\Delta u_\epsilon|^2 \leq C_\alpha + \epsilon.$$

It can be checked that the same argument as above also yields that there exists $\epsilon_i \downarrow 0$ $u_{\epsilon_i} \rightarrow v$ in $C^k(M, N)$ for any $k \geq 1$. Hence $v \in C^\infty(M, N)$ is a biharmonic map such that $v \in \alpha$ and $\int_M |\Delta v|^2 = C_\alpha$. Therefore, there exists a minimizing biharmonic map in α . This completes the proof of Theorem B. \blacksquare

REFERENCES

- [BN] H. Brezis, L. Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*. Selecta Math. (N.S.) 1 (1995) no. 2, 197–263.
- [CS] Y. Chen, M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*. Math. Z. 201 (1989), no. 1, 83–103.
- [CWY] S. Y. Chang, L. Wang, P. Yang, *A regularity theory of biharmonic maps*. Comm. Pure Appl. Math. 52 (1999) 1113–1137.
- [EL] J. Eells, L. Lemaire, *Selected topics in harmonic maps*. CBMS Regional Conference Series in Mathematics, 50. American Mathematical Society, Providence, RI, 1983.
- [G] A. Gastel, *The extrinsic polyharmonic map heat flow in the critical dimension*. To appear, Adv. Geom.
- [GT] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*. Springer, 1983.

- [KS1] E. Kuwert, R. Schätzle, *The Willmore flow with small initial energy*. J. Differential Geom. 57 (2001) 409–441.
- [KS2] E. Kuwert, R. Schätzle, *Gradient flow for the Willmore functional*. Comm. Anal. Geom. 10(2002), 307–339.
- [L1] T. Lamm, *Biharmonischer Wärmefluss*. Diplomarbeit, Universität Freiburg (2001)
- [L2] T. Lamm, *Heat flow for extrinsic biharmonic maps with small initial energy*. Ann. Global Anal. Geom. 26 (2004), no. 4, 369–384.
- [L3] T. Lamm, *Biharmonic map heat flow into manifolds of nonpositive curvature*. Calc. Var. Partial Differential Equations 22 (2005), no. 4, 421–445.
- [S1] L. Simon, *Existence of surfaces minimizing the Willmore functional*. Comm. Anal. Geom. 1 (1993), no. 2, 281–326.
- [Sm1] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*. Comm. Math. Helv. 60 (1985) 558–581.
- [Sm2] M. Struwe, *Heat-flow methods for harmonic maps of surfaces and applications to free boundary problems*. Lecture Notes in Math., 1324, 293–319, Springer, Berlin, 1988.
- [Sm3] M. Struwe, *Geometric Evolution Equations*. Nonlinear partial differential equations in differential geometry (Park City, UT, 1992), 257–339. IAS/Park City Math. Ser. 2, Amer. Math. Soc., Providence, RI (1996).
- [Sp] P. Strzelecki, *On biharmonic maps and their generalizations*. Calc. Var. 18 (2003), 401–432.
- [Sr] R. Schoen, *Analytic aspects of the harmonic map problem*. Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), 321–358, Math. Sci. Res. Inst. Publ., 2, Springer, New York, 1984.
- [SU] J. Sacks, K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*. Ann. of Math. (2) 113 (1981), no. 1, 1–24.
- [W1] C. Y. Wang, *Biharmonic maps from \mathbf{R}^4 into a Riemannian manifold*. Math. Z. 247 (2004), no. 1, 65–87.
- [W2] C. Y. Wang, *Stationary biharmonic maps from \mathbf{R}^m into a Riemannian manifold*. Comm. Pure Appl. Math. 57 (2004), no. 4, 419–444 .
- [W3] C. Y. Wang, *Remarks on biharmonic maps into spheres*. Calc. Var. Partial Differential Equations 21 (2004), no. 3, 221–242.

[W4] C. Y. Wang, *Heat flow of biharmonic maps for small data in dimensions at most 8*. In preparation.

Changyou Wang
Department of Mathematics
University of Kentucky, Lexington, KY 40506
E-mail: cywang@ms.uky.edu