Lecture Notes of the Mini-Course

# Introduction of the Navier-Stokes equations

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**Abstract**

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Lecture One: December 19, 2012

# 1 The Background

Let $u(x,t)$ denote the velocity field of the underlying fluid, $x \in \Omega \subset \mathbb{R}^n$ $(n \geq 2)$, and $\Omega$ is a domain representing the container of fluid. Consider the deformation

$$x = x(\alpha, t),$$

where $x$ is the Eulerian coordinate and $\alpha$ is the Lagrangian coordinate. Then

$$\begin{cases} \dfrac{dx}{dt} = u(x,t), \\ x(\alpha, 0) = \alpha. \end{cases}$$

Thus the time-dependent accelerations is given by

$$a = \frac{d^2}{dx^2}x = \frac{d}{dt}u(x(\alpha,t),t) = u_t + \frac{\partial u}{\partial x_i}\frac{dx_i}{dt} = u_t + u^i\frac{\partial u}{\partial x_i} = u_t + (u \cdot \nabla)u.$$

From now on, we denote the material derivative as

$$\frac{Du}{Dt} := u_t + (u \cdot \nabla)u,$$

the second term is called convective acceleration term. Let $\rho$ denote the density of fluid. Then by the conservation law of mass, for any $O \subset \Omega$, the rate of change of mass of fluid over $O$ is equal to the mass flux over $O$, that is,

$$\frac{d}{dt}\int_O \rho\, dx = -\int_{\partial O} \rho\, u \cdot \nu\, d\sigma.$$

Using divergence theorem, we have

$$\frac{d}{dt}\int_O \rho\, dx = -\int_O \mathrm{div}(\rho\, u) dx.$$

Then by the arbitrary of $O$, we have

$$\frac{d\rho}{dt} + \mathrm{div}(\rho\, u) = 0.$$

This is called the continuity equation.

By the conversation of linear momentum (Newton's second law: $F = ma$), the external body force

$$f = \rho\, a = \rho(u_t + (u \cdot \nabla)u) = \rho\frac{Du}{Dt}.$$

There is a problem, as the fluid has friction property (resistance of flow of fluid). The "thin" the fluid is, the less frictional it acts; the "thick" the fluid is, the more frictional it acts. So viscosity is a measurement of the frictional property of a given fluid. Newtonian fluid is a simple fluid that only has viscous property, no other properties (e.g. elasticity).

The Cauchy stress tensor can be described as follows. $\int_{\partial O} \tau_{ij} \nu_j d\sigma$, where $\tau_{ij} = \tau_{ji}$ is a tensor of order $n$. For a fluid in steady state, we have

$$\int_O f + \int_{\partial O} \tau \nu \, d\sigma = 0.$$

This implies that

$$f + \operatorname{div} \tau = 0.$$

There are two forms of $\tau$, for an ideal fluid (inviscid):

$$\tau = -pI_n,$$

where $p = p(\rho)$ is the pressure and $I_n$ is the identity $n \times n$-matrix. For a viscous fluid, where the viscous stress exists, we have

$$\tau = -pI_n + \sigma,$$

where $\sigma = (\sigma_{ij})$ is the viscous stress given by

$$\sigma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{\nabla u + (\nabla u)^T}{2}.$$

This symmetric part of velocity gradient also represents the deformation stretching, and the antisymmetric part of velocity gradient

$$\frac{1}{2}(u_{i,j} - u_{j,i}) = \frac{\nabla u - (\nabla u)^T}{2}$$

represents the rigid rotation. There is another characterization of a simple, Newtonian fluid that the shear stress depends linearly on the rate of strain $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. That is,

$$\sigma = L(e),$$

where $L$ is independent of $x$. Moreover, for any $Q \in SO(3)$, $L$ satisfies the property:

$$L(QeQ^T) = QL(e)Q.$$

It follows that

$$\sigma_{ij} = 2\mu \, e_{ij} + \lambda \delta_{ij} e_{kk} = 2\mu \frac{\nabla u + (\nabla u)^T}{2} + \lambda(\operatorname{div} u) I_n,$$

where $\mu$ is the shear viscosity, which is a measurement of the frictional property of fluid or the thickness of the fluid.

So the equation of steady states is

$$f + \operatorname{div}\left(-pI_n + \mu\left(\nabla u + (\nabla u)^T\right) + \lambda(\operatorname{div} u)I_n\right) = 0.$$

While the dynamical equation is

$$\rho(u_t + u \cdot \nabla u) = f + \operatorname{div}\left(-pI_n + \mu\left(\nabla u + (\nabla u)^T\right) + \lambda(\operatorname{div} u)I_n\right).$$

If the fluid is incompressible, then divergence of $u$ is free and hence

$$\rho(u_t + u\cdot\nabla u) = f - \nabla p + \mu\Delta u.$$

Here is the reason why an incompressible fluid has its velocity field being divergence free. Consider the transformation $x = \phi^t(\alpha, t)$,

$$\begin{cases} \dfrac{d\phi^t}{dt} = u(\phi^t(\alpha, t)) \\ \phi(\alpha, 0) = \alpha \in \mathbb{R}^n, \end{cases}$$

which transforms any open set $O$ to another open set $O_t$. Then we have

$$\mathrm{vol}(O_t) = \mathrm{vol}(O).$$

Since

$$\mathrm{vol}(O_t) = \int_O \det(\nabla\phi^t)d\alpha$$

is constant in $t$, we have

$$0 = \frac{d}{dt}\Big|_{t=0}\mathrm{vol}(O_t) = \frac{d}{dt}\Big|_{t=0}\int_O \det(\nabla\phi^t)d\alpha = \int_O \mathrm{tr}(\nabla\, u)d\alpha = \int_O \mathrm{div}\, u\, d\alpha.$$

In fact, we have

$$\frac{d}{dt}\Big|_{t=0}\det(\nabla\phi^t) = \sum_{i,j} A_{ij}\frac{d}{dt}\Big|_{t=0}\frac{\partial\phi^i}{\partial\alpha_j},$$

where $A_{ij}$ is the co-factor of $\frac{\partial\phi^i}{\partial\alpha_j}$ in the Jacobian matrix $(\nabla\phi)$. Using the factor

$$\sum_j A_{ij}\frac{\partial\phi^k}{\partial\alpha_j} = \delta_{ik}\det(\nabla\phi),$$

we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\det(\nabla\phi^t) &= \sum_{i,j} A_{ij}\frac{d}{dt}\Big|_{t=0}\frac{\partial\phi^i}{\partial\alpha_j} \\
&= \sum_{i,j} A_{ij}\frac{\partial}{\partial\alpha_j}\Big(\frac{d\phi^i}{dt}\Big|_{t=0}\Big) \\
&= \sum_{i,j} A_{ij}\frac{\partial}{\partial\alpha_j}u^i(\phi(\alpha, t)) \\
&= \sum_{i,j} A_{ij}\frac{\partial u^i}{\partial\phi^k}\frac{\partial\phi^k}{\partial\alpha_j} \\
&= \sum_i \frac{\partial u^i}{\partial\phi^k}\delta_{ij}\det(\nabla\phi) \\
&= (\mathrm{div}\, u)\det(\nabla\phi),
\end{aligned}$$

Since $O$ is arbitrary, we have

$$\mathrm{div}\, u = 0.$$

## 1.1 The incompressible Euler equation

When $\mu = 0$, the fluid is ideal or inviscid and we have the incompressible, Euler equation

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = f \\ \nabla \cdot u = 0. \end{cases}$$

Observe that we have

$$\frac{d}{dt}\int_{O^t} f\,dx = \int_{O^t} (f_t + \mathrm{div}(fu))dx.$$

Indeed,

$$\begin{aligned} \frac{d}{dt}\int_{O^t} f\,dx &= \int_O f(\phi^t(x,t),t)\det(\nabla\phi^t)dx \\ &= \int_O (f_t + \nabla f \cdot\, u)\det(\nabla\phi^t)dx + \int_O f(\phi^t(x,t),t)(\mathrm{div}\,u)\det(\nabla\phi^t)dx \\ &= \int_O (f_t + \mathrm{div}(fu))\det(\nabla\phi^t)dx \\ &= \int_{O_t} (f_t + \nabla f \cdot\, u)dx. \end{aligned}$$

Next, we give some properties of divergence free vector fields, e.g. translation, rigid rotation and stretching.

$$u(x_0 + h, t_0) = u(x_0, t_0) + \nabla u(x_0, t_0)h + O(h^2),$$

For

$$E = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad \Omega = \frac{1}{2}(\nabla u - (\nabla u)^T),$$

if $\mathrm{div}\, u = 0$, then $\mathrm{tr}E = 0$. Recalling

$$\omega = \mathrm{curl}\, u = (u_2^3 - u_3^2, u_3^1 - u_1^3, u_1^2 - u_2^1)^T,$$

we have

$$\Omega h = \frac{1}{2}\omega \times h.$$

On the other hand,

$$u(x, t_0) \doteq u(x_0, t_0) + E(x_0, t_0)(x - x_0) + \frac{1}{2}\omega \times (x - x_0).$$

Solving the equation

$$\dot{x}(\alpha, t) = u(x_0, t_0);\ x(\alpha, 0) = \alpha,$$

we have

$$x(\alpha, t) = \alpha + u(x_0, t_0)(t - t_0).$$

This corresponds to the translational motion.

**Example 1.1.** *If* $\omega_0 = 0$ *and* $E = (-r_1, -r_2, r_1 + r_2)$ *for some* $r_1, r_2 > 0$*, then*

$$u(x,t) = (-r_1 x_1, -r_2 x_2, (r_1 + r_2)x_3)^T.$$

*So*

$$x(\alpha,t) = \begin{pmatrix} e^{-r_1 t} & 0 & 0 \\ 0 & e^{-r_2 t} & 0 \\ 0 & 0 & e^{(r_1+r_2)t} \end{pmatrix} \alpha$$

$$(x_1^2 + x_2^2)(\alpha,t) = e^{-2(r_1+r_2)t}(\alpha_1^2 + \alpha_2^2) \to 0.$$

**Example 1.2.** *If* $\omega_0 = 0$ *and* $E = (-r, r, 0)$ *for some* $r > 0$*, then*

$$u(x,t) = (-rx_1, rx_2, 0)^T.$$

$$\begin{cases} \begin{pmatrix} x_1(\alpha,t) \\ x_2(\alpha,t) \end{pmatrix} = \begin{pmatrix} e^{-rt} & 0 \\ 0 & e^{rt} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \\ x_3(\alpha,t) = \alpha_3. \end{cases}$$

**Example 1.3.** *If* $E = 0$ *and* $\omega_0 = (0, 0, \omega_0)^T$*, then*

$$u(x,t) = (-\frac{1}{2}\omega_0 x_2, \frac{1}{2}\omega_0 x_1, 0)^T.$$

$$\begin{cases} \begin{pmatrix} x_1(\alpha,t) \\ x_2(\alpha,t) \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \Big|_{\phi = \frac{1}{2}\omega_0 t} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \\ x_3(\alpha,t) = \alpha_3. \end{cases}$$

Vorticity stretching: For Euler equation

$$\begin{cases} u_t + \nabla p + u \cdot \nabla u = 0, \\ \nabla \cdot u = 0, \end{cases}$$

we have that

$$\begin{cases} \frac{D\omega}{Dt} = \omega \cdot \nabla u, & n = 3, \\ \frac{D\omega}{Dt} = 0, & n = 2. \end{cases}$$

Let $x(\alpha,t)$ express the smooth particle trajectory corresponding to a divergence free vector field $u$. Then we have that

$$\begin{cases} \omega(x(\alpha,t),t) = \nabla_x u(x(\alpha,t),t)\omega_0(\alpha), & n = 3 \\ \omega(x(\alpha,t),t) = \omega_0(\alpha), & n = 2. \end{cases}$$

## 1.2 Leray's reformulation of the Navier-Stokes equation

By

$$\begin{cases} \frac{Du}{Dt} = -\nabla p + \mu \Delta u, \\ \nabla \cdot u = 0, \end{cases}$$

we have

$$-\Delta p = \mathrm{tr}(\nabla u)^2 = \sum_{i,j} u^i_{x_j} u^j_{x_i}$$

so that the pressure $p$ solves the Poisson equation:

$$p(x) = \int_{\mathbb{R}^n} N(x-y)\mathrm{tr}(\nabla u)^2(y,t)\,dy,$$

provided that $\nabla p$ vanishes sufficiently fast as $|x| \to +\infty$, where

$$N(x) = \begin{cases} \dfrac{1}{2\pi}\log|x|, & n = 2, \\ \dfrac{1}{(2-n)\omega_n}|x|^{2-n}, & n \geq 3, \end{cases}$$

is the Newtonian potential. It follows that

$$\nabla p(x,t) = -c_n \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}\mathrm{tr}(\nabla u)^2(y,t)\,dy,$$

so that the material derivative of $u$ is given by

$$\frac{Du}{Dt} = -c_n \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}\mathrm{tr}(\nabla u)^2(y,t)\,dy + \mu\Delta u.$$

Next, we will prove $\mathrm{div}\, u = 0$. Taking divergence on both sides of the Euler equation, we have

$$\begin{cases} \frac{D}{Dt}\mathrm{div} u = \mu\Delta(\mathrm{div} u), \\ \nabla\cdot u|_{t=0} = 0. \end{cases}$$

Multiplying $\mathrm{div}\, u$ and integrating by parts yields

$$\int_{\mathbb{R}^n} \frac{D}{Dt}\mathrm{div}\, u \mathrm{div}\, u = -\mu \int_{\mathbb{R}^n} |\nabla \mathrm{div}\, u|^2.$$

$$\text{LHS} = \frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}|\mathrm{div}\, u|^2 + \int_{\mathbb{R}^n} u\cdot\nabla\frac{(\mathrm{div}\, u)^2}{2} = \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^n}|\mathrm{div}\, u|^2\right) - \int_{\mathbb{R}^n}\nabla\cdot u\frac{(\mathrm{div}\, u)^2}{2} \leq 0.$$

This implies

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n}(\mathrm{div}\, u)^2\right) \leq c\int_{\mathbb{R}^n}(\mathrm{div}\, u)^2.$$

By the Gronwall inequality, we have

$$\int_{\mathbb{R}^n}(\mathrm{div}\, u)^2(t) \leq e^{ct}\int_{\mathbb{R}^n}(\mathrm{div}\, u)^2(0) = 0.$$

Therefore, we have

$$\mathrm{div}\, u(t) = 0.$$

We have proved the following proposition

**Proposition 1.1.** *The Navier-Stokes equation*

$$\begin{cases} \frac{Du}{Dt} = -\nabla p + \mu\Delta u, \\ \nabla\cdot u = 0, \\ u\Big|_{t=0} = u_0( \quad \mathrm{div}\, u_0 = 0), \end{cases}$$

*is equivalent to*

$$\begin{cases} \frac{Du}{Dt} = -c_n\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}\mathrm{tr}(\nabla u)^2(y,t))\,dy + \mu\Delta u, \\ u|_{t=0} = u_0, \\ p \text{ is determined by } -\Delta p = \mathrm{tr}(\nabla u)^2. \end{cases}$$

## 1.3 Vorticity formulation of Navier-Stokes equation in dimension two

From

$$\operatorname{div} u = 0, \quad \operatorname{curl} u = \omega,$$

it follows that

$$u = \nabla^T \psi = (-\psi_{x_2}, \psi_{x_1})$$

and

$$\operatorname{curl} u = (\psi_{x_2 x_2} + \psi_{x_1 x_1}) = \Delta\psi = \omega.$$

So

$$\psi(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \omega(y,t) dy.$$

Recalling

$$\frac{D\omega}{dt} = \mu\Delta\omega, \quad \omega_{t=t_0} = \omega_0$$

we have

$$u(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)dy,$$

where

$$K_2(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)^T.$$

This is Boit-Savart law. We can also recover the pressure function $p$ through the Poisson equation:

$$-\Delta p = \sum_{i,j} u^i_{x_j} u^j_{x_i}.$$

Lecture 2, December 20, 2012

# 2 Introduction (continued)

Recall that the Navier-Stokes equation is given by

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \nu \Delta u \\ \nabla u = 0. \end{cases} \tag{2.1}$$

The fundamentally open question is

> **Given a smooth, compactly supported, divergence free vector field** $u_0(x)$ **in** $\mathbb{R}^3$**, are there smooth solutions of** (2.1) **with** $u|_{t=0} = v_0$**?**

## 2.1 Another word on NSE's derivation

By the momentum balance law, we have

$$\frac{\partial}{\partial t}\int_O \rho u dx = -\int_{\partial O} (\rho u) u \cdot \nu \, dS + \int_{\partial O} \tau \cdot \nu \, dS,$$

where

$$\tau = -pI + \sigma = -pI + 2\mu \frac{\nabla u + (\nabla u)^T}{2}.$$

It follows from divergence theorem that

$$\frac{d}{dt}(\rho u) + \mathrm{div}(\rho u \otimes u) = \mathrm{div}\tau = -\nabla p + 2\frac{\mu}{2}(\Delta u + \nabla \mathrm{div} u)$$

Combining with mass conservation law

$$\frac{d\rho}{dt} + \mathrm{div}(\rho u) = 0,$$

we have

$$\begin{cases} \dfrac{d}{dt}(\rho u) + \mathrm{div}(\rho u \otimes u) = \mathrm{div}\tau = -\nabla p + 2\dfrac{\mu}{2}(\Delta u + \nabla \mathrm{div} u), \\ \\ \dfrac{d\rho}{dt} + \mathrm{div}(\rho u) = 0. \end{cases}$$

## 2.2 Vorticity formulation in dimension 3

We first review the vorticity formulation in dimension 2

$$\frac{D\omega}{Dt} = \mu \, \Delta\omega,$$

where $\omega = \mathrm{curl} u$.

If $\mu = 0$, for Euler equation, then

$$\frac{D\omega}{Dt} = 0. \tag{2.2}$$

That is,

$$\omega(x(\alpha, t)) = \omega_0(\alpha),$$

The vorticity, as a scalar function, is transported along the flow trajectory.

If $\mu > 0$, for the Navier-Stokes equation, then $\omega$ solves the convective heat equation. Here is a fact. In the smooth case, if $\omega$ solves

$$\begin{cases} \dfrac{D\omega}{Dt} = \mu\,\Delta\omega, \\ \omega|_{t=0} = \text{curl}\ u_0, \quad (\text{div} u_0 = 0) \end{cases}$$

with

$$u(x, t) = \int_{\mathbb{R}^2} K_2(x - y)\omega(y, t)dy$$

where

$$K_2(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right),$$

then $u$ solves (2.1), with

$$-\Delta\, p = \text{tr}(\nabla\, u)^2.$$

## 2.3 Construction of steady solutions to the Euler equation in dimension 2

By (2.2), we have

$$\omega_t + u \cdot \nabla\omega = 0.$$

Let $u = \nabla^\perp\psi$. Then

$$\omega = \text{curl} u = \Delta\psi,$$

and

$$u \cdot \nabla\omega = \nabla^\perp\psi \cdot \nabla\Delta\psi = \det\begin{pmatrix} \psi_{x_1} & \psi_{x_2} \\ \Delta\psi_{x_1} & \Delta\psi_{x_2} \end{pmatrix}.$$

This means that

$$\omega_t + \det\begin{pmatrix} \psi_{x_1} & \psi_{x_2} \\ \Delta\psi_{x_1} & \Delta\psi_{x_2} \end{pmatrix} = 0$$

Now we have the following Lemma

**Lemma 2.1.** *A function $\psi$ defines a steady solution to Euler equation in dimension 2 if and only if $\Delta\psi = F(\psi)$ for some function $F$.*

*Proof.* It follows from $\omega_t = 0$ that

$$\det\begin{pmatrix} \nabla\psi \\ \nabla\Delta\psi \end{pmatrix} = 0.$$

So we have $\nabla\psi \parallel \nabla\Delta\psi$. This means that $\psi$ and $\Delta\psi$ has level curves. Therefore,

$$\Delta\,\psi = F(\psi).$$

□

**Lemma 2.2.** *For a steady flow, $\psi$ is constant along the particle trajectories.*

*Proof.* Recalling that

$$\begin{cases} \dfrac{dx^1}{dt} = -\psi_{x_2}(x(\alpha, t)), \\ \dfrac{dx^2}{dt} = \psi_{x_1}(x(\alpha, t)), \end{cases}$$

we have

$$\frac{d}{dt}\psi(x(\alpha, t)) = \psi_{x_1}\frac{dx^1}{dt} + \psi_{x_2}\frac{dx^2}{dt} = -\psi_{x_1}\psi_{x_2} + \psi_{x_1}\psi_{x_2} = 0.$$

So $\psi$ = constant. □

Here we give two simple examples.

**Example 2.1** (Steady eddies)**.** *If $\omega_0$ is radial, i.e. $\omega_0 = \omega_0(|x|)$, then it follows from $\Delta\psi_0 = \omega_0$ that $\psi_0$ is also radial, that is $\psi_0 = \psi_0(|x|)$. By*

$$\det(\nabla\psi_0, \nabla\Delta\psi_0) = 0,$$

*$\omega_0$ produce a steady, radially symmetric solution to the Euler equation in dimension 2. By*

$$u_0(x) = \nabla^\perp\psi_0 = \left(-\frac{x_2}{r}, \frac{x_1}{r}\right)^T \psi_0'(r),$$

*and*

$$\psi_0''(r) + \frac{1}{r}\psi_0'(r) = \omega_0(r),$$

*we have*

$$u_0(x) = \left(-\frac{x_2}{r^2}, \frac{x_1}{r^2}\right)^T \int_0^r s\omega_0(s)ds.$$

*This means that the streamlines of the flow are circles. The fluid rotates depending on the sign of $\omega_0$.*

**Example 2.2** (Time-dependent viscous eddies)**.** *Let $\omega_0 = \omega_0(r)$. If $\omega(x, t)$ is radially symmetric, then $\psi(x, t)$ is radially symmetric. So*

$$\det(\nabla\psi_0, \nabla\Delta\psi_0) = 0.$$

*Since $u \cdot \nabla\omega = 0$, we have*

$$u(x, t) = \left(-\frac{x_2}{r^2}, \frac{x_1}{r^2}\right)^T \int_0^r s\omega(s, t)ds.$$

*Solving the heat equation*

$$\begin{cases} \omega_t = \mu\Delta\omega, \\ \omega|_{t=0} = \omega_0, \end{cases}$$

*we have*

$$\omega(x, t) = \frac{1}{4\pi\mu t}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4\mu t}}\omega_0(|y|)dy.$$

**Proposition 2.3.** *Let $\omega_0(r)$ satisfies $|\omega_0| + |\nabla\omega_0| \leq M$, $u_0(r)$ is the invisicid radial eddies solution. Then*

$$|\omega(x,t) - \omega_0(r)| \lesssim \sqrt{\mu t}, \qquad |u(x,t) - u_0(r)| \lesssim |x|\sqrt{\mu t}.$$

*Proof.* We fist recall that $\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}}|z|dz = 1$. Let $x - y = \sqrt{\mu t}z$, then

$$\begin{aligned}|\omega(x,t) - \omega_0(r)| &\leq \left|\frac{1}{4\pi\mu t}\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}}(\omega_0(x + \sqrt{\mu t}z) - \omega_0(|z|))dz\right| \\ &\lesssim \|\nabla\omega_0\|_{L^\infty}\sqrt{\mu t}\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}}|z|dz \\ &\lesssim \sqrt{\mu t}.\end{aligned}$$

So

$$\begin{aligned}|u(x,t) - u_0(r)| &\leq \frac{1}{r}\left|\int_0^r s(\omega(s,t) - \omega_0)ds\right| \\ &\lesssim \frac{1}{r}\int_0^r s\sqrt{\mu t}ds \\ &\lesssim r\sqrt{\mu t}.\end{aligned}$$

□

Now we return to 3D vorticity formulation of NSE. Consider

$$\begin{cases}\text{curl}u = \omega, \\ \text{div}u = 0.\end{cases} \tag{2.3}$$

**Lemma 2.4.** *Let $\omega \in L^2 \cap L^\infty(\mathbb{R}^3, \mathbb{R}^3)$, and $\omega \to 0$ sufficiently fast as $|x| \to 0$. Then*

($i$) (2.3) *has a solution $u$ vanishing at $\infty$ if and only if* $\text{div}\omega = 0$.

($ii$) *If* $\text{div}\omega = 0$*, then $u = -\text{curl}\psi$, where $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ solves the Poisson equation:*

$$\Delta\psi = \omega.$$

*Proof.* ($i$) Note that $\text{div}\,\text{curl}f = 0$, for all $f \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$. So

$$\text{div}\,\text{curl}f = \frac{\partial}{\partial x_i}(\text{curl}f)_i = \frac{\partial}{\partial x_i}(\epsilon_{ijk}f_j^k)_i = \sum_{i,j,k}\epsilon_{ijk}f_{ij}^k = 0.$$

It suffices to establish ($ii$). Let $\psi$ solve $\Delta\psi = \omega$. Note

$$\text{curl}\,\text{curl}\,\psi = \nabla \times (\nabla \times \psi) = \nabla(\nabla \cdot \psi) - \nabla \cdot (\nabla\psi) = \nabla(\text{div}\,\psi) - \Delta\psi.$$

Hence

$$-\text{curl}\,\text{curl}\,\psi + \nabla(\text{div}\,\psi) = \omega.$$

Multiplying $\nabla(\operatorname{div}\psi)$ on both sides of this equality, and integrating by parts, we have

$$RHS = \int_{\mathbb{R}^3} \omega \nabla \operatorname{div}\psi = -\int_{\mathbb{R}^3} \nabla\omega \operatorname{div}\psi = 0.$$

So

$$LHS = \int_{\mathbb{R}^3} |\nabla \operatorname{div}\psi|^2 = 0.$$

Hence

$$\nabla \operatorname{div}\psi = 0.$$

This implies that

$$\operatorname{curl}(-\operatorname{curl}\psi) = \omega.$$

Set $u = -\operatorname{curl}\psi$, then

$$u = -\operatorname{curl}\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|}\omega(y)dy.$$

That is,

$$u^i = \epsilon_{ijk}\left(\int_{\mathbb{R}^3} \frac{\omega^k(y)}{4\pi|x-y|}dy\right)_j = \epsilon_{ijk}\int_{\mathbb{R}^3} \frac{(x-y)^j\omega^k(y)}{4\pi|x-y|^3}dy = \left(\int_{\mathbb{R}^3} \frac{(x-y)\omega^k(y)}{4\pi|x-y|^3}dy\right)^i.$$

Thus,

$$u(x) = \int_{\mathbb{R}^3} K_3(x-y)\omega(y)dy, \quad \text{where} \quad K_3(x-y)h = \frac{1}{4\pi}\frac{(x-y)h}{|x-y|^3}.$$

The above is the Boit-Savarat law in dimension 3. □

## 2.4 Vorticity equations

Apply $\partial_j$ to the Navier-Stokes equation, we obtain

$$(u^k_j)_t + u^l_j u^k_l + u^l u^k_{jl} = \mu\Delta u^k_j.$$

Then $\omega^i = \epsilon_{ijk}u^k_j$ satisfies

$$(\omega^i)_t + u^l \cdot \nabla\omega^i + \epsilon_{ijk}u^l_j u^k_l = \mu\Delta\omega^i.$$

For $i = 1$, it follows from $\operatorname{div} u = 0$ that

$$\begin{aligned}
\epsilon_{1jk}u^l_j u^k_l &= \epsilon_{123}u^l_2 u^3_l + \epsilon_{132}u^l_3 u^2_l \\
&= u^l_2 u^3_l - u^l_3 u^2_l \\
&= u^1_2 u^3_1 + u^2_2 u^3_2 + u^3_2 u^3_3 - u^1_3 u^2_1 - u^2_3 u^2_2 - u^3_3 u^2_3 \\
&= u^1_2 u^3_1 - u^1_1 u^3_2 - u^1_3 u^2_1 + u^2_3 u^1_1 \\
&= -(u^3_2 - u^2_3)u^1_1 - (u^1_3 - u^3_1)u^1_2 - (u^2_1 - u^1_2)u^1_3 \\
&= -(\omega \cdot \nabla u)^1.
\end{aligned}$$

Hence

$$\epsilon_{ijk}u^l_j u^k_l = -\omega \cdot \nabla u^i.$$

Therefore,
$$\frac{D\omega}{Dt} = \omega \cdot \nabla u + \mu\Delta \omega. \tag{2.4}$$
Denote
$$\Omega = \frac{1}{2}\left(\nabla u - (\nabla u)^T\right),$$
then
$$\Omega h = \frac{1}{2}\omega \times h.$$
Indeed, for $i = 1$,
$$(\Omega h)^1 = \frac{1}{2}(u^1_j - u^j_1)h^j = \frac{1}{2}(u^1_2 - u^2_1)h^2 + \frac{1}{2}(u^1_3 - u^3_1)h^3,$$
and
$$\frac{1}{2}(\omega \times h)^1 = \frac{1}{2}(\omega^2 h^3 - \omega^3 h^2) = \frac{1}{2}(u^1_3 - u^3_1)h^3 - \frac{1}{2}(u^2_1 - u^1_2)h^2.$$
For $i = 2$ and $i = 3$, it is similar.

There is another way to derive (2.4). Denote
$$V = \left(\frac{\partial u^i}{\partial x_k}\right), \quad \text{and} \quad P = (p_{x_i x_k}).$$
Then
$$\frac{DV}{Dt} + V^2 = -P + \mu\Delta V.$$
Recalling $V = \mathcal{D} + \Omega$, $V^2 = \mathcal{D}^2 + \Omega^2 + \mathcal{D}\Omega + \Omega\mathcal{D}$, we have
$$\frac{D\mathcal{D}}{Dt} + \mathcal{D}^2 + \Omega^2 = -P + \mu\Delta\mathcal{D}, \quad \text{and} \quad \frac{D\Omega}{Dt} + \mathcal{D}\Omega + \Omega\mathcal{D} = \mu\Delta\Omega.$$
We claim that
$$(\Omega\mathcal{D} + \mathcal{D}\Omega)_{21} = -(\Lambda\omega)^3.$$
*Proof.* Note that $\Omega_{31} = -\omega^2$, $\Omega_{21} = \omega^3$, $\Omega_{23} = -\omega^1$, and $\mathcal{D}_{11} + \mathcal{D}_{22} + \mathcal{D}_{33} = \operatorname{tr}(\nabla u) = \operatorname{div} u = 0$, and
$$\begin{aligned}
(\Omega\mathcal{D} + \mathcal{D}\Omega)_{21} &= \mathcal{D}_{21}\Omega_{11} + \mathcal{D}_{22}\Omega_{21} + \mathcal{D}_{23}\Omega_{31} + \Omega_{21}\mathcal{D}_{11} + \Omega_{22}\mathcal{D}_{21} + \Omega_{23}\mathcal{D}_{31} \\
&= \mathcal{D}_{23}\Omega_{31} + (\mathcal{D}_{11} + \mathcal{D}_{22})\Omega_{21} + \mathcal{D}_{31}\Omega_{23} \\
&= -\mathcal{D}_{23}\omega^2 - \mathcal{D}_{33}\omega^3 - \mathcal{D}_{31}\omega^1 \\
&= -(\mathcal{D}\omega)^3.
\end{aligned}$$
□

So that
$$\frac{D\omega}{Dt} = \mathcal{D}\omega + \mu\Delta\omega.$$

**Proposition 2.5.** *Let $\mathcal{D}(t)$ be $3\times 3$, symmetric, traceless real matrix. Let $\omega(t)$ solve*

$$\begin{cases} \dfrac{d\omega}{dt} = \mathcal{D}(t)\omega, \\ \omega|_{t=0} = \omega_0, \\ \Omega\, h = \dfrac{1}{2}\omega\times h, \quad h\in\mathbb{R}^3. \end{cases}$$

*Define*

$$u = \frac{1}{2}\omega\times x + \mathcal{D}\,x, \quad p = -\frac{1}{2}\Big(\frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2\Big)x\cdot x.$$

*Then $v, p$ solves the Navier-Stokes equations in dimension 3.*

*Proof.* If $u = \frac{1}{2}\omega(t)\times x + \mathcal{D}(t)\,x$, then $\operatorname{curl} u = \omega(t)$, $\Delta\omega = u\cdot\nabla\omega = 0$. Now the vorticity equation reduces to

$$\frac{\partial\omega}{\partial t} = \mathcal{D}(t)\omega, \quad \Delta\mathcal{D} = v\cdot\nabla\mathcal{D} = 0.$$

So we have

$$\frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2 = -p(t).$$

This implies $p(t)$ is a symmetric matrix. Hence

$$P(t) = \nabla^2\Big(\frac{1}{2}p(t)x\cdot x\Big).$$

□

**Definition 2.1.** For $n = 2, 3$,

$$p.v.\int_{\mathbb{R}^n} f(x)dx = \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} f(x)dx,$$

provided that the limit exists.

**Theorem 2.6** (3D vorticity-stream formulation of Navier-Stokes equation)**.** *For 3D smooth flows that vanish sufficiently rapidly as $|x|\to\infty$, the Navier-Stokes equation is equivalent to*

$$\begin{cases} \dfrac{D\omega}{Dt} = \omega\cdot\nabla\, u + \mu\Delta\omega, \quad \mathbb{R}^3\times\mathbb{R}_+, \\ \omega|_{t=0} = \omega_0 = \operatorname{curl} u_0 \end{cases}$$

*where $u$ is given by the Biot-Savart Law:*

$$u(x,t) = \int_{\mathbb{R}^3} K_3(x-y)\omega(y,t)dy, \quad K_3(x)h = \frac{1}{4\pi}\frac{x\times h}{|x|^3}, \quad h\in\mathbb{R}^3,$$

*and*

$$\nabla\, u(x)h = -p.v.\int_{\mathbb{R}^3}\left[\frac{\omega(y)\times h}{4\pi|x-y|^3} + \frac{3}{4\pi}\frac{((x-y)\times\omega(y))\otimes(x-y)}{|x-y|^5}h\right]dy + \frac{1}{3}\omega(x)\times h.$$

**Lemma 2.7.** *If*

$$u(x,t) = \int_{\mathbb{R}^3} K_3(x-y)\omega(y,t)dy, \quad K_3(x)h = \frac{1}{4\pi}\frac{x\times h}{|x|^3}, \quad h\in\mathbb{R}^3,$$

*then*

$$\nabla u(x)h = -p.v. \int_{\mathbb{R}^3}\left[\frac{\omega(y)\times h}{4\pi|x-y|^3} + \frac{3}{4\pi}\frac{((x-y)\times\omega(y))\otimes(x-y)}{|x-y|^5}h\right]dy + \frac{1}{3}\omega(x)\times h.$$

*Proof.* First we need to calculate the distributional derivative of $K_3$. For $\varphi\in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}
\langle \partial_{x_i}K_3,\varphi\rangle_{L^2} &= -\langle K_3, \partial_{x_i}\varphi\rangle_{L^2}\\
&= -\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} K_3\partial_{x_i}\varphi\\
&= -\lim_{\epsilon\to 0}\left(-\int_{|x|\geq\epsilon}\partial_{x_i}K_3\varphi + \int_{|x|=\epsilon}K_3\varphi\frac{x_i}{|x|}\right)\\
&= p.v.\int_{\mathbb{R}^3}\partial_{x_i}K_3\varphi - \lim_{\epsilon\to 0}\int_{|y|=1}K_3(y)\varphi(\epsilon y)\frac{y_i}{|y|}dy\\
&= p.v.\int_{\mathbb{R}^3}\partial_{x_i}K_3\varphi - \varphi(0)c_i, \quad c_i = \int_{|y|=1}K_3(y)y_i d\sigma.
\end{aligned}$$

Then

$$\begin{aligned}
\nabla u(x)h = &- p.v.\int_{\mathbb{R}^3}\left[\frac{\omega(y)\times h}{4\pi|x-y|^3} + \frac{3}{4\pi}\frac{((x-y)\times\omega(y))\otimes(x-y)}{|x-y|^5}h\right]dy\\
&-\frac{1}{4\pi}\int_{|y|=1}[y\times\omega(y)]y\cdot\ hd\sigma, \qquad (2.5)
\end{aligned}$$

where

$$\frac{1}{4\pi}\int_{|y|=1}[y\times\omega(y)]y\cdot\ hd\sigma = -\frac{1}{3}\omega(x)\times h,$$

we have used

$$\int_{|y|=1}y_iy_j = \begin{cases}\frac{4\pi}{3}, & i=j,\\ 0, & i\neq j.\end{cases}$$

□

*Proof.* Formally, since $u = -\operatorname{curl}\psi$ and $\Delta\psi = \omega$, we have $\operatorname{div} u = 0$, we have $\operatorname{div} u = 0$.

Rigorously, one need to use (2.5) to verify $\operatorname{div} u = 0$, but we leave it to the reader.

First, we use $\operatorname{div} u = 0$ to show that

$$\frac{D}{Dt}(\operatorname{div} u) = \mu\Delta\operatorname{div} u.$$

$$\frac{\partial\omega_i^i}{\partial t} + u^j\partial_j\omega_i^i + u_i^j\partial_j\omega^i = \omega^j\partial_j(u_i^i) + \omega_i^j\partial_j u^i + \mu\Delta(\omega_i^i) = \mu\Delta(\omega_i^i).$$

By

$$\begin{cases}\dfrac{\partial\operatorname{div}\omega}{\partial t} + u\cdot\nabla\operatorname{div}\omega = \mu\Delta\operatorname{div}\omega,\\ \operatorname{div}\omega|_{t=0} = \operatorname{div}\operatorname{curl}u_0 = 0,\end{cases}$$

we have

$$\operatorname{div} \omega = 0, \quad \text{for all } t \geq 0.$$

On the other hand, by

$$\frac{\partial}{\partial t}(\operatorname{curl} u) + u \cdot \nabla \operatorname{curl} u = \operatorname{curl} u \cdot \nabla u + \mu \Delta \operatorname{curl} u,$$

we have

$$\operatorname{curl} \left( \frac{Du}{Dt} - \mu \Delta u \right) = 0.$$

So that

$$\frac{Du}{Dt} - \mu \Delta u = -\nabla p,$$

for some scalar function $p$. □

**Lemma 2.8.** *If $K_3$ is a homogeneous of degree -2 function, then*

$$\int_{|x|=1} \partial_{x_i} K_3 d\sigma = 0.$$

*Proof.* Let $\rho \in C_0^\infty(\mathbb{R})$, $\rho \geq 0$, $\rho(r) = \begin{cases} 1, & r \leq A, \\ 0, & r \geq B, \end{cases}$ for some $0 < A < B$. Then

$$\int_0^\infty \rho'(r) dr = 0, \quad \int_0^\infty \frac{\rho(r)}{r} dr = c > 0.$$

So

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} \partial_{x_i}(\rho(|x|)K(x))dx \\ &= \int_{\mathbb{R}^3} \rho'(r) \frac{x_i}{|x|} K(x)dx + \int_{\mathbb{R}^3} \rho(r) \partial_{x_i} K(x)dx \\ &= \int_0^\infty \rho'(r)dr \int_{|x|=1} x_i K(x)d\sigma + \int_0^\infty \frac{\rho(r)}{r} dr \int_{|x|=1} \partial_{x_i} K(x)d\sigma \\ &= c \int_{|x|=1} \partial_{x_i} K(x)d\sigma. \end{aligned}$$

The proof is completed. □

Lecture 3, December 21, 2012

# 3 Basic properties of the Navier-Stokes equation

If $u$ satisfies Navier-Stokes equation

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \mu \Delta u, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{3.1}$$

then

- translation invariance: for any $c \in \mathbb{R}^3$,

$$\begin{cases} u_c(x,t) = u(x - \vec{c}t, t) + \vec{c}, \\ p_c(x,t) = p(x - \vec{c}t, t), \end{cases}$$

  also solves (3.1).

- rotation invariance: for any $Q \in O(3)$,

$$\begin{cases} u_\theta(x,t) = \theta^T u(\theta x, t), \\ p_\theta(x,t) = p(\theta x, t), \end{cases}$$

  is also a solution.

- scaling invariance: for any $\lambda > 0$,

$$\begin{cases} u_\lambda(x,t) = \frac{1}{\lambda} u(\lambda^{-1} x, \lambda^{-2} t), \\ p_\lambda(x,t) = \frac{1}{\lambda^2} p(\lambda^{-1} x, \lambda^{-2} t) \end{cases}$$

  is also a solution.

Dimension in Navier-Stokes equation:

$$x \to 1, \quad t \to 2;$$
$$\frac{\partial}{\partial x} \to -1 \quad \frac{\partial}{\partial t} \to -2;$$
$$u \to -1, \quad p \to -2;$$
$$\Delta_x \to -2.$$

## 3.1 Helmholtz decomposition and Leray projection operator

**Finite dimensional analog:** Suppose $\Sigma \subset \mathbb{R}^3$ is a plane, $x$ is a particle in $\Sigma$. Then

$$F = F^{\perp} + F^{\parallel},$$

where $F^{\perp}$ has no effect on the particle's acceleration, while $F^{\parallel}$ cause the particle to accelerate. That is,

$$F^{\parallel} = ma^{\parallel}.$$

**Infinite dimensional case:** Consider the linearization of Navier-Stokes equation at $u = 0$, $\rho = \rho_0 =$ constant. Applying infinitesimally small force $f(x,t)$ to it, we have

$$\begin{cases} \rho_0 u_t + \nabla p = f, & \text{in } \Omega, \\ u \cdot \nu = 0, & \text{on } \partial\Omega, \\ \operatorname{div} u = 0, \end{cases}$$

where $f$ can be decomposed into two special force: a gradient force, and a divergence free force

$$g = \rho_0 u_t, \quad g \cdot \nu = 0, \quad \text{on } \partial\Omega.$$

Now we consider Helmholtz decomposition.

Define

$$X = \left\{ g : \Omega \to \mathbb{R}^3 \middle| \, g \in C^{\infty}, \operatorname{div} g = 0, \ g \cdot \nu = 0 \text{ on } \partial\Omega \right\},$$

and

$$Y = \left\{ \nabla\varphi \middle| \, \varphi \in C^{\infty}(\mathbb{R}^3) \right\},$$

then

$$X \perp Y,$$

that is,

$$\langle g, \nabla\varphi \rangle_{L^2} = \int_{\Omega} g \nabla\varphi = \int_{\Omega} \operatorname{div}(g\varphi) = \int_{\partial\Omega} \varphi\, g \cdot \nu = 0.$$

Set

$$\overline{X} = \text{closure of } X \text{ in } L^2(\Omega, \mathbb{R}^3), \quad \overline{Y} = \text{closure of } Y \text{ in } L^2(\Omega, \mathbb{R}^3),$$

then

$$\overline{X} \perp \overline{Y}.$$

**Theorem 3.1.** *(Helmholtz decomposition)* $L^2(\Omega, \mathbb{R}^3) = \overline{X} \oplus \overline{Y}$.

*Proof.* For any $f \in L^2(\Omega, \mathbb{R}^3)$, let

$$\begin{cases} \Delta g = \nabla \cdot f, & \text{in } \Omega, \\ \frac{\partial g}{\partial \nu} = f \cdot \nu, & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

then

$$h = f - \nabla g$$

is divergence free and

$$h \cdot \nu = f \cdot \nu - \frac{\partial g}{\partial \nu} = 0.$$

So

$$f = (f - \nabla g) + \nabla g$$

is the desired decomposition, provided that (3.2) is solvable.

(3.2) can be solved by the following minimization process:

$$\min_{u \in H^1(\Omega)} \int_\Omega |\nabla u - f|^2. \tag{3.3}$$

Suppose (3.3) is attained by a $u$, then for any $v \in H^1(\Omega)$,

$$0 = \frac{d}{dt}\Big|_{t=0} \int_\Omega |\nabla(u + tv) - f|^2 = \int_\Omega (\nabla u - f, \nabla v).$$

Hence

$$\begin{cases} \nabla \cdot (\nabla u - f) = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = f \cdot \nu. \end{cases}$$

(Existence). Let $\{u_k\} \subset H^1(\Omega)$ be a minimizing sequence, that is,

$$\int_\Omega |\nabla u_k - f|^2 \to \inf_{u \in H^1} \int_\Omega |\nabla u - f|^2 = c \in [0, +\infty),$$

then

$$\int_\Omega |\nabla \left(\frac{u_k - u_l}{2}\right)|^2 + \int_\Omega |\nabla \left(\frac{u_k + u_l}{2}\right) - f|^2 = \frac{1}{2} \int_\Omega |\nabla u_k - f|^2 + \frac{1}{2} \int_\Omega |\nabla u_l - f|^2.$$

$$RHS \to \frac{c}{2} + \frac{c}{2} = c, \quad \text{as } k, l \to \infty,$$

while

$$\int_\Omega |\nabla \left(\frac{u_k + u_l}{2}\right) - f|^2 \geq c,$$

we conclude that

$$\lim_{k,l \to \infty} \int_\Omega |\nabla \left(\frac{u_k - u_l}{2}\right)|^2 = 0,$$

and hence $\{\nabla u_k\}$ is a Cauchy sequence in $L^2(\Omega)$. Since we can replace $u_k$ by $\widetilde{u_k} = u_k - \fint_\Omega u_k$, we may assume that $\int_\Omega u_k = 0$. By the Poincaré inequality, we have

$$\int_\Omega |u_k - u_l|^2 \lesssim \int_\Omega |\nabla (u_k - u_l)|^2 \to 0, \quad \text{as } k, l \to \infty.$$

Hence we may assume that there exists a $u \in H^1(\Omega)$ with $\int_\Omega u = 0$, so that $u_k \to u$ strongly in $H^1(\Omega, \mathbb{R}^3)$. It is easy to see that

$$\int_\Omega |\nabla u - f|^2 = \lim_{k \to \infty} \int_\Omega |\nabla u_k - f|^2 = c,$$

that is, $u$ achieves the infimum.

It turn out that the decomposition is unique. Suppose that there are $f_1, f_2 \in L^2(\Omega, \mathbb{R}^3)$, $\varphi_1, \varphi_2 \in H^1(\Omega)$ such that

$$\operatorname{div} f_1 = \operatorname{div} f_2 = 0,$$

and

$$f_1 \cdot \nu = f_2 \cdot \nu = 0 \quad \text{on } \partial\Omega,$$
$$f = f_1 + \nabla\varphi_1 = f_2 + \nabla\varphi_2,$$

then

$$f_1 - f_2 = \nabla(\varphi_2 - \varphi_1)$$

and

$$\int_\Omega |f_1 - f_2|^2 = \langle \nabla(\varphi_2 - \varphi_1), f_1 - f_2\rangle_{L^2} = \langle \varphi_2 - \varphi_1, \nabla(f_1 - f_2)\rangle_{L^2} = 0.$$

This implies that $f_1 = f_2$. Of course $\varphi_1, \varphi_2$ are possibly different. □

Let $\mathbb{P} : L^2(\Omega, \mathbb{R}^3) \to \overline{X}$. Then $\mathbb{P}$ is called the Leray projection operator. It turns out

**Proposition 3.2.** *$\mathbb{P}$ is a bounded operator from $L^2(\Omega, \mathbb{R}^3)$ to $L^2(\Omega, \mathbb{R}^3)$:*

$$\|\mathbb{P}f\|_{L^2} \lesssim \|f\|_{L^2(\Omega)}. \tag{3.4}$$

*Proof.* i) Since $\mathbb{P}f = f - \nabla u$, where $u \in H^1(\Omega)$ achieves

$$\int_\Omega |\nabla u - f|^2 = \inf_{v \in H^1(\Omega)} \int_\Omega |\nabla v - f|^2 \leq \int_\Omega |f|^2,$$

we obtain

$$\int_\Omega |\mathbb{P}f|^2 \lesssim \int_\Omega |f|^2,$$

so (3.4) holds with the coefficient 1.

ii) If $\operatorname{div}(\nabla u - f) = 0$, $\frac{\partial u}{\partial \nu} = f \cdot \nu$ on $\partial\Omega$, then by elliptic estimate, we also have

$$\|\nabla u\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$

So

$$\|\nabla u - f\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)},$$

but without optimal bound. □

**Representation of Leray projection operator in the case $\Omega = \mathbb{R}^n$:** For $f \in L^2(\mathbb{R}^n, \mathbb{R}^n)$, let $u \in H^1(\mathbb{R}^n)$ solve

$$\Delta u = \operatorname{div} f \quad \text{in } \mathbb{R}^n,$$

then

$$\mathbb{P}f = f - \nabla u$$

satisfies the condition that

$$\operatorname{div}(\mathbb{P}f) = 0 \quad \text{in } \mathbb{R}^n.$$

Recall that

$$u = (\Delta^{-1})\operatorname{div} f,$$

we have

$$\nabla u = \nabla(\Delta^{-1})\operatorname{div} f,$$

so

$$(\mathbb{P}f)^i = f^i - \nabla_i(\Delta^{-1})(f^j_j) = f^i - \nabla_i(\Delta^{-1})^{\frac{1}{2}}(\Delta^{-1})^{\frac{1}{2}}\nabla_j f^j = f^i - R_i R_j f^j,$$

where $R_i = \nabla_i(\Delta^{-1})^{\frac{1}{2}}$ denotes the $i^{\text{th}}$ Riesz transform. Therefore

$$(\mathbb{P}f)^i = f^i - R_i R_j f^j,$$

is the Leray projection operator.

## 3.2 The Steady Stokes equation

Now we consider the steady Stokes equation

$$\begin{cases} -\mu\Delta u + \nabla p = f, & \text{in } \Omega, \quad \mu > 0, \\ \nabla\cdot u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

**Basic function spaces:** Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain: $\partial\Omega \in C^{0,1}$, that is, for any $y \in \partial\Omega$, there exists $r > 0$ such that $\partial\Omega \cap B_r(y)$ is the graph of a Lipschitz function. For $1 \le p \le +\infty$, define

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \left(\int_\Omega |f|^p\right)^{1/p} = \|f\|_{L^p} < +\infty \right\}$$

Recalling the Poincaré's inequality: for any $1 \le p \le +\infty$,

$$\|f\|_{L^p} \le C(\Omega, p)\|\nabla f\|_{L^p}, \quad \forall f \in W_0^{1,p}(\Omega).$$

Define

$$H = \overline{\{u \in C_c^\infty(\Omega) \mid \operatorname{div} u = 0\}}_{L^2},$$

and

$$V = \overline{\{u \in C_c^\infty(\Omega) \mid \operatorname{div} u = 0\}}_{H_0^1} = H_0^1(\Omega) \cap \{\operatorname{div} u = 0\}.$$

$$E(\Omega) = \{u \in L^2(\Omega) \mid \operatorname{div} u \in L^2(\Omega)\} \supset H^1(\Omega),$$

with

$$\langle u, v\rangle_E = \int_\Omega uv + \operatorname{div} u \operatorname{div} v.$$

Here is a fact: $C_c^\infty(\overline{\Omega})$ is dense in $E(\Omega)$, provided $\Omega$ is Lipschitz.

By trace theorem, we know

$$\gamma_0 : H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\partial\Omega).$$

Now here is a question: Do we have

$$\gamma_\mu : E(\Omega) \hookrightarrow H^{-\frac{1}{2}}(\partial\Omega)?$$

Indeed,

$$\operatorname{Ker}\gamma_0 = H_0^1(\Omega), \quad \operatorname{Im}\gamma_0 = H^{\frac{1}{2}}(\partial\Omega)$$

and

$$H^{-\frac{1}{2}}(\partial\Omega) = (H^{\frac{1}{2}}(\partial\Omega))^*.$$

For any $u \in C_c^\infty(\overline{\Omega})$, define

$$\gamma_\nu u = u \cdot \nu.$$

Then Stokes' formula holds in $E(\Omega)$.

**Proposition 3.3.** *For $u \in E(\Omega)$, $w \in H^1(\Omega)$,*

$$\langle u, \nabla w\rangle + \langle \operatorname{div} u, w\rangle = \langle \gamma_\nu u, \gamma_0 w\rangle$$

*Proof.* Let $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ and let $w \in H^1(\Omega)$ such that $\gamma_0 w = \phi$. For $u \in E(\Omega)$, define

$$X_u(\phi) = \int_\Omega [\langle u, \nabla w\rangle + \langle \operatorname{div} u, w\rangle].$$

Then $X_u(\phi)$ is well defined. Let $\widetilde{w} \in H^1(\Omega)$ be such that $\gamma_0\widetilde{w} = \phi$. Now need to show

$$\int_\Omega [\langle u, \nabla w\rangle + \langle \operatorname{div} u, w\rangle] = \int_\Omega [\langle u, \nabla \widetilde{w}\rangle + \langle \operatorname{div} u, \widetilde{w}\rangle].$$

Since

$$\gamma_0(w - \widetilde{w}) = 0,$$

it follows that there exists a sequence $w_k \in H_0^1(\Omega)$ such that $w - \widetilde{w} = \lim_{k\to\infty} w_k$. Then

$$\int_\Omega [\langle u, \nabla w - \widetilde{w}\rangle + \langle \operatorname{div} u, w - \widetilde{w}\rangle] = \lim_{k\to\infty} \int_\Omega [\langle u, \nabla w_k\rangle + \langle \operatorname{div} u, w_k\rangle] = \lim_{k\to\infty} \int_\Omega \operatorname{div}\langle u, w_k\rangle = 0.$$

□

Since

$$|X_u(\phi)| \le \|u\|_{E(\Omega)}\|w\|_{H^1(\Omega)} \lesssim \|u\|_{E(\Omega)}\|\phi\|_{H^{\frac{1}{2}}(\partial\Omega)},$$

it follows that

$$\phi \to X_u(\phi)$$

is a linear continuous map. So there exists $g = g(u) \in H^{-\frac{1}{2}}(\partial\Omega)$ such that

$$X_u(\phi) = \langle g, \phi\rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}}.$$

Hence

$$u \to g(u) = \gamma_\nu u$$

is linear, and

$$\|g(u)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \lesssim \|u\|_{E(\Omega)}.$$

By Stokes' formula, we have

$$\gamma_\nu u = u \cdot \nu, \quad \text{if } u \in C_c^\infty(\overline{\Omega}).$$

If $\partial\Omega \in C^2$, then the map

$$\gamma_\nu : E(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is onto.

$$\operatorname{Ker}\gamma_\nu = E_0(\Omega) = \overline{C_c^\infty(\Omega)}_{E(\Omega)}.$$

For any $\phi \in H^{-\frac{1}{2}}(\partial\Omega)$, let

$$\psi = \phi - \frac{\langle \phi, 1\rangle}{|\partial\Omega|},$$

then $\langle \psi, 1\rangle = 0$. Recalling $\gamma_0$ is onto, it follows that there exists $p \in H^1(\Omega)$ such that

$$\begin{cases} \Delta p = 0, & \text{in } \Omega, \\ \dfrac{\partial p}{\partial \nu} = \psi, & \text{on } \partial\Omega. \end{cases}$$

Let $u = \nabla p$, then $u \in E(\Omega)$, $\gamma_\nu u = \psi$. Hence

$$\phi = \gamma_\nu u + \frac{\langle \phi, 1\rangle}{|\partial\Omega|}$$

there exists $u_0 \in H^1(\Omega)$ such that $\gamma_\nu u_0 = 1$.

Denote $\mathcal{D}'(\Omega)$ as the space of distribution. Then for $f \in \mathcal{D}'(\Omega)$, if $f = \nabla p$ for some $p \in \mathcal{D}'(\Omega)$ if and only if $\langle f, v\rangle = 0$ for any $v \in \mathcal{V}$, where

$$\mathcal{V} = \left\{ v \in C_c^\infty(\Omega) \,\middle|\, \mathrm{div} v = 0 \right\}.$$

Denote

$$H = \left\{ u \in L^2(\Omega) \,\middle|\, \mathrm{div}\, u = 0,\ \gamma_\nu u = 0 \right\},$$

then the orthogonal component of $H$ in $L^2(\Omega)$,

$$H^\perp = \left\{ u \in L^2(\Omega) \,\middle|\, u = \nabla p,\ p \in H^1(\Omega) \right\}$$

Next, we consider the variational formulation of Stokes equation (3.5). Let $f \in L^2(\Omega)$ and $p \in L^2(\Omega)$. Then for any $v \in \mathcal{V}$, we have

$$\mu\langle \nabla u, \nabla v\rangle + \langle \nabla p, v\rangle = \langle f, v\rangle.$$

Denote

$$((u, v)) = \langle \nabla u, \nabla v\rangle_{L^2}.$$

Then for $u \in V$ satisfies

$$\mu((u, v)) = (f, v), \quad \forall\, v \in \mathcal{V}.$$

Here is a fact: $u \in V$ solves (3.5) if and only if

$$\mu((u, v)) = (f, v), \quad \forall\, v \in \mathcal{V}.$$

**Theorem 3.4.** *Assume that $\Omega \subset \mathbb{R}^n$ is bounded Lipschitz. Then for any $f \in H^{-1}(\Omega)$, there exists a unique solution $u \in V = H_0^1 \cap \{\mathrm{div}\, u = 0\}$ of (3.5).*

*Proof.* Method 1. (Lax-Milgram) Since $\|u\|_V = \|\nabla u\|_{L^2}$, define

$$a(u, v) = \mu((u, v)), \quad \forall\, u, v \in V,$$

then $a$ is a bounded bilinear form, and

$$a(u, u) = \mu((u, v)) \geq \mu\|u\|_V^2,$$

that is, $a$ is coercive. Hence by Lax-Milgram theorem, for any $f \in L^2$, there exists a unique $u \in V$ such that

$$a(u, v) = (f, v).$$

Method 2. (Garlekin's method) Let $\{w_m\}$ be an complete orthogonal base of $V$. Let

$$V_m = \mathrm{span}\{w_1, \cdots, w_m\}, \quad m \geq 1,$$

and

$$u_m = \sum_{i=1}^{m} \xi_i^m w_i \in V_m$$

solves

$$a(u_m, v) = (f, v), \quad \forall\, v \in V_m.$$

Then

$$\sum_{i=1}^{m} \xi_i^m a(w_i, w_j) = \langle f, w_j \rangle$$

$$\begin{pmatrix} a(w_1, w_1) & \cdots & a(w_m, w_1) \\ a(w_1, w_2) & \cdots & a(w_m, w_2) \\ \cdots & \cdots & \cdots \\ a(w_1, w_m) & \cdots & a(w_m, w_m) \end{pmatrix} \begin{pmatrix} \xi_1^m \\ \xi_2^m \\ \vdots \\ \xi_m^m \end{pmatrix} = \begin{pmatrix} \langle f, w_1 \rangle \\ \langle f, w_2 \rangle \\ \vdots \\ \langle f, w_m \rangle \end{pmatrix}$$

So

$$(a(w_i, w_j))_{1 \le i, j \le m}$$

is a nonsingular matrix. This implies that

$$\sum_{i=1}^{m} \xi_i^m a(w_i, w_j) = 0, \quad 1 \le j \le m$$

has only trivial solution. Hence, by

$$a\left( \sum_{i=1}^{m} \xi_i^m w_i, (\sum_{i=1}^{m} \xi_i^m w_i \right) = 0,$$

we have

$$(\sum_{i=1}^{m} \xi_i^m w_i = 0,$$

that is,

$$(\xi_1, \cdots, \xi_m) = (0, \cdots, 0).$$

On the other hand, from

$$a(u_m, u_m) = \langle f, u_m \rangle$$

it follows that

$$\|u_m\|_V^2 \lesssim \frac{1}{\mu} \|f\|_{L^2} \|u_m\|_V,$$

that is,

$$\|u_m\|_V \lesssim \frac{1}{\mu} \|f\|_{L^2}.$$

So there exists $u \in V$ such that

$$u_m \rightharpoonup u \quad \text{in } V.$$

Hence

$$a(u, v) = (f, v) \quad \forall\, v \in V_m.$$

Therefore,

$$a(u, v) = (f, v).$$

Uniqueness: If there are two solutions $u$ and $\bar{u}$ such that

$$\begin{cases} a(u,v) = (f,v), \\ a(\bar{u},v) = (f,v), \end{cases}$$

then

$$a(u-\bar{u},v) = 0.$$

Especially,

$$a(u-\bar{u},u-\bar{u}) = 0.$$

So $u = \bar{u}$. □

**Minimization principle** Let

$$E(u) = \mu\|u\|^2 - 2(f,u).$$

Then

**Theorem 3.5.** *$u \in V$ solves* (3.5) *if and only if*

$$E(u) \leq E(\tilde{u}), \quad \forall\, \tilde{u} \in V.$$

*Proof.* ($\Leftarrow$) For any $v \in V$,

$$\left.\frac{d}{dt}\right|_{t=0} E(u+tv) = 0,$$

then

$$2\mu(\nabla u, \nabla v) - 2(f,v) = 0.$$

($\Rightarrow$) If

$$\mu((u,\widetilde{v})) = (f,\widetilde{v}), \quad \forall \widetilde{v} \in V.$$

then for $v \in V$, letting $\widetilde{v} = u - v$, we have

$$\mu((u,u-v)) = (f,u-v).$$

That is,

$$\mu((u,u)) - \mu((u,v)) = (f,u) - (f,v).$$

Then

$$\mu\|u\|^2 \leq \frac{\mu}{2}\|u\|^2 + \frac{\mu}{2}\|v\|^2 + (f,u) - (f,v),$$

that is,

$$\frac{1}{2}E(u) \leq \frac{1}{2}E(v).$$

□

## 3.3 Nonhomogeneous Stokes problem

**Theorem 3.6.** *Let $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega \in C^2$. Let $f \in H^{-1}(\Omega)$, $g \in L^2(\Omega)$, $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ such that $\int_\Omega g = \int_{\partial\Omega} \phi \cdot \nu$. Then there exists a unique $u \in H^1(\Omega)$, $p \in L^2(\Omega)$ (unique up to a constant) such that*

$$\begin{cases} -\mu\Delta\, u + \nabla\, p = f, & in\ \Omega, \quad \mu > 0, \\ \nabla \cdot\ u = g, & in\ \Omega, \\ \gamma_0 u = \phi, & on\ \partial\Omega. \end{cases} \tag{3.6}$$

*Proof.* (Uniqueness) Suppose that there exist $u_1, u_2 \in H^1(\Omega)$, $p_1, p_2 \in L^2(\Omega)$ such that

$$\begin{cases} -\mu\Delta\, u_i + \nabla\, p_i = f, & \text{in } \Omega, \quad \mu > 0, \\ \nabla \cdot\ u_i = g, & \text{in } \Omega, \\ \gamma_0 u_i = \phi, & \text{on } \partial\Omega, \quad i = 1, 2. \end{cases}$$

Let $w = u_1 - u_2$, $p = p_1 = p_2$, then

$$\begin{cases} -\mu\Delta\, w + \nabla\, p = 0, & \text{in } \Omega, \quad \mu > 0, \\ \nabla \cdot\ w = 0, & \text{in } \Omega, \\ \gamma_0 w = 0, & \text{on } \partial\Omega. \end{cases}$$

So that, by

$$\mu(\nabla\, w, \nabla\, w) = 0,$$

we have $w =$ constant. Further by $\gamma_0 w = 0$ on $\partial\Omega$, we have $w = 0$ in $\Omega$. By $\nabla\, p = 0$, we obtain $p_1 - p_2 =$ const..
(Existence). Let $u_0 \in H^1(\Omega)$ such that $\gamma_0 u_0 = \phi$, then

$$\int_\Omega (\operatorname{div} u_0 - g) = 0.$$

Hence there exists $u_1 \in H_0^1(\Omega)$ such that

$$\operatorname{div} u_1 = -\operatorname{div} u_0 + g.$$

Let $v = u - u_0 - u_1$, then

$$\begin{cases} -\mu\Delta\, v + \nabla\, p = f - \mu\Delta(u_0 + u_1) \in H^{-1}, & \text{in } \Omega, \quad \mu > 0, \\ \nabla \cdot\ v = 0, & \text{in } \Omega, \\ \gamma_0 v = 0, & \text{on } \partial\Omega \end{cases}$$

has a unique solution $v$ and $p$. Hence the original problem is also solvable. □

**Lemma 3.7.** div : $H_0^1(\Omega) \to L^2(\Omega)/\mathbb{R} = \{g \in L^2(\Omega) \mid \int_\Omega g = 0\}$ *is an onto map.*

*Proof.* $\nabla$ : $L^2(\Omega) \cap \{\int_\Omega g = 0\} \to H^{-1}(\Omega)$ is isomorphism onto its range $R(\nabla)$. Hence $A^* = -\operatorname{div} \in \mathcal{L}(H_0^1(\Omega), L^2(\Omega))$ is onto $L^2(\Omega)/\mathbb{R}$. □

For the regularity of the weak solutions, we have

**Theorem 3.8.** *Let $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega \in C^\gamma$, $\gamma = \max\{2, m+2\}$, $m \geq 0$. Let $u \in W^{2,q}$, $p \in W^{1,q}$, $1 < q < +\infty$, solves* (3.6). *If $f \in W^{m,q}$, $g \in W^{m+1,q}$, $\phi \in W^{m+2-\frac{1}{q},q}(\partial\Omega)$, then $u \in W^{m+2,q}$, $p \in W^{m+1,q}$ and*

$$\|u\|_{W^{m+2,q}} + \|p\|_{W^{m+1,q}/\mathbb{R}} \leq C(q,\gamma,m,\Omega)\left(\|f\|_{W^{m,q}} + \|g\|_{W^{m+1,q}} + \|\phi\|_{W^{m+2-\frac{1}{q},q}} + c_q\|u\|_{L^q}\right),$$

*where*

$$c_q = \begin{cases} 0, & q \geq 2, \\ 1, & 1 < q < 2. \end{cases}$$

**Theorem 3.9.** *(Existence) ($n = 2, 3$) Under the same assumption on $f, g, \phi$ and $\int_\Omega g = \int_{\partial\Omega} \phi \cdot \nu$. Then there exist unique $u \in W^{m+2,q}$, $p \in W^{m+1,q}$ solving the system and satisfying the above estimates.*

*Proof.* We will only present the proof for simply connected domain in $\mathbb{R}^2$. First we claim that there exists $v \in W^{m+1,q}(\Omega)$ such that

$$\begin{cases} \operatorname{div} v = g & \text{in } \Omega \\ v = \phi & \text{on } \partial\Omega. \end{cases}$$

To see it, let $\theta \in W^{m+3,q}(\Omega)$ such that

$$\begin{cases} \Delta\theta = g & \text{in } \Omega \\ \frac{\partial\theta}{\partial\nu} = \phi \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Write $v = \nabla\theta + w$. Then $w$ satisfies

$$\operatorname{div} w = 0 \text{ in } \Omega;\ w \cdot \nu = 0 \text{ on } \partial\Omega.$$

Hence we may write $w = (\frac{\partial\sigma}{\partial x_2}, -\frac{\partial\sigma}{\partial x_1})$ for an unknown function $\sigma$. The boundary condition on $w$ yields that $\sigma$ satisfies

$$w \cdot \nu = \frac{\partial\sigma}{\partial x_2}\nu_2 - \frac{\partial\sigma}{\partial x_1}\nu_1 = \nabla_{\tan}\sigma = 0 \text{ on } \partial\Omega,$$

and

$$w \cdot \tau = \frac{\partial\sigma}{\partial\nu} = (v - \nabla\theta) \cdot \tau = \phi \cdot \tau - \frac{\partial\theta}{\partial\tau} \in W^{m+2-\frac{1}{q},q}(\partial\Omega).$$

The existence of $\sigma$ is guaranteed by the following biharmonic equation: there exists $\sigma \in W^{m+3,q}(\Omega)$ that solves

$$\begin{cases} \Delta^2\sigma = 0 & \text{in } \Omega \\ \sigma = 0 & \text{in } \Omega \\ \frac{\partial\sigma}{\partial\nu} = \phi \cdot \tau - \frac{\partial\theta}{\partial\tau} \in W^{m+2-\frac{1}{q},q}(\partial\Omega). \end{cases}$$

With the help of $v$, we can consider $w = u - v$. Then $u$ solves the original equation if and only if $w$ solves

$$\begin{cases} -\mu\Delta w + \nabla p = f' \equiv f + \mu\Delta v \in W^{m,q}(\Omega) & \text{in } \Omega \\ \operatorname{div} w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The solvability of $w$ can be done by solving another biharmonic equation as follows: since we can write $w = (\frac{\partial\rho}{\partial x_2}, -\frac{\partial\rho}{\partial x_1})$ for some unknown function $\rho$ in $\Omega$. $w = 0$ on $\partial\Omega$ yields that $\rho = \frac{\partial\rho}{\partial\nu} = 0$ on $\partial\Omega$. The equation of $w$ yields an equation for $\rho$:

$$-\mu\Delta\rho_{x_2} + p_{x_1} = f^{',1} \tag{3.7}$$

$$\mu\Delta\rho_{x_1} + p_{x_2} = f^{',2}. \tag{3.8}$$

Taking $\frac{\partial}{\partial x_2}$ of the first equation and $\frac{\partial}{\partial x_1}$ of the second equation and then subtracting the two resulting equations, we would obtain

$$-\mu\Delta^2\rho = \mathrm{curl(f')} \text{ in } \Omega, \ \rho = \frac{\partial\rho}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{3.9}$$

Since $\mathrm{curl}(f') \in W^{m-1,q}(\Omega)$, it follows from the linear theory that there exists $\rho \in W^{m+3,q}(\Omega)$. This implies the equation for $w$ is solvable for $w \in W^{m+2,q}(\Omega)$. The proof is now complete. □

Lecture 4, December 24, 2012

# 4 The Steady Navier-Stokes equation

## 4.1 Eigenvalues and eigenfunctions of the Stokes operator

.

Consider

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

From Lecture 3, we know that for any $f \in L^2(\Omega)$, there exists a unique $u \in V$ solving the equation (4.1). Define

$$\Lambda(f) = \frac{1}{\mu} u : L^2(\Omega, \mathbb{R}^n) \to H_0^1(\Omega, \mathbb{R}^n) \subset L^2(\Omega, \mathbb{R}^n).$$

Then $\Lambda : L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)$ is compact. $\Lambda$ is also self-adjoint:

$$(\Lambda f_1, f_2)_{L^2} = (f_2, \Lambda f_1)_{L^2}.$$

Therefore there exist $0 < \lambda_1 < \lambda_2 \le \cdots \lambda_j \uparrow +\infty$ and $0 \ne w_i \in V$ such that

$$\Lambda w_i = \lambda_i w_i, \ \forall i \ge 1,$$

and

$$(w_i, w_j)_{L^2} = \delta_{ij}, \ (w_i, w_j)_V = \lambda_i \delta_{ij}.$$

There also exist $p_i \in L^2(\Omega)$ such that

$$\begin{cases} -\mu\Delta w_i + \nabla p_i = \lambda_i w_i & \text{in } \Omega \\ \nabla \cdot w_i = 0 & \text{in } \Omega \\ w_i = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

By the regularity theory of Stokes' equation from Lecture 3, we have

$$\Omega \in C^m \Rightarrow w_i \in H^m(\Omega), \ p_i \in H^{m-1}(\Omega),$$

and

$$\Omega \in C^\infty \Rightarrow w_i \in C^\infty(\overline{\Omega}), \ p_i \in C^\infty(\overline{\Omega}).$$

## 4.2 Steady Navier-Stokes equation

For $f \in L^2(\Omega, \mathbb{R}^n)$, a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, seek $u : \Omega \to \mathbb{R}^n, \ p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\mu\Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3}$$

**<u>Weak formulation of</u>** (4.3): Find $u \in V$ such that

$$\mu(u, v)_V + B[u, u, v] = (f, v)_{L^2}, \ \forall v \in \mathcal{V}, \tag{4.4}$$

where $B$ is the trilinear form defined by

$$B[u, v, w] = \int_\Omega u \cdot \nabla v \cdot w, \ u, v \in V, w \in \mathcal{V}.$$

*Remark* 4.1. For $n \leq 4$, $B : V \times V \times V \to \mathbb{R}$ is a well-defined trilinear form. For $n \geq 5$, $B : V \times V \times (V \cap L^n(\Omega)) \to \mathbb{R}$ is well-defined.

To see it, recall by the Sobolev embedding inequality we have

$$H_0^1(\Omega) \subset \begin{cases} L^{\frac{2n}{n-2}}(\Omega) & n \geq 3 \\ L^p(\Omega) \ \forall \ p < +\infty & n = 2. \end{cases}$$

By Hölder's inequality, we have

$$\begin{aligned} \Big| \int_\Omega u \cdot \nabla v \cdot w \Big| \ &\leq \ \begin{cases} ||u||_{L^4(\Omega)} ||\nabla v||_{L^2(\Omega)} ||w||_{L^4(\Omega)} & n \leq 4 \\ ||u||_{L^{\frac{2n}{n-2}}(\Omega)} ||\nabla v||_{L^2(\Omega)} ||w||_{L^n(\Omega)} & n \geq 5 \end{cases} \\ &\leq \ \begin{cases} C||u||_{H_0^1(\Omega)} ||\nabla v||_{L^2(\Omega)} ||w||_{H_0^1(\Omega)} & n \leq 4 \\ ||u||_{H_0^1(\Omega)} ||\nabla v||_{L^2(\Omega)} ||w||_{L^n(\Omega)} & n \geq 5 \end{cases} \end{aligned}$$

From this discussion, we have obtained

**Lemma 4.1.** *$B : V \times V \times (V \cap L^n(\Omega)) \to \mathbb{R}$ is continuous.*

Define $\widetilde{V}$ = closureof$\mathcal{V}$ in $H_0^1 \cap L^n(\Omega)$, with the norm

$$||v||_{\widetilde{V}} = ||v||_{H_0^1(\Omega)} + ||v||_{L^n(\Omega)}.$$

Then we have

**Lemma 4.2.** *(i) For $n \leq 4$, $B : V \times V \times V \to \mathbb{R}$ is a continuous, trilinear operator.*
*(ii) For $n \geq 5$, $B : V \times V \times \widetilde{V} \to \mathbb{R}$ is a continuous, trilinear operator.*

For the trilinear form $B$, we have

**Lemma 4.3.** *For $u \in V, v \in \widetilde{V}$, it holds $B[u, v, v] = 0$. In particular, for $u \in V, v, w \in \widetilde{V}$, $B[u, v, w] = -B[u, w, v]$.*

*Proof.* Assume $u, v \in C_0^\infty(\Omega)$ and $\text{div} u = 0$. Then

$$\int_\Omega u \cdot \nabla v \cdot v = \int_\Omega u \cdot \nabla (\frac{|v|^2}{2}) = -\int_\Omega (\nabla \cdot u) \frac{|v|^2}{2} = 0.$$

Now by the density argument, we see that $B[u, v, v] = 0$ for all $u \in V$ and $v \in \widetilde{V}$.

Since $B[u, v + w, v + w] = 0$, it follows that

$$B[u, v, v] + B[u, w, w] + B[u, v, w] + B[u, w, v] = 0.$$

Hence $B[u, v, w] + B[u, w, v] = 0$. □

For $u, v \in W$, we also define the bilinear form $B[u, v]$ by

$$\langle B[u, v], w\rangle = B[u, v, w],\ \forall w \in \widetilde{V}.$$

**Theorem 4.4.** *For any $f \in L^2(\Omega)(or H^{-1}(\Omega))$, there exists at least one solution $u \in V$ and $p \in L^1_{\mathrm{loc}}(\Omega)$ of the steady Navier-Stokes equation (4.4).*

*Proof.* (Galerkin's method): Let $\{w_i\}_{i=1}^{\infty}$ be a complete orthogonal base of $V$ formed by the eigenfunctions of the Stokes operator. Let $V_m = \mathrm{span}\{w_1, \cdots, w_m\}$, $m \geq 1$. Let $u_m = \sum_{i=1}^m \xi_i^m w_i$, $\xi_i^m \in \mathbb{R}$, solve

$$\mu(u_m, w_i)_V + B[u_m, u_m, w_i] = (f, w_i)_{L^2},\ i = 1, \cdots, m. \tag{4.5}$$

In terms of $(\xi_i^m)$, this becomes

$$\xi_k^m + A_{ijk}\xi_i^m\xi_j^m = c_k,\ k = 1, \cdots, m, \tag{4.6}$$

where

$$A_{ijk} = B[w_i, w_j, w_k],\ c_k = (f, w_k)_{L^2}.$$

We will need to apply the fixed point lemma below to find a solution of (4.6). To do it, set $X = V_m$ and define the inner product $[u, v]_X = (u, v)_V$ and the induced norm $|u|_X = \sqrt{[u, u]}$. Define $P : X \to X$ by

$$[P(u), v]_X = \mu(u, v)_V + B[u, u, v] - (f, v),\ u, v \in X.$$

Then we have

$$\begin{aligned} [P(u), u]_X &= \mu(u, u)_V + B[u, u, u] - (f, u) \\ &= \mu(u, u)_V - (f, u) \\ &\geq \mu|u|_X^2 - \|f\|_{L^2}|u|_X \\ &\geq |u|_X(\mu|u|_X - \|f\|_{L^2}), \end{aligned}$$

so that if we choose $r > 0$ such that $\mu r - \|f\|_{L^2} > 0$, then

$$[P(u), u]_X > 0,\ \forall u \in X \text{ with } |u|_X = r.$$

Hence by lemma 4.5, there exists $u_m \in X$ such that $P(u_m) = 0$. Furthermore, we have the estimate

$$\mu|u_m|_X - \|f\|_{L^2} \leq 0$$

or

$$|u_m|_X \leq \frac{1}{\mu}\|f\|_{L^2}. \tag{4.7}$$

We may assume that $u_m \to u$ weakly in $V$ and $u_m \to u$ strongly in $L^2(\Omega)$. We need to verify that $u$ satisfies (4.4). It is easy to see that for any $m_0 \geq 1$ fixed,

$$\mu(u_m, v)_V \to \mu(u, v)_V,\ \forall v \in V_{m_0}.$$

For $v \in V_{m_0}$,

$$B[u_m, u_m, v] = -B[u_m, v, u_m] = -\int_\Omega u_m \cdot \nabla v \cdot u_m$$
$$\to -\int_\Omega u \cdot \nabla v \cdot u = -B[u, v, u] = B[u, u, v].$$

Therefore we have

$$\mu(u, v)_V + B[u, u, v] = (f, v), \ \forall v \in V_{m_0}.$$

Since $\cup_{m_0 \geq 1} V_{m_0} = \mathcal{V}$, (4.4) holds. □

**Lemma 4.5.** *Let $X$ be a finite dimensional Hilbert space with inner product $[\cdot, \cdot]$ and norm $|\cdot|$. Let $P : X \to X$ be a continuous map and satisfy*

$$[P(\xi), \xi] > 0, \ \forall |\xi| = k > 0.$$

*Then there exists a $\xi \in X$, with $|\xi| \leq k$, such that $P(\xi) = 0$.*

*Proof.* Suppose that the conclusion were false, Then $P(\xi) \neq 0$ for any $|\xi| \leq k$. Define a continuous map $\Phi : B_k \to B_k$ by letting

$$\Phi(\xi) = -k \frac{P(\xi)}{|P(\xi)|}.$$

Hence by the Browder fixed point theorem, there exists a $\xi_0 \in B_k$ such that $\Phi(\xi_0) = \xi_0$. However,

$$0 \leq |\xi_0|^2 = [\xi_0, \Phi(\xi_0)] = [\xi_0, -k \frac{P(\xi_0)}{|P(\xi_0)|}] = -k \frac{[P(\xi_0), \xi_0]}{|P(\xi_0)|} < 0.$$

This is impossible. The proof is complete. □

For the uniqueness of steady Navier-Stokes equations, we have the following

**Theorem 4.6.** *For $n \leq 4$, if $\mu > 0$ satisfies*

$$\mu^2 \geq c(n) \|f\|_{L^2(\Omega)},$$

*then there exists a unique solution $u$ of (4.4).*

*Proof.* Assume that $u_1$ is the solution constructed by the above theorem so that it satisfies

$$\|u_1\|_V \leq \frac{1}{\mu} \|f\|_{L^2(\Omega)}.$$

Let $u_2$ be an arbitrary solution of (4.4). Define $w = u_1 - u_2$. Then, since $n \leq 4$, we have

$$\mu(w, v)_V + B[u_1, u_1, v] - B[u_2, u_2, v] = 0, \ \forall v \in V.$$

Notice that

$$B[u_1, u_1, v] - B[u_2, u_2, v] = B[u_2, w, v] + B[w, u_1, v].$$

Hence by substituting $v = w$, we obtain

$$\mu(w, w)_V + B[u_2, w, w] + B[w, u_1, w] = 0,$$

which implies

$$\mu\|w\|_V^2 = -B[w,u_1,w] \le c(n)\|w\|_V^2\|\nabla u_1\|_{L^2} \le c(n)\frac{\|f\|_{L^2(\Omega)}}{\mu}\|w\|_V^2.$$

Hence

$$\left(\mu - \frac{c(n)\|f\|_{L^2(\Omega)}}{\mu}\right)\|w\|_V^2 \le 0.$$

Thus $\|w\|_V = 0$ and hence $u_1 \equiv u_2$. □

## 4.3 Regularity in dimensions $n \le 4$

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**Theorem 4.7.** *For $n = 2,3$, any weak solution $u \in V$ of (4.3) is smooth in $\overline{\Omega}$, provided that $f, \partial\Omega \in C^\infty$.*

*Proof.* i) $n = 2$: $u \in V$ implies that $u \in L^q$ for all $q < +\infty$. Hence $u\cdot\nabla u = \nabla\cdot(u\otimes u) \in W^{-1,q}$. Therefore, by the regularity of Stokes equations, we have that $u \in W^{1,q}(\Omega)$ and $p \in L^q(\Omega)$. This in turn implies $u \cdot \nabla u \in L^q$ and hence $u \in W^{2,q}(\Omega)$ and $p \in W^{1,q}(\Omega)$. Repeating this argument eventually yields $u, p \in C^\infty(\overline{\Omega})$.
ii) $n = 3$: $u \in L^6$ so that $u \cdot \nabla u = \nabla \cdot (u \otimes u) \in W^{-1,3}(\Omega)$. Thus $u \in W^{1,3}(\Omega)$. By Sobolev's embedding, this implies $u \in L^q(\Omega)$ for any $q < +\infty$. Now we can repeat the same argument as in the case $n = 2$. □

*Remark* 4.2. For $n = 4$, the solution is still smooth. But the proof requires a different argument. Since in this case $u \in L^4(\Omega)$ and hence $u \cdot \nabla u = \nabla \cdot (u \otimes u) \in W^{-1,2}(\Omega)$. Hence the regularity theory of Stokes equation implies $u \in H^1(\Omega)$ so that there is no improvement. However, the size does get an improvement:

$$\|\nabla u\|_{L^2(\Omega)} \lesssim \|u \otimes u\|_{L^2(\Omega)} \lesssim \|u\|_{L^4(\Omega)}^2 \lesssim \|\nabla u\|_{L^2(\Omega)}^2.$$

It turns out that this observation, after suitable localization, can imply the regularity.

## 4.4 The time-dependent Navier-Stokes equation

For $f \in L^2(\Omega \times [0,T])$ and $u_0 \in H$, consider the Navier-Stokes equation:

$$\begin{cases} u_t + u\cdot\nabla u - \mu\Delta u + \nabla p = f & \text{in } \Omega\times(0,T) \\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{4.8}$$

For (4.8), we have the following existence theorem, due to E. Hopf and J. Leray. Denote $Q_T = \Omega \times [0,T]$. Then we have

**Theorem 4.8.** *For any $T > 0$, there exists at least one weak solution $u \in L_t^\infty L_x^2(Q_T) \cap L^2H^1(Q_T)$ of (4.8) that satisfies the energy inequality: for any $0 < t \le T$,*

$$\int_\Omega |u(t)|^2 + 2\mu\int_0^t\int_\Omega |\nabla u|^2 \le \int_\Omega |u_0|^2 + 2\int_0^t\int_\Omega (f,u). \tag{4.9}$$

*Proof.* (Galerkin's method): As in the steady case, let $V_m = \text{span}\{\mathrm{w_1}, \cdots, \mathrm{w_m}\}$, where $\{w_i\}$ is the family of eigenfunctions of the Stokes operator, which forms a complete base of $V$. Look for $v : [0, T] \to V_m$ such that

$$\int_\Omega (u_t v + u \cdot \nabla u v + \mu \nabla u \nabla v - f v) = 0, \ \forall v \in V_m, \forall t \in (0, T).$$

Write $u_m(x, t) = \sum_{i=1}^{m} \xi_i^m(t) w_i(x)$. Then we have

$$\dot{\xi_i^m} = -\mu a_{ij}\xi_j^m + b_{jki}\xi_j^m \xi_k^m + c_i, \ \xi_i^m(0) = \langle u_0, w_i \rangle, \tag{4.10}$$

where

$$a_{ij} = (\nabla w_i, \nabla w_j)_{L^2}, \ b_{jki} = B[w_j, w_k, w_i], \ c_i = (f, w_i)_{L^2}.$$

Observe that

$$a_{ij}\eta_i\eta_j = (\nabla(\eta_i w_i), \nabla(\eta_j w_j))_{L^2} = \sum_{i=1}^{m} \lambda_i \eta_i^2 \geq \lambda_1 |\eta|^2,$$

so that $(a_{ij})$ is a positive-definite matrix. Also notice that $(b_{jki})$ is skew-symmetric in the last two indices:

$$b_{jki} = -b_{jik}.$$

Notice that (4.10) is locally uniquely solvable: there exists $T_0 > 0$ and a unique solution $\xi^m = (x_1^m, \cdots, x_m^m)^t : [0, T_0] \to \mathbb{R}^m$ to the ODE (4.10).

Now we want to derive a priori energy estimate. Multiplying $(4.10)_1$ by $\xi_i^m$ and summing over $1 \leq i \leq m$, we obtain

$$\begin{aligned} \frac{d}{dt}\Big(\sum_{i=1}^{m}(\xi_i^m)^2\Big) &\leq -2\lambda_1\Big[\sum_{i=1}^{m}(\xi_i^m)^2\Big] + c(t)\Big|\sum_{i=1}^{m}(\xi_i^m)^2\Big|^{\frac{1}{2}} \\ &\leq -\lambda_1\Big[\sum_{i=1}^{m}(\xi_i^m)^2\Big] + \frac{|c(t)|^2}{4\lambda_1}. \end{aligned}$$

Here

$$|c(t)| = \|f(t)\|_{L^2(\Omega)} \in L^2([0, T]).$$

Therefore we obtain

$$\frac{d}{dt}\left(e^{\lambda_1 t}|\xi^m|^2\right) \leq e^{\lambda_1 t}\frac{|c(t)|^2}{4\lambda_1}$$

so that

$$|\xi^m(t)|^2 \leq |\xi^m(0)|^2 e^{-\lambda_1 t} + \int_0^t e^{\lambda_1(s-t)}\frac{|c(s)|^2}{4\lambda_1}\, ds. \tag{4.11}$$

It follows from the energy estimate (4.11) that the solution $\xi^m$ can be extended to $[0, T]$. Moreover, the estimate on $\xi^m$ translates into estimates of $u_m$:

$$\frac{d}{dt}\int_\Omega |u_m|^2 + 2\mu\int_\Omega |\nabla u_m|^2 = 2(f, u_m). \tag{4.12}$$

By Hölder's inequality, this implies that

$$\frac{d}{dt}\int_\Omega |u_m|^2 + \mu\int_\Omega |\nabla u_m|^2 \leq \frac{C}{\mu}\int_\Omega |f|^2. \tag{4.13}$$

After integrating over $[0, T]$, we have achieved

$$\sup_{0 \le t \le T} \|u_m(t)\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega |\nabla u_m|^2 \le C\left(\|f\|_{L^2(Q_T)}, \|u_0\|_{L^2(\Omega)}\right). \tag{4.14}$$

**Goal**: To show that, up to possible subsequences, $u_m$ converges weakly to some function $u$ in suitable spaces, which solves the Navier-Stokes equation in the weak sense. □

# 5 The Galerkin method for the Navier-Stokes equation

From Lecture 4, we have that

$$u^m(x,t) = \sum_{i=1}^{m} \xi_i^m(t) w_i(x)$$

solves

$$\begin{cases} \partial_t u^m + u^m \cdot \nabla u^m - \mu \Delta u^m + \nabla p^m = f^m \\ \nabla \cdot u^m = 0 \\ u^m \Big|_{t=0} = u_0^m \\ u^m \Big|_{\partial\Omega} = 0 \end{cases} \tag{5.1}$$

where

$$f^m = \sum_{i=1}^{m} (f, w_i)_{L^2} w_i, \quad u_0^m = \sum_{i=1}^{m} (u_0, w_i)_{L^2} w_i.$$

Note that the equation (5.1) should be understood as the follows: for any $\eta \in C^\infty([0,T])$ and $v(x) \in V^m$, if we set $V(x,t) = v(x)\eta(t)$, then for any $[t_1, t_2] \subset [0,T]$ it holds

$$\int_\Omega u^m V \Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \int_\Omega \Big[ -u^m V_t - u^m \otimes u^m : \nabla V + \mu \nabla u^m \cdot \nabla V - fV \Big] \, dxdt = 0. \tag{5.2}$$

The following energy bound also holds:

$$\sup_{0 \le t \le T} \int_\Omega |u^m|^2 \, dx + \mu \int_0^T \int_\Omega |\nabla u^m|^2 \, dxdt \le C\Big( \|f\|_{L^2(\Omega \times [0,T])}, \ \|u_0\|_{L^2(\Omega)} \Big). \tag{5.3}$$

Hence $\{u^m\} \subset L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$ is a bounded sequence. We may assume, after passing to a subsequence, that

$$u^m \to u \text{ weak}^* \text{ in } L_t^\infty L_x^2(Q_T); \ \ u^m \to u \text{ weakly in } L_t^2 H_x^1(Q_T)$$

for some $u \in L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$.

**Claim**. $u$ is a weak solution of the Navier-Stokes equation. This amounts to showing that for any $[t_1, t_2] \subset [0,T]$, it holds

$$\int_\Omega uV \Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \int_\Omega \Big[ -uV_t - u \otimes u : \nabla V + \mu \nabla u \cdot \nabla V - fV \Big] \, dxdt = 0 \tag{5.4}$$

for any $V = v(x)\eta(t)$, with $\eta \in C^\infty([0,T])$ and $v(x) \in V^m$.

There are two main difficulties that we encounter when taking the limit process, namely,

$$\int_\Omega u^m V \to \int_\Omega uV, \ \forall t \in [0,T]; \ \int_{t_1}^{t_2} \int_\Omega u^m \otimes u^m : \nabla V \to \int_{t_1}^{t_2} \int_\Omega u \otimes u : \nabla V \ ??$$

A key step to overcome these difficulties is to show that, after taking possible subsequences,

$$u^m \to u \text{ strongly in } L^2(Q_T). \tag{5.5}$$

First we recall the Sobolev-interpolation inequality.

**Lemma 5.1.** *For $n \geq 3$, assume that $u \in L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$. Then, for any $2 \leq q \leq 2^* \equiv \frac{2n}{n-2}$ and $p \geq 2$ satisfying*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2},$$

*we have $u \in L_t^p L_x^q(Q_T)$. Moreover, it holds*

$$\|u\|_{L_t^p L_x^q(Q_T)} \leq C\|u\|_{L_t^\infty L_x^2(Q_T)}^{1-\frac{2}{p}} \|u\|_{L_t^2 H_x^1(Q_T)}^{\frac{2}{p}}. \tag{5.6}$$

*Proof.* For $2 \leq q \leq 2^*$, by both the interpolation inequality and the Sobolev inequality we have

$$\|u\|_{L_x^q(\Omega)} \leq \|u\|_{L_x^2(\Omega)}^\alpha \|u\|_{L_x^{2^*}(\Omega)}^{1-\alpha} \leq C\|u\|_{L_x^2(\Omega)}^\alpha \|u\|_{H_x^1(\Omega)}^{1-\alpha},$$

where $0 \leq \alpha \leq 1$ satisfies $\frac{1}{q} = \frac{\alpha}{2} + \frac{1-\alpha}{2^*}$. Integrating over $t \in [0, T]$, we obtain

$$\begin{aligned} \int_0^T \|u\|_{L_x^q}^p \, dt &\leq C \int_0^T \|u\|_{L_x^2(\Omega)}^{p\alpha} \|u\|_{H_x^1(\Omega)}^{p(1-\alpha)} \, dt \\ &\leq C\|u\|_{L_t^\infty L_x^2(Q_T)}^{p\alpha} \int_0^T \|u\|_{H_x^1(\Omega)}^{p(1-\alpha)} \, dt \end{aligned}$$

Set $p(1-\alpha) = 2$. Then $1 - \alpha = \frac{2}{p}$ and $\alpha = 1 - \frac{2}{p}$. Hence $\frac{1}{q} = \frac{1-\frac{2}{p}}{2} + \frac{2}{2^* p}$ is equivalent to $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$. It is clear that (5.6) follows directly from this inequality. □

**Corollary 5.2.** *For $n = 3$, if $u \in L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$, then $u \in L^{\frac{10}{3}}(Q_T)$ and*

$$\|u\|_{L^{\frac{10}{3}}(Q_T)} \leq C\|u\|_{L_t^\infty L_x^2(Q_T)}^{\frac{2}{5}} \|u\|_{L_t^2 H_x^1(Q_T)}^{\frac{3}{5}}. \tag{5.7}$$

*Proof.* Set $p = q$ and $n = 3$ in the equality $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, one has $p = q = \frac{10}{3}$. Hence the conclusion follows directly from the lemma. □

Now we need to prove

**Claim**. For any $V = v(x)\eta(t)$ with $\eta \in C^\infty([0, T])$ and $v \in V^{m_0}$, $\int_\Omega u^m(t)V(x, t)\, dx : [0, T] \to \mathbb{R}$ is equicontinuous for all $m \geq m_0$.

In order to show this claim, for any $0 \leq t_1 < t_2 \leq T$ let's define

$$I_V^m(t_1, t_2) := \int_{t_1}^{t_2} \int_\Omega [-u^m V_t - u^m \otimes u^m : \nabla V + \mu \nabla u^m \cdot \nabla V - fV] \, dxdt.$$

Observe that it follows from the equation (5.2) that for any $m \geq m_0$,

$$\int_\Omega u^m V\Big|_{t=t_2} - \int_\Omega u^m V\Big|_{t=t_1} = -I_V^m(t_1, t_2). \tag{5.8}$$

Now we want to show that

$$\sup_{m\geq m_0} |I_V^m(t_1,t_2)| \leq C(m_0,T)|t_2-t_1|^{\frac{1}{4}}. \tag{5.9}$$

In fact,

$$|\int_{t_1}^{t_2}\int_\Omega u^m V_t| \lesssim \|V_t\|_{L^\infty(Q_T)}|\Omega|^{\frac{1}{2}}|t_2-t_1|^{\frac{1}{2}}\|u^m\|_{L_t^\infty L_x^2(Q_T)},$$

$$|\int_{t_1}^{t_2}\int_\Omega u^m\otimes u^m : \nabla V| \lesssim \|\nabla V\|_{L^\infty(Q_T)}\|u^m\|^2_{L_t^\infty L_x^2(Q_T)}|t_2-t_1|,$$

$$|\int_{t_1}^{t_2}\int_\Omega \nabla u^m\cdot\nabla V| \lesssim \|\nabla V\|_{L^\infty(Q_T)}|\Omega|^{\frac{1}{2}}|t_2-t_1|^{\frac{1}{2}}\|\nabla u^m\|_{L^2H^1(Q_T)},$$

$$|\int_{t_1}^{t_2}\int_\Omega fV| \lesssim \|V\|_{L^\infty(Q_T)}|\Omega|^{\frac{1}{2}}|t_2-t_1|.$$

Putting these estimates together yield (5.9). It is easy to see that (5.9) yields that $\int_\Omega u^m(x,t)\cdot V(x,t)\,dx : [0,T]\to\mathbb{R}$ is equip-continuous for $m\geq m_0$.

It is clear that for any $V=\eta(t)v(x)\in C^\infty([0,T],V^{m_0})$ and $[t_1,t_2]\subset(0,T)$, we have

$$0=\int_\Omega u^m V\Big|_{t=t_1}^{t=t_2}+\int_{t_1}^{t_2}\int_\Omega[-u^mV_t-u^m\otimes u^m:\nabla V+\mu\nabla u^m\cdot\nabla V-fV]\,dxdt.$$

Since $u^m\to u$ weakly in $L^2(Q_T)\cap L^2H^1(Q_T)$, we have

$$\int_{t_1}^{t_2}-u^mV_t\to\int_{t_1}^{t_2}-uV_t,\ \int_{t_1}^{t_2}\int_\Omega\nabla u^m\cdot\nabla V\to\int_{t_1}^{t_2}\int_\Omega\nabla u\cdot\nabla V.$$

For $t\in[0,T]$, set

$$h^m(t)=\int_\Omega u^m(x,t)v(x)\,dx,$$

and

$$h(t)=\int_\Omega u(x,t)v(x)\,dx$$

provided that it exists. By the weak convergence of $u^m$ to $u$ in $L^2(Q_T)$, we have

$$\int_0^T h^m(t)\eta(t)\,dt\to\int_0^T h(t)\eta(t)\,dt.$$

Since $h^m\in C([0,T])$ is equi-continuous for $m\geq m_0$, by the Arzela-Ascoli theorem, $h^m$ is precompact in the topology of uniform convergence. Hence we may assume that

$$\|h^m-h\|_{C([0,T])}\to 0.$$

This implies that for any $v\in V^{m_0}$,

$$\int_\Omega u^m(x,t)v(x)\,dx\to\int_\Omega u(x,t)v(x)\,dx$$

uniformly in $t\in[0,T]$.

Since $\cup_{m\geq m_0} V^{m_0} = V$, it is not hard to see that for any $v \in V$,

$$\int_\Omega u^m(x,t)v(x)\,dx \to \int_\Omega u(x,t)v(x)\,dx$$

uniformly in $t \in [0,T]$.

Denote

$$L^2_{\rm div}(\Omega) = \left\{a \in L^2(\Omega,\mathbb{R}^n) :\ {\rm div} a = 0,\ \gamma_\nu a = 0 \text{ on } \partial\Omega\right\}.$$

**Claim**. $V$ is dense in $L^2_{\rm div}(\Omega)$ with respect to $L^2$-norm.

Suppose that this were false. Then there exists $0 \neq a \in L^2_{\rm div}(\Omega)$ such that

$$\int_\Omega a \cdot v = 0,\ \forall v \in V.$$

This implies that $a = \nabla\phi$ for some $\phi \in H^1(\Omega)$. Since ${\rm div}(a) = 0$ and $\gamma_\nu(a) = 0$ on $\partial\Omega$, we have

$$\Delta\phi = 0 \text{ in } \Omega;\ \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega.$$

It is easy to see that $\phi$ is constant and hence $a = \nabla\phi \equiv 0$. This is impossible.

By the density and approximation, it follows that for any $v \in L^2_{\rm div}(\Omega)$,

$$\int_\Omega u^m(x,t)v(x)\,dx \to \int_\Omega u(x,t)v(x)\,dx$$

uniformly in $t \in [0,T]$. On the other hand, by the Helmholtz decomposition we have that any $v \in L^2(\Omega,\mathbb{R}^n)$ can be written as

$$v = v_1 + \nabla\phi_1$$

for some $v_1 \in L^2_{\rm div}(\Omega)$ and $\phi \in H^1(\Omega)$ so that

$$\begin{aligned}\int_\Omega u^m(x,t)v(x) &= \int_\Omega u^m(x,t)v_1(x) + \int_\Omega u^m(x,t)\nabla\phi_1 = \int_\Omega u^m(x,t)v_1(x)\\ &\to \int_\Omega u(x,t)v_1(x) = \int_\Omega u(x,t)(v_1(x) + \nabla\phi_1(x)),\end{aligned}$$

as ${\rm div}(u^m) = {\rm div}(u) = 0$ yields

$$\int_\Omega u^m(x,t)\nabla\phi_1(x) = \int_\Omega u(x,t)\nabla\phi_1(x) = 0.$$

This implies that for any $v \in L^2(\Omega,\mathbb{R}^n)$, $\int_\Omega u(x,t)v(x) : [0,T] \to \mathbb{R}$ is continuous. This is equivalent to say that $u(\cdot,t) : [0,T] \to L^2(\Omega,\mathbb{R}^n)$ is continuous with respect to the weak topology of $L^2(\Omega,\mathbb{R}^n)$.

Now we return to prove that

$$\int_{t_1}^{t_2}\int_\Omega u^m \otimes u^m : \nabla V \to \int_{t_1}^{t_2}\int_\Omega u \otimes u : \nabla V.$$

This amounts to proving that $u^m \to u$ strongly in $L^2(Q_T)$. We present three approaches due to E. Hopf, J. Leray, and T. Aubin and J. Lions respectively.

**Lemma 5.3.** *(E. Hopf, 1951). Let $Q_T = \Omega \times [0,T]$. Assume $w^m : Q_T \to \mathbb{R}^n$ is bounded in $L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$ and converges weak* in $L_t^\infty L_x^2(Q_T)$ to a function $w : Q_T \to \mathbb{R}^n$. In addition, assume*

$$w^m(\cdot,t) \to w(\cdot,t) \text{ weakly in } L^2(\Omega)$$

*for all $t \in [0,T]$. Then*

$$w^m \to w \text{ strongly in } L^2(Q_T).$$

*Proof.* Recall the Friedrichs inequality: for any $\epsilon > 0$ there exist $r \in \mathbb{N}$ and functions $a_i \in C_c^\infty(\Omega, \mathbb{R}^n)$, $1 \le i \le r$ such that for any $z \in H^1(\Omega, \mathbb{R}^n)$ can be estimated by

$$\int_\Omega |z|^2 \le \sum_{i=1}^r |\int_\Omega a_i z|^2 + \epsilon \int_\Omega |\nabla z|^2.$$

Applying this inequality to $z = w^m - w$, we obtain

$$\int_0^T \int_\Omega |w^m - w|^2 \, dxdt \le \int_0^T \sum_{i=1}^r |\int_\Omega a_i(w^m - w)|^2 \, dt + \epsilon \int_0^T \int_\Omega |\nabla(w^m - w)|^2 \, dxdt.$$

Since $w^m(\cdot,t) \to w(\cdot,t)$ weakly in $L^2(\Omega)$ for all $t \in [0,T]$, it follows that

$$\lim_{m\to\infty} \int_0^T \sum_{i=1}^r |\int_\Omega a_i(w^m - w)|^2 \, dt = 0.$$

Hence we have

$$\lim_{m\to\infty} \int_0^T \int_\Omega |w^m - w|^2 \, dxdt \le C\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $w^m \to w$ strongly in $L^2(Q_T)$. □

There is another approach by J. Leray (1934's).

**Lemma 5.4.** *$u^m \to u$ strongly in $L^2(Q_T)$.*

*Proof.* Set

$$e^m(t) = \int_\Omega |u^m|^2 \, dx, \quad e(t) = \int_\Omega |u|^2 \, dx.$$

By the energy inequality for $u^m$ and the Poincaré inequality, we have

$$\frac{d}{dt} e^m(t) = -\mu \int_\Omega |\nabla u^m|^2 + \int_\Omega f \cdot u^m \le -\frac{\mu}{2} \int_\Omega |\nabla u^m|^2 + \frac{C}{\mu} \int_\Omega |f|^2,$$

and

$$\frac{d}{dt} e^m(t) = -\mu \int_\Omega |\nabla u^m|^2 + \int_\Omega f \cdot u^m \ge -(\mu + 1) \int_\Omega |\nabla u^m|^2 - \int_\Omega |f|^2.$$

Since

$$\int_0^T \int_\Omega |\nabla u^m|^2 \, dxdt \le C\Big(\|f\|_{L^2(Q_T)}, \|u_0\|_{L^2(\Omega)}\Big),$$

it follows that $\int_0^T |\frac{d}{dt} e^m(t)\|, dt$ is uniformly bounded. Hence $e^m \in BV([0,T])$ is a bounded sequence. Since $BV([0,T]) \subset L^1([0,T])$ is precompact, we may assume that there exists $e^* \in L^1([0,T])$ such that

$$e^m \to e^* \text{ in } L^1([0,T]).$$

It suffices to verify that $e^*(t) = e(t)$ for $L^1$ a.e. $t \in [0,T]$. Define $D^m(t) = \int_\Omega |\nabla u^m(t)|^2\,dx$ and

$$D^*(t) = \liminf_{m\to\infty} \int_\Omega |\nabla u^m(t)|^2\,dx = \liminf_{m\to\infty} D^m(t).$$

By the Fatou lemma, we have

$$\int_0^T D^*(t)\,dt \le \liminf_{m\to\infty} \int_0^T \int_\Omega |\nabla u^m|^2\,dxdt < +\infty.$$

Hence for $L^1$ a.e. $t \in [0,T]$, $D^*(t) < +\infty$, i.e.,

$$\liminf_{m\to\infty} \int_\Omega |\nabla u^m(t)|^2\,dx < +\infty,$$

which implies that $u^m(\cdot,t)$ is bounded in $H_0^1(\Omega)$. Thus $u^m(\cdot,t) \to u(\cdot,t)$ strongly in $L^2(\Omega)$ by the Rellich compactness Theorem and the fact that $u^m(\cdot,t) \to u(\cdot,t)$ weakly in $L^2(\Omega)$. Therefore we have for $L^1$ a.e. $t \in [0,T]$, $e^*(t) = e(t)$. As a consequence, we will have

$$\int_0^T \int_\Omega |u^m|^2\,dxdt \to \int_0^T \int_\Omega |u|^2\,dxdt.$$

This implies that

$$\int_0^T \int_\Omega |u^m - u|^2\,dxdt \to 0$$

as $m \to \infty$. □

Putting these estimates together, we can conclude that for any $v \in C^\infty(Q_T)$, with $\mathrm{div}(v) = 0$ and $v = 0$ on $\partial\Omega \times [0,T]$, it holds that for any $0 \le t_1 \le t_2 \le T$,

$$\int_\Omega u \cdot v\Big|_{t=t_1}^{t_2} + \int_{t_1}^{t_2} [-u \cdot v_t - u \otimes u : \nabla v + \mu \nabla u \cdot \nabla v - fv]\,dxdt = 0. \tag{5.10}$$

**Definition 5.1.** For an initial data $u_0 \in L^2(\Omega, \mathbb{R}^n)$ with $\mathrm{div}(u_0) = 0$, and $f \in L^2(Q_T)$, a function $u \in L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$ is called a Leray-Hopf tye of weak solution of the Navier-Stokes equation, if

- $u$ satisfies the equation in the sense of distribution, i.e., (5.10) holds.
- $u(\cdot,t) \to u_0$ in $L^2(\Omega)$ as $t \downarrow 0^+$.
- $t \to u(\cdot,t)$ is continuous from $[0,T]$ to $(L^2, \text{weak} - L^2)$.
- it satisfies the weak version of the energy inequality:

$$\int_\Omega |u|^2(t)\,dx + 2\mu \int_0^t \int_\Omega |\nabla u|^2\,dxdt \le \int_\Omega |u_0|^2 + 2\int_0^t \int_\Omega fu \tag{5.11}$$

  for any $0 < t \le T$.

**Theorem 5.5.** *For any bounded domain $\Omega \subset \mathbb{R}^n$ and $0 < T \le \infty$, $u_0 \in L^2(\Omega, \mathbb{R}^n)$ with $\mathrm{div}(u_0) = 0$, and $f \in L^2(\Omega \times [0,T])$, there exists at least one Leray-Hopf type of weak solution to the initial-boundary value problem of the Navier-Stokes equation.*

**Open problems**.

- Whether the energy inequality (5.12) is an equality for any Leray-Hopf type of weak solution?

- Whether the following stronger version of the energy inequality holds for a Leray-Hopf weak solution:

$$\int_\Omega |u|^2(t_2)\,dx + 2\mu\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\,dxdt \le \int_\Omega |u|^2(t_1)\,dx + 2\int_{t_1}^{t_2}\int_\Omega fu \tag{5.12}$$

  for any pair $0 \le t_1 < t_2 \le T$.

- Whether the uniqueness holds for the class of Leray-Hopf type of weak solutions.

- Whether the class of Leray-Hopf type of weak solution is smooth.

Now we outline the Aubin-Lions compactness.

**Lemma 5.6.** *(Aubin-Lions). Let $X_0, X, X_1$ be three Banach spaces such that $X_0 \subset X \subset X_1$ are continuous injections. Assume $X_0, X_1$ are self-reflective, and $X_0 \subset X$ is compact. For $0 < T < +\infty$, $\alpha_0, \alpha_1 \in (1, +\infty)$, consider*

$$Y = Y(0, T, \alpha_0, \alpha_1, X_0, X_1) := \{f \in L^{\alpha_0}([0,T], X) : \partial_t f \in L^{\alpha_1}([0,T], X_1)\}$$

*equipped with the norm*

$$\Big\|f\Big\|_Y = \Big\|f\Big\|_{L^{\alpha_0}([0,T];X_0)} + \Big\|\partial_t f\Big\|_{L^{\alpha_1}([0,T],X_1)}.$$

*Then $Y \subset L^{\alpha_0}([0,T], X)$ is compact.*

*Proof.* First we claim that for any $\epsilon > 0$ there exists $c(\epsilon) > 0$ such that

$$\|x\|_X \le \epsilon\|x\|_{X_0} + c(\epsilon)\|x\|_{X_1},\ \forall x \in X_0. \tag{5.13}$$

For, otherwise, there exist $\epsilon_0 > 0$ and $x_k \in X_0$ such that

$$\|x_k\|_X \ge \epsilon_0\|x_k\|_{X_0} + k\|x_k\|_{X_1}.$$

Without loss of generality, we may assume that $\|x_k\|_X = 1$, for all $k \ge 1$. Hence we have

$$\|x_k\|_{X_0} \le {\epsilon_0}^{-1},\ \|x_k\|_{X_1} \le k^{-1}.$$

Since $X_0 \subset X$ is compact, we may assume that $x_k \to x$ in $X \cap X_1$. This yields that $\|x\|_X = 1$. On the other hand, $\|x_k\|_{X_1} \to 0$ implies that $\|x\|_{X_1} = 0$ and hence $x = 0$. We get the desired contradiction.

Since $1 < \alpha_0, \alpha_1 < +\infty$, $X_0$ and $X_1$ are self-reflective, we have that $L^{\alpha_0}([0,T], X_0)$ and $L^{\alpha_1}([0,T], X_1)$ are self-reflective. Let $\{u^m\} \subset Y$ be a bounded sequence. Then we may assume, after passing to subsequences,

$$u^m \to u \text{ weakly in } L^{\alpha_0}([0,T], X_0),\ \partial_t u^m \to \partial_t u \text{ weakly in } L^{\alpha_1}([0,T], X_1).$$

By considering $v^m = u^m - u$, we may assume that $u \equiv 0$. Applying (5.13) and integrating over $t \in [0, T]$, we have

$$\|u^m\|_{L^{\alpha_0}([0,T],X)} \le \epsilon \|u^m\|_{L^{\alpha_0}([0,T],X_0)} + c(\epsilon)\|u^m\|_{L^{\alpha_0}([0,T],X_1)}.$$

It suffices to show that $\|u^m\|_{L^{\alpha_0}([0,T],X_1)} \to 0$. Since $Y \subset C([0,T], X_1)$ is continuous, it suffices to show that $u^m(t) \to 0$ in $X_1$ for $L^1$-a.e. $t \in [0,T]$ by the Lebesgue Dominated Convergence Theorem.

Since

$$u^m(0) = u^m(t) - \int_0^t (u^m)'(\tau)\, d\tau,$$

we have that for any $0 < s < T$,

$$u^m(0) = \frac{1}{s}\int_0^s u^m(t)\, dt - \frac{1}{s}\int_0^s \int_0^t (u^m)'(\tau)\, d\tau\, dt = a_m + b_m.$$

For any $s \in (0, T)$ fixed, it is easy to see that

$$a_m \to 0 \text{ weakly in } X_0,$$

and

$$\|b_m\|_{X_1} = |\frac{1}{s}\int_0^s (s-t)(u^m)'(t)\, dt| \le \int_0^s \|(u^m)'(t)\|_{X_1}\, dt \le \frac{\epsilon}{2}$$

provided that $s > 0$ is chosen to be sufficiently small. Since $X_0 \subset X_1$ is compact, it follows that

$$\|a_m\|_{X_1} \to 0$$

Putting these two estimates together yields that $\|u^m(t)\|_{X_1} \to 0$ for a.e. $t \in [0, T]$. □

Finally we indicate how to apply the Aubin-Lions lemma to show that $u^m \to u$ in $L^2(Q_T)$ when $n = 3$.

Choose $X_0 = H^1(\Omega)$, $X = L^2(\Omega)$, and $X_1 = W^{-2,2}(\Omega) = (W_0^{2,2}(\Omega))'$. It is clear that $X_0 \subset X \subset X_1$ are continuous injections, $X_0 \subset X$ is compact, and $X_0, X_1$ are self-reflective Hilbert spaces.

**Claim**. $\{\partial_t u_m\} \subset L^2([0,T], X_1)$ is bounded.

This claim is non-trivial, and we leave it for the readers to verify as a challenging homework problem. Then we can apply Aubin-Lions' lemma directly to conclude that $u_m \to u$ strongly in $L^2(Q_T)$. □

Lecture 6, December 26, 2012

# 6 Uniqueness question on the Navier-Stokes equation

We begin with the uniqueness result on the Leray-Hopf weak solution in dimension two, while the similar result in dimension three is completely open.

**Theorem 6.1.** *For $n = 2$, the class of Leray-Hopf weak solutions with respect to the initial boundary value problem enjoys the uniqueness property.*

A key step to obtain this uniqueness is the Ladyzhenskaya inequality:

$$\left\|v\right\|_{L^4(\Omega)} \leq c\left\|v\right\|^{\frac{1}{2}}_{L^2(\Omega)}\left\|\nabla v\right\|^{\frac{1}{2}}_{L^2(\Omega)} \tag{6.1}$$

holds for any $v \in H^1_0(\Omega)$, with $\Omega \subset \mathbb{R}^2$ a bounded domain.

**Lemma 6.2.** *For $n = 2$ and a bounded domain $\Omega \subset \mathbb{R}^2$, we have*

$$\left|B[u, v, w]\right| \leq C\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|w\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla w\|^{\frac{1}{2}}_{L^2(\Omega)} \tag{6.2}$$

*holds for any $u, v, w \in H^1_0(\Omega, \mathbb{R}^2)$.*

*Proof.* Since

$$B[u, v, w] = \int_\Omega u \cdot \nabla v \cdot w,$$

it follows from the Hölder inequality that

$$|B[u, v, w]| \leq \|u\|_{L^4(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}.$$

Applying the inequality (6.1) to both $u$ and $w$ immediately yields (6.2). □

**Proof of Theorem 6.1**: Let $u_1, u_2 \in L^\infty_t L^2_x(Q_T) \cap L^2_t H^1_x(Q_T)$ be two Leray-Hopf type of weak solutions. Set $w = u_1 - u_2$. Then we have

$$w = 0 \text{ on } \partial_p(Q_T).$$

Since $w$ satisfies

$$\partial_t w - \mu\Delta w + u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2 + \nabla p = 0 \text{ in } \Omega \times (0, T),$$

we can multiply the equation by $w$ and integrate over $\Omega$ to get

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + 2\mu\|\nabla w\|^2_{L^2(\Omega)} &= 2B[u_2, u_2, w] - 2B[u_1, u_1, w] = -2B[w, u_2, w] \\ &\lesssim \|w\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\|\nabla u_2\|_{L^2(\Omega)} \\ &\leq \mu\|\nabla w\|^2_{L^2(\Omega)} + c\mu^{-1}\|w\|^2_{L^2(\Omega)}\|\nabla u_2\|^2_{L^2(\Omega)}.\end{aligned}$$

This implies

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} \leq c\mu^{-1}\|w\|^2_{L^2(\Omega)}\|\nabla u_2\|^2_{L^2(\Omega)}.$$

Hence we obtain

$$\frac{d}{dt}\left(e^{-c\int_0^t \|\nabla u_2(s)\|^2_{L^2(\Omega)}\,ds}\|w(t)\|^2_{L^2(\Omega)}\right) \le 0.$$

In particular, we have

$$\|w(t)\|_{L^2(\Omega)} \le e^{c\int_0^t \|\nabla u_2(s)\|^2_{L^2(\Omega)}\,ds}\|w(0)\|_{L^2(\Omega)} = 0.$$

This completes the proof. □

Next we present Serrin's weak-strong uniqueness in higher dimensions.

First we indicate that under higher integrability condition, Leray-Hopf's weak solutions do enjoy the energy equality property.

**Lemma 6.3.** *If $u \in L^\infty_t L^2_x(Q_T) \cap L^2_t H^1_x(Q_T) \cap L^4(Q_T)$ is a Leray-Hopf weak solution, then the energy inequality becomes an equality. In fact, one has that for any $0 \le t_1 < t_2 \le T$, it holds*

$$\int_\Omega |u(t_2)|^2 + 2\mu \int_{t_1}^{t_2} \int_\Omega |\nabla u|^2\,dxdt = \int_\Omega |u(t_1)|^2. \tag{6.3}$$

*Proof.* One can view the Navier-Stokes equation as a perturbed Stokes equation:

$$u_t - \mu\Delta u + \nabla p = -\nabla \cdot (u \otimes u).$$

Since $u \in L^4(Q_T)$, we see that $u \otimes u \in L^2(Q_T)$ and hence $\nabla \cdot (u \otimes u) \in L^2([0,T]; H^{-1}(\Omega))$. It follows that $\nabla \cdot (u \otimes u) \cdot u \in L^1(Q_T)$ and

$$B[u,u,u] = \int_{t_1}^{t_2} \int_\Omega \nabla \cdot (u \otimes u) \cdot u = 0,$$

as $\nabla \cdot u = 0$. It is clear that this fact easily implies (6.3). □

In general, we will show that the class of Serrin's weak solutions enjoy the above energy equality property. First, we introduce Serrin's weak solutions.

**Lemma 6.4.** *A nonzero function $f \in L^p_t L^q_x(\mathbb{R}^n \times \mathbb{R}_+)$ is scaling invariant, i.e.*

$$\|f_\lambda\|_{L^p_t L^q_x(\mathbb{R}^n \times \mathbb{R}_+)} = \|f\|_{L^p_t L^q_x(\mathbb{R}^n \times \mathbb{R}_+)}, \ \forall \lambda > 0,$$

*iff*

$$\frac{2}{p} + \frac{n}{q} = 1. \tag{6.4}$$

*Here $f_\lambda(x,t) = \lambda f(\lambda x, \lambda^2 t)$.*

*Proof.* By direct calculations,we have

$$\left\|f_\lambda\right\|_{L^p_t L^q_x(\mathbb{R}^n \times \mathbb{R}_+)} = \lambda^{1-\frac{2}{p}-\frac{n}{q}}\left\|f\right\|_{L^p_t L^q_x(\mathbb{R}^n \times \mathbb{R}_+)}. \tag{6.5}$$

It is readily seen that the conclusion follows from this identity. □

**Lemma 6.5.** *Suppose that $v, w \in L^\infty_t L^2_x(Q_T) \cap L^2_t H^1_x(Q_T)$ and $u \in L^p_t L^q_x(Q_T)$ for a pair of exponents $(p,q)$ satisfying (6.4). Then*

$$\int_0^T \int_\Omega |v \cdot \nabla w \cdot u|\,dxdt \lesssim \|\nabla w\|_{L^2(Q_T)}\|\nabla v\|^{\frac{n}{q}}_{L^2(Q_T)}\left(\int_0^T \|u\|^p_{L^q_x(\Omega)}\|v\|^2_{L^2(\Omega)}\right)^{\frac{1}{p}}. \tag{6.6}$$

*Proof.* By Hölder's inequality we have

$$\int_0^T \int_\Omega |v \cdot \nabla w \cdot u| \, dxdt \le \|u\|_{L^q(\Omega)} \|v\|_{L^r(\Omega)} \|\nabla w\|_{L^2(\Omega)}, \tag{6.7}$$

where $r$ is given by

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}.$$

Now by the Sobolev and the interpolation inequalities we have

$$\|v\|_{L^r(\Omega)} \le \|v\|^{\theta}_{L^2(\Omega)} \|v\|^{1-\theta}_{L^{2^*}(\Omega)} \lesssim \|v\|^{\theta}_{L^2(\Omega)} \|\nabla v\|^{1-\theta}_{L^2(\Omega)},$$

where

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{2^*}.$$

The conditions on $(p, q, r, \theta)$ imply

$$\theta = \frac{2}{p}, \quad 1 - \theta = \frac{n}{q}.$$

Hence

$$\|v\|_{L^r(\Omega)} \lesssim \|v\|^{\frac{2}{p}}_{L^2(\Omega)} \|\nabla v\|^{\frac{n}{q}}_{L^2(\Omega)}.$$

Substituting this inequality into (6.7) and integrating the resulting inequality, we obtain

$$\begin{aligned} \int_0^T \int_\Omega |v \cdot \nabla w \cdot u| \, dxdt &\lesssim \int_0^T \|u\|_{L^q_x(\Omega)} \|v\|^{\frac{2}{p}}_{L^2(\Omega)} \|\nabla v\|^{\frac{n}{q}}_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \\ &\lesssim \|\nabla w\|_{L^2(Q_T)} \|\nabla v\|^{\frac{n}{q}}_{L^2(Q_T)} \left( \int_0^T \|u\|^p_{L^q(\Omega)} \|v\|^2_{L^2(\Omega)} \right)^{\frac{1}{p}}, \end{aligned}$$

where we have used the fact $\frac{1}{2} + \frac{n}{2q} + \frac{1}{p} = 1$ in the last step. □

**Theorem 6.6.** *Let $u \in L^\infty_t L^2_x(Q_T) \cap L^2_t H^1_x(Q_T)$ be a weak solution of the initial value problem of the Navier-Stokes equation. If, in addition, $u \in L^p_t L^q_x(Q_T)$ for a pair of exponents $(p, q)$ satisfying (6.4). Then for any $0 \le t \le T$, it holds*

$$\|u(t)\|^2_{L^2(\Omega)} + 2\mu \int_0^t \|\nabla u\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega)}. \tag{6.8}$$

*Proof.* Let $K \in C_c^\infty(\mathbb{R})$ be an even mollifier function. For $h > 0$ define $K_h(t) = h^{-1} K(\frac{t}{h})$. Let $\{u^k\} \subset \widetilde{\mathcal{V}} \equiv \left\{ v \in C^\infty(Q_T) : \ \mathrm{div} v = 0, \ v = 0 \text{ on } \partial\Omega \times [0, T] \right\}$ be a sequence of maps approximating $u$. For $\tau \in (0, T]$ fixed, let $h \in (0, \tau)$ and define

$$u^k_h(x, t) = \int_0^\tau K_h(t - t') u^k(x, t') \, dt', \quad u_h(x, t) = \int_0^\tau K_h(t - t') u(x, t') \, dt'.$$

First testing the Navier-Stokes equation by $u^k_h$ and then sending $k \to \infty$ yields

$$\int_0^\tau \{(u, \partial_t u_h) - \mu(\nabla u, \nabla u_h) + (u, u \cdot \nabla u_h)\} \, dt = (u, u_h)|_{t=\tau} - (u_0, u_h(0)).$$

Note that

$$\int_0^\tau (u, \partial_t u_h) = \int_0^\tau \int_0^\tau \partial_t K_h(t-t')(u(t), u(t'))\, dt'\, dt = 0,$$

$$\int_0^\tau \mu(\nabla u, \nabla u_h) \to \mu \int_0^\tau \int_\Omega |\nabla u|^2,$$

$$(u, u_h)|_{t=\tau} = \int_0^h K(t)(u(\tau), u(\tau - t))\, dt \to \frac{1}{2}\|u(\tau)\|^2_{L^2(\Omega)},$$

$$(u_0, u_h(0)) \to \frac{1}{2}\|u_0\|^2_{L^2(\Omega)},$$

and

$$\int_0^\tau (u, u\cdot\nabla u_h) \to \int_0^\tau (u, u \cdot \nabla u) = 0,$$

as $h \to 0$, where we have used lemma 6.5 and divu $= 0$ in the last step. Putting these together yields (6.8). □

Now we present the weak-strong uniqueness theorem, due to J. Serrin.

**Theorem 6.7.** *Let $u, v \in L^\infty_t L^2_x(Q_T) \cap L^2_t H^1_x(Q_T)$ be two Leray-Hopf weak solutions of the initial and boundary value problem of the Navier-Stokes equation. Suppose also that $u \in L^p_t L^q_x(Q_T)$ for a pair of exponents satisfying (6.4) and $n \le q < +\infty$. Then*

$$\|u(t) - v(t)\|_{L^2(\Omega)} \le \|u_0 - v_0\|_{L^2(\Omega)} \exp(c \int_0^t \|u(t)\|^p_{L^q(\Omega)}\, dt). \tag{6.9}$$

*In particular, if $u_0 = v_0$, then $u \equiv v$ on $Q_T$.*

*Proof.* Let $u_h$ and $v_h$ be defined as in the above lemma. Then we have

$$\int_0^\tau \{(u, \partial_t v_h) - \mu(\nabla u, \nabla v_h) + (u, u\cdot\nabla v_h)\}\ dt = (u, v_h)|_{t=\tau} - (u_0, v_h(0)), \tag{6.10}$$

and

$$\int_0^\tau \{(v, \partial_t u_h) - \mu(\nabla v, \nabla u_h) - (u_h, v\cdot\nabla v)\}\ dt = (v, u_h)|_{t=\tau} - (v_0, u_h(0)). \tag{6.11}$$

Observe that $\int_0^\tau (u, \partial_t v_h) = -\int_0^\tau (v, \partial_t u_h)$. Adding (6.10) and (6.11) yields

$$\begin{aligned} &-\int_0^\tau \{\mu[(\nabla u, \nabla v_h) + (\nabla v, \nabla u_h)] + (u_h, v\cdot\nabla v) - (u, u\cdot\nabla v_h)\} \\ &= (u(\tau), v_h(\tau)) + (v(\tau), u_h(\tau)) - (u_0, v_h(0)) - (v_0, u_h(0)). \end{aligned} \tag{6.12}$$

It is easy to see that

$$(u(\tau), v_h(\tau)) + (v(\tau), u_h(\tau)) - (u_0, v_h(0)) - (v_0, u_h(0)) \to (u(\tau), v(\tau)) - (u_0, v_0),$$

and

$$\begin{aligned} &-\int_0^\tau \{\mu[(\nabla u, \nabla v_h) + (\nabla v, \nabla u_h)] + (u_h, v\cdot\nabla v) - (u, u\cdot\nabla v_h)\} \\ &\to -\int_0^\tau \{2\mu(\nabla u, \nabla v) + (u, (v-u)\cdot\nabla v)\} \end{aligned}$$

as $h \to 0$. Hence we have

$$-\int_0^\tau \{4\mu(\nabla u, \nabla v) + (u, (v-u)\cdot\nabla v)\} = 2(u(\tau), v(\tau)) - 2(u_0, v_0). \tag{6.13}$$

Since

$$\|v(\tau)\|^2_{L^2(\Omega)} + 2\mu \int_0^\tau \|\nabla v\|^2_{L^2(\Omega)}\, dt \le \|v_0\|^2_{L^2(\Omega)}, \tag{6.14}$$

and

$$\|u(\tau)\|^2_{L^2(\Omega)} + 2\mu \int_0^\tau \|\nabla u\|^2_{L^2(\Omega)}\, dt = \|u_0\|^2_{L^2(\Omega)}, \tag{6.15}$$

by adding (6.13), (6.14), and (6.15), we have

$$\begin{aligned}
&\|(u-v)(\tau)\|^2_{L^2(\Omega)} + 2\mu \int_0^\tau \int_\Omega |\nabla(u-v)|^2 \\
&\le \|u_0 - v_0\|^2_{L^2(\Omega)} + 2\int_0^\tau (u, (u-v)\cdot\nabla(v-u)) + (u, (u-v)\cdot\nabla u)\, dt \\
&= \|u_0 - v_0\|^2_{L^2(\Omega)} + 2\int_0^\tau (u, (u-v)\cdot\nabla(v-u))\, dt \text{ (since } (u, (u-v)\cdot\nabla u) = 0) \\
&\le \|u_0 - v_0\|^2_{L^2(\Omega)} + C\|\nabla(u-v)\|^{1+\frac{n}{q}}_{L^2(Q_\tau)} \left(\int_0^\tau \|u\|^p_{L^q(\Omega)} \|u-v\|^2_{L^2(\Omega)}\, dt\right)^{\frac{1}{p}} \\
&\le \|u_0 - v_0\|^2_{L^2(\Omega)} + \mu\|\nabla(u-v)\|^2_{L^2(Q_\tau)} + C\int_0^\tau \|u(t)\|^p_{L^q(\Omega)} \|u-v\|^2_{L^2(\Omega)}\, dt.
\end{aligned}$$

Therefore we have

$$\|(u-v)(\tau)\|^2_{L^2(\Omega)} \le \|u_0 - v_0\|^2_{L^2(\Omega)} + C\int_0^\tau \|u(t)\|^p_{L^q(\Omega)} \|u-v\|^2_{L^2(\Omega)}\, dt.$$

This, with the help of Gronwall's inequality, implies (6.9). □

*Remark* 6.1. For the end point case $p = \infty, q = n$, the reader can check that the same argument also works if we assume that $\|u\|_{L^\infty_t L^n_x(Q_T)}$ is sufficiently small.

Now we want to discuss the existence of local and global strong solutions in low dimensions.

**Theorem 6.8.** *(*Kiselev-Ladyzhenskaya*). For $n = 2$ or 3 and $f = 0$. For any $u_0 \in H^2(\Omega) \cap V$, there exists a weak solution $u \in L^\infty_t L^2_x(Q_T) \cap L^2_t H^1_x(Q_T)$ of the initial and boundary value problem of the Navier-Stokes equation, and a $T > 0$ such that $\|\nabla u\|_{L^2(\Omega)}$ and $\|\partial_t u\|_{L^2(\Omega)}$ are uniformly bounded for $0 \le t < T$. Furthermore, $T = +\infty$ if $n = 2$ or $\|u_0\|_{H^2(\Omega)}$ is sufficiently small when $n = 3$.*

*Proof.* Here we sketch the argument for the solution $u$. Rigorously speaking, one needs to first work with the Galerkin's approximate solution $u^m$ and then taking $m \to \infty$.

Taking $\partial_t$ of the equation, we have

$$u_{tt} - \mu\Delta u_t + (u\cdot\nabla u)_t + \nabla p_t = 0.$$

Multiplying this equation by $u_t$ and integrating over $\Omega$, we obtain

$$\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} = -2\mu\|\nabla u_t\|^2_{L^2(\Omega)} - 2(u_t, u_t\cdot\nabla u),$$

where we have used

$$\int_\Omega \nabla p_t \cdot u_t = -\int_\Omega p_t \cdot \nabla \cdot u_t = 0,$$

and

$$(u_t, u \cdot \nabla u_t) = B[u, u_t, u_t] = 0.$$

Observe that

$$|(u_t, u_t \cdot \nabla u)| \le \|\nabla u\|_{L^2(\Omega)} \|u_t\|^2_{L^4(\Omega)}.$$

By the Sobolev inequality, we then have

$$\begin{aligned} |(u_t, u_t \cdot \nabla u)| &\le \begin{cases} C\|u_t\|_{L^2(\Omega)} \|\nabla u_t\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} & n = 2 \\ C\|u_t\|^{\frac{1}{2}}_{L^2(\Omega)} \|\nabla u_t\|^{\frac{3}{2}}_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} & n = 3 \end{cases} \\ &\le \begin{cases} \mu \int_\Omega |\nabla u_t|^2 + \frac{C}{\mu} \|u_t\|^2_{L^2(\Omega)} \|\nabla u\|^2_{L^2(\Omega)} & n = 2 \\ \mu \int_\Omega |\nabla u_t|^2 + \frac{C}{\mu^3} \|u_t\|^2_{L^2(\Omega)} \|\nabla u\|^4_{L^2(\Omega)} & n = 3. \end{cases} \end{aligned}$$

Therefore we have

$$\frac{d}{dt} \|u_t\|^2_{L^2(\Omega)} \le \begin{cases} \frac{C}{\mu} \|u_t\|^2_{L^2(\Omega)} \|\nabla u\|^2_{L^2(\Omega)} & n = 2 \\ \frac{C}{\mu^3} \|u_t\|^2_{L^2(\Omega)} \|\nabla u\|^4_{L^2(\Omega)} & n = 3. \end{cases} \tag{6.16}$$

Now we proceed as follows.
(i) $n = 2$: By Gronwall's inequality, we have

$$\|u_t\|_{L^2(\Omega)} \le \|u_t(0)\|_{L^2(\Omega)} \exp\left(c \int_0^t \|\nabla u\|^2_{L^2(\Omega)} \, dt\right).$$

Since

$$u_t(0) = \mu \Delta u_0 - u_0 \cdot \nabla u_0 - \nabla p,$$

and $\nabla \cdot u_t(0) = 0$ and $u_t(0) = 0$ on $\partial\Omega$, we have $\int_\Omega \nabla p \cdot u_t(0) = 0$ and hence

$$\|u_t(0)\|^2_{L^2(\Omega)} = \|\mu \Delta u_0 - u_0 \cdot \nabla u_0\|^2_{L^2(\Omega)} \lesssim \|u_0\|^2_{H^2(\Omega)}.$$

Therefore we have

$$\|u_t\|_{L^2(\Omega)} \le C(\|u_0\|_{H^2(\Omega)}) \; \forall 0 \le t \le T.$$

Since the energy inequality implies that

$$2\mu \|\nabla u\|^2_{L^2(\Omega)} = -\frac{d}{dt} \|u\|^2_{L^2(\Omega)} \le 2\|u\|_{L^2(\Omega)} \|u_t\|_{L^2(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} \|u_t\|_{L^2(\Omega)}$$

is also uniformly bounded for all $0 \le t \le T$. From this argument, one also sees that the maximal time interval $T$ is $+\infty$.
(ii) $n = 3$: Since

$$\mu \|\nabla u\|^2_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)} \|u_t\|_{L^2(\Omega)},$$

we have

$$\frac{d}{dt} \|u_t\|_{L^2(\Omega)} \le \frac{C}{\mu^4} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|u_t\|^2_{L^2(\Omega)}.$$

Thus we have
$$\|u_t\|_{L^2(\Omega)} \le \frac{\|u_t(0)\|_{L^2(\Omega)}}{1 - C\mu^{-4}\|u_t(0)\|_{L^2(\Omega)}A(t)} \tag{6.17}$$
where
$$A(t) = \int_0^t \|u\|_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\,dt = \|u_0\|^3_{L^2(\Omega)} - \|u(t)\|^3_{L^2(\Omega)}.$$
Note that if $u_0$ satisfies
$$\|u_0\|^3_{L^2(\Omega)}\|u_t(0)\|_{L^2(\Omega)} < \frac{\mu^4}{C}, \tag{6.18}$$
then
$$1 - \frac{C}{\mu^4}\|u_0\|^3_{L^2(\Omega)}\|u_t(0)\|_{L^2(\Omega)} > 0$$
so that $\|u_t(t)\|_{L^2(\Omega)}$ is uniformly bounded for all $0 \le t < T = \infty$ . Since $\|\nabla u(t)\|^2_{L^2(\Omega)} \le \|u(t)\|_{L^2(\Omega)}\|u_t(t)\|_{L^2(\Omega)}$, it follows that $\|\nabla u(t)\|_{L^2(\Omega)}$ is also uniformly bounded for all $0 \le t < T = \infty$.

If (6.18) doesn't hold, then since
$$\frac{d}{dt}\|u_t\|_{L^2(\Omega)} \le \frac{C}{\mu^4}\|u\|^2_{L^2(\Omega)}\|u_t\|^3_{L^2(\Omega)},$$
we have
$$\left\|u_t\right\|^2_{L^2(\Omega)} \le \frac{\|u_t(0)\|^2_{L^2(\Omega)}}{1 - C\mu^{-4}\|u_0\|^2_{L^2(\Omega)}\|u_t(0)\|_{L^2(\Omega)}t}. \tag{6.19}$$
Therefore if
$$T < \frac{\mu^4}{C\|u_0\|^2_{L^2(\Omega)}\|u_t(0)\|^2_{L^2(\Omega)}},$$
then the estimates on $\|u_t\|_{L^2(\Omega)}$ and $\|\nabla u(t)\|_{L^2(\Omega)}$ hold for all $0 \le t < T$. □

It turns out that the above theorem also holds for small initial data in dimension $n = 4$. Namely, we have

**Theorem 6.9.** *For $n = 4$, and $u_0 \in H^2_0(\Omega)$ with $\nabla \cdot u_0 = 0$ and $\|u_0\|_{H^2(\Omega)}$ sufficiently small, then there is a solution which is strongly differentiable with respect to x and t, and $\|u_t\|_{L^2(\Omega)}, \|\nabla u(t)\|_{L^2(\Omega)}$ is uniformly bounded for all $0 \le t < +\infty$.*

*Proof.* The idea is similar to the above Theorem, but the argument is different. As in the above theorem, we first have
$$\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} + \mu\|\nabla u_t\|^2_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}\|u_t\|^2_{L^4(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}\|\nabla u_t\|^2_{L^2(\Omega)},$$
so that we have
$$\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} + (\mu - C\|\nabla u\|_{L^2(\Omega)})\|u_t\|^2_{L^2(\Omega)} \le 0. \tag{6.20}$$
Now we have

**Claim**. If $u_0$ satisfies
$$\|u_0\|_{L^2(\Omega)}\|u_t(0)\|_{L^2(\Omega)} < \frac{\mu^2}{C^2} \tag{6.21}$$

then for all $0 \le t < +\infty$ it holds that

$$\|\nabla u(t)\|_{L^2(\Omega)} < \frac{\mu}{C}, \ \forall\ 0 \le t < +\infty. \tag{6.22}$$

To see this, we first observe that the condition (6.21) and the energy equality of the Navier-Stokes equation imply

$$\|\nabla u_0\|^2_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}\|u_t(0)\|_{L^2(\Omega)} < \frac{\mu^2}{C^2},$$

which clearly implies that there exists $\delta > 0$ such that (6.22) holds for $0 \le t \le \delta$. Assume $T_0 \le T$ is the maximal time such that (6.22) holds. If $T_0 < +\infty$, then we would have

$$\|\nabla u(t)\|_{L^2(\Omega)} < \frac{\mu}{C}, \ \forall 0 \le t < T_0; \ \|\nabla u(T_0)\|_{L^2(\Omega)} = \frac{\mu}{C}. \tag{6.23}$$

Substituting (6.23) into the inequality (6.20), we would obtain

$$\|u_t\|_{L^2(\Omega)} \le \|u_t(0)\|_{L^2(\Omega)} \ \forall 0 \le t \le T_0.$$

At $t = T_0$, we would then have

$$\frac{\mu^2}{C^2} = \|\nabla u(T_0)\|^2_{L^2(\Omega)} \le \|u_t(T_0)\|_{L^2(\Omega)}\|u(T_0)\|_{L^2(\Omega)} \le \|u_t(0)\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)} < \frac{\mu^2}{C^2}.$$

This is impossible. Thus $T_0 = \infty$. The proof is complete. □

## 6.1 The Ossen Kernel

The Ossen kernel plays very important roles in the study of mild solutions to the Navier-Stokes equation in the entire $\mathbb{R}^n$. It is the fundamental solution of the time-dependent linear Stokes system on $\mathbb{R}^n$: For $f \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ and $u_0 \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$, consider

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \mathbb{R}^n \times (0, +\infty) \\ \nabla \cdot u = 0 & \mathbb{R}^n \times (0, \infty) \\ u|_{t=0} = u_0 & \mathbb{R}^n. \end{cases} \tag{6.24}$$

It is not hard to see that by the superposition principle that $u = u_1 + u_2$, where

$$u_1(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)u_0(y)\,dy$$

is the solution to the heat equation with $u_0$ is the initial data, while

$$u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-s)\mathbb{P}f(y,s)\,dyds$$

is the solution to the Stokes system with zero initial data. Here $\mathbb{P} : L^2(\mathbb{R}^n) \to L^2_{\mathrm{div}}(\mathbb{R}^n)$ is the Leray projection operator, and

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-\frac{|x|^2}{4t}),$$

is the fundamental solution to the heat equation on $\mathbb{R}^n$.

Recall from Lecture 2 that for $1 \le i \le n$,

$$(\mathbb{P}f)^i(x) = f^i(x) - \frac{\partial}{\partial x_i}(\Delta^{-1}\nabla \cdot f) = f^i(x) + \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j} G(x-y) f^j(y)\, dy,$$

where $G$ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$.

Define

$$\Phi(x,t) = \int_{\mathbb{R}^n} G(y)\Gamma(x-y,t)\, dy.$$

Then the Duhamel formula for the Stokes equation will be given by

$$u^i(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t) u_0^i(y)\, dy + \int_0^t \int_{\mathbb{R}^n} k_{ij}(x-y,t-s) f^j(y,s)\, dyds, \tag{6.25}$$

where

$$k_{ij}(x,t) = \Big(\delta_{ij}\Delta + \frac{\partial^2}{\partial x_i \partial x_j}\Big)\Phi(x,t) \tag{6.26}$$

is called the Ossen kernel.

For the Ossen kernel, we have the following property.

**Lemma 6.10.** *Let $k_{ij}$ be the Ossen kernel defined by (6.26). Then it holds*

$$\Big|k_{ij}(x,t)\Big| \lesssim \frac{1}{(|x|^2+t)^{\frac{n}{2}}}, \ \Big|\nabla_t^l \nabla_x^k k_{ij}(x,t)\Big| \lesssim \frac{1}{(|x|^2+t)^{\frac{n+k+2l}{2}}}, \ \forall (x,t) \in \mathbb{R}^n \times (0,+\infty). \tag{6.27}$$

*Proof.* It is straightforward from the definition of $\Phi$. □

# 7 Leray's construction of local classical solutions and BKM criterion

## 7.1 Hölder estimates for the Stokes system

. Assume that $f = \operatorname{div}(F)$ for some $F \in L^\infty(\mathbb{R}^n \times (0,+\infty), \mathbb{R}^{n\times n})$. Assume $u_0 \in L^\infty(\mathbb{R}^n)$. Then it is easy to see that $u_1 = \Gamma(t) * u$, the solution to the heat equation with $u_0$ as the initial data, satisfies

$$\left\|\partial_t^l \nabla_x^k u_1(x,t)\right| \lesssim \frac{1}{t^{\frac{k}{2}+l}}\left\|u_0\right\|_{L^\infty(\mathbb{R}^n)},\ \forall (x,t) \in \mathbb{R}^n \times (0,+\infty). \tag{7.1}$$

Since

$$u_2^i(x,t) = \int_0^t \int_{\mathbb{R}^n} K_{ijl}(x-y,t-s)F^{lj}(y,s)\,dyds,$$

where $K_{ijl} = \frac{\partial k_{ij}}{\partial x_l}$ is the partial derivative of the Ossen kernel $k_{ij}$.

We want to estimate $|u_2(x_1,t_1) - u_2(x_2,t_2)|$ by estimating $|u_2(x_1,t_1) - u_2(x_2,t_1)|$ and $|u_2(x_2,t_1) - u_2(x_2,t_2)|$ separately. Since we are interested in the interior estimate, we may assume $t_1 \approx 4$. By translation invariance, we can assume $x_1 = 0$ and $x_2 = \alpha e$ for some $e \in \mathbb{S}^{n-1}$. Observe that $K_{ijl}$ enjoys the following homogeneity property:

$$K_{ijl}(\lambda x, \lambda^2 t) = \lambda^{-n-1}K(x,t),\ \forall \lambda > 0.$$

Thus we have

$$\begin{aligned}
|u_2(x_1,t_1) - u_2(x_2,t_1)| &\le \|F\|_{L^\infty(\mathbb{R}^n\times[0,t_1])}\int_0^t \int_{\mathbb{R}^n} |K(-y,s) - K(\alpha e - y, s)|\,dyds \\
&\le \alpha\|F\|_{L^\infty}\int_0^{\frac{t_1}{\alpha^2}} \int_{\mathbb{R}^n} |K(-z,\tau) - K(e-z,\tau)|\,dzd\tau \\
&\le \alpha\|F\|_{L^\infty}\Big\{\int_0^2 + \int_2^{\frac{t_1}{\alpha^2}}\Big\}\int_{\mathbb{R}^n} |K(-z,\tau) - K(e-z,\tau)|\,dzd\tau = I + II.
\end{aligned}$$

Here $K = (K_{ijl})$ for $t \ge 0$ and $K = 0$ for $t < 0$. Since

$$|K(-z,\tau) - K(e-z,\tau)| \le |K(-z,\tau)| + |K(e-z,\tau)|,\ 0 \le \tau \le 2,$$

and

$$|K(-z,\tau) - K(e-z,\tau)| \lesssim \frac{1}{(|z|^2+\tau)^{\frac{n+2}{2}}},\ 2 \le \tau \le \frac{t_1}{\alpha^2},$$

we see that

$$|I| \le C\alpha\|F\|_{L^\infty},$$

and

$$\begin{aligned}
|II| &\lesssim \alpha\|F\|_{L^\infty}\int_2^{\frac{t_1}{\alpha^2}}\int_{\mathbb{R}^n}\frac{dzd\tau}{(|z|^2+\tau)^{\frac{n+2}{2}}} \\
&\lesssim \alpha\|F\|_{L^\infty}\int_2^{\frac{t_1}{\alpha^2}}\frac{d\tau}{\tau} \lesssim \alpha\log(\frac{1}{\alpha})\|F\|_{L^\infty}.
\end{aligned}$$

Therefore we have

$$|u_2(x_1,t_1) - u_2(x_2,t_1)| \lesssim |x_1 - x_2| \left(1 + \log(\frac{1}{|x_1 - x_2|})\right) \|F\|_{L^\infty}. \tag{7.2}$$

To estimate $|u_2(x_2,t_1) - u(x_2,t_2)|$, we assume $x_2 = 0$ and $t_2 = t_1 - \alpha^2$. Then we have

$$\begin{aligned}
|u_2(0,t_1) - u_2(0,t_1-\alpha)| &\lesssim \|F\|_{L^\infty} \int_0^{t_1} \int_{\mathbb{R}^n} |K(-y,\tau) - K(-y,\tau-\alpha^2)\, dyd\tau \\
&\lesssim \alpha \|F\|_{L^\infty} \int_0^{\frac{t_1}{\alpha^2}} \int_{\mathbb{R}^n} |K(-y,\tau) - K(-y,\tau-1)\, dyd\tau \\
&\lesssim \alpha \|F\|_{L^\infty} \left\{ \int_0^2 + \int_2^{\frac{t_1}{\alpha^2}} \right\} \int_{\mathbb{R}^n} |K(-y,\tau) - K(-y,\tau-1)|\, dyd\tau \\
&\lesssim \alpha \|F\|_{L^\infty} \left\{ 1 + \int_2^{\frac{t_1}{\alpha^2}} \right\} \int_{\mathbb{R}^n} |K(-y,\tau) - K(-y,\tau-1)|\, dyd\tau \\
&\lesssim \alpha \|F\|_{L^\infty} \left\{ 1 + \int_2^{\frac{t_1}{\alpha^2}} \right\} \int_{\mathbb{R}^n} \frac{1}{(|z|^2+\tau)^{\frac{n+3}{2}}}\, dyd\tau \\
&\lesssim \alpha \|F\|_{L^\infty} \left( 1 + \int_2^{\frac{t_1}{\alpha^2}} \tau^{-\frac{3}{2}}\, d\tau \right) \lesssim \|F\|_{L^\infty} \sqrt{|t_1 - t_2|}.
\end{aligned}$$

Combining these estimates on $u_1$ and $u_2$ together, we would obtain

**Theorem 7.1.** *Suppose that $u_0 \in L^\infty(\mathbb{R}^n)$ and $F \in L^\infty(\mathbb{R}^n \times [0,T])$. Then for any $\theta \in (0,1)$ and $R > 0$, $\delta > 0$, $u \in C^alpha(B_R \times [\delta,T], \mathbb{R}^n)$ and*

$$\|u\|_{C^\theta(B_R \times [\delta,T])} \leq C\left(R, \delta, \|u_0\|_{L^\infty}, \|F\|_{L^\infty}\right). \tag{7.3}$$

## 7.2 Mild Solutions to the Navier-Stokes equation

Consider the initial value problem for the Navier-Stokes equation in $\mathbb{R}^n$:

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{7.4}$$

**Definition 7.1.** For $u_0 \in L^\infty(\mathbb{R}^n)$ and $0 < T \leq +\infty$, $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is called a mild solution of (7.4) if

$$u(t) = \Gamma(t) * u_0 + \int_0^t K(t-s) * (-u(s) \otimes u(s))\, ds, \ 0 < t \leq T, \tag{7.5}$$

where $K = (\nabla k_{ij})$. Definition

$$U(t) = F(t) * u_0, \ B[u,v] = \int_0^t K(t-s) * (-u(s) \otimes v(s))\, ds.$$

Then (7.5) can be written as

$$u = U + B[u,u]. \tag{7.6}$$

**Lemma 7.2.** *Let $X$ be a Banach space and $B : X \times X \to X$ be a continuous bilinear form with*

$$\|B[x, y]\| \leq \gamma \|x\| \|y\|, \ \ x, y \in X.$$

*For $a \in X$, consider the equation*

$$x = a + B(x, x). \tag{7.7}$$

*Suppose $4\gamma\|a\| < 1$. Then (7.7) has a unique solution*

$$\bar{x} \in \left\{ x \in X : \ \Big| \ \|x\| < \frac{1 + \sqrt{1 - 4\gamma\|a\|}}{2\gamma} \right\}.$$

*Moreover,*

$$\|\bar{x}\| < \frac{1 - \sqrt{1 - 4\gamma\|a\|}}{2\gamma}.$$

*Proof.* Since $4\gamma\|a\| < 1$, there are two real roots

$$r_{\pm} = \frac{1 \pm \sqrt{1 - 4\gamma\|a\|}}{2\gamma}$$

of $\|a\| + \gamma r^2 = r$. First observe that there exists no solution of (7.7) in the annulus $\{x \in X : r_- < \|x\| < r_+\}$. For, otherwise, there exists $x_1$ in this annulus such that $x_1 = a + B(x_1, x_1)$. Hence we have $\|x_1\| = \|a + B(x_1, x_1)\| \leq \|a\| + \gamma\|x_1\|^2$. This is impossible, as for any $r \in (r_{-1}, r_+)$, $\|a\| + \gamma r^2 < r$. Therefore, it suffices to look for a fixed point of the map

$$\Phi(x) = a + B(x, x) : \{x \in X : \|x\| \leq r_-\} \to \{x \in X : \|x\| \leq r_-\}.$$

In fact, since

$$\|\Phi(x)\| \leq \|a\| + \gamma\|x\|^2 \leq \|a\| + \gamma r_-^2 = r_-,$$

we see that the map is well-defined. Also, since

$$\|\Phi(x) - \Phi(y)\| \geq \gamma(\|x\| + \|y\|)\|x - y\| \leq 2\gamma r_- \|x - y\| < \|x - y\|$$

for $x, y$ in the ball. Hence $\Phi$ is a contraction map. Thus there exists $\bar{x}$, with $\|\bar{x}\| \leq \gamma_-$, such that $\bar{x} = a + B(\bar{x}, \bar{x})$. This completes the proof. □

Now we apply this abstract lemma to obtain the short time smooth solution to the Navier-Stokes equation as follows.

**Theorem 7.3.** (Leray) *For any $u_0 \in L^\infty(\mathbb{R}^n)$, there exists a $T_0 > 0$ depending on $\|u_0\|_{L^\infty}$ and a unique solution $u \in C^\infty(\mathbb{R}^n \times (0, T_0], \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \times [0, T_0], \mathbb{R}^n)$ to the initial value problem of the Navier-Stokes equation.*

*Proof.* For $T > 0$, set $X = X_T = L^\infty(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$. Then we have

$$\|\Gamma(t) * u_0\|_{X_T} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)}, \tag{7.8}$$

and

$$\begin{aligned}
\|B[u,v]\|_{X_T} &\leq \|u\|_{X_T}\|v\|_{X_T}\int_0^T\int_{\mathbb{R}^n}|K(x,t)|\,dxdt \\
&\leq C\|u\|_{X_T}\|v\|_{X_T}\int_0^T\int_{\mathbb{R}^n}\frac{dxdt}{(|x|^2+t)^{\frac{n+1}{2}}} \\
&\leq C\|u\|_{X_T}\|v\|_{X_T}\int_0^T\frac{dt}{\sqrt{t}} \leq C\|u\|_{X_T}\|v\|_{X_T}\sqrt{T} \qquad (7.9)
\end{aligned}$$

for any $u, v \in X_T$.

If $4C\sqrt{T_0}\|u_0\|_{L^\infty} < 1$, then we can apply the abstract lemma to conclude that there exists a unique $u \in X_{T_0}$ that solves

$$u = \Gamma(t) * u_0 + B[u,u],$$

which is equivalent to that $u$ solves the initial value problem of the Navier-Stokes equation.

*Remark* 7.1. i) In general, the solution $u(t)$ doesn't converge to $u_0$ in $L^\infty(\mathbb{R}^n)$, since $\Gamma(t) * u \nrightarrow u_0$ in $L^\infty(\mathbb{R}^n)$.
ii) If $T_* > 0$ is the maximal interval for the solution $u$ and $T_* < +\infty$, then according to Leray's theorem it holds

$$\Big\|u(t)\Big\|_{L^\infty(\mathbb{R}^n)} \geq \frac{\epsilon_1}{\sqrt{T_* - t}}, \quad 0 < t < T_*, \tag{7.10}$$

for some $\epsilon_1 > 0$.
iii) For any $0 < T \leq +\infty$, the uniqueness holds for solutions to the Navier-Stokes equation in $X_T$. The proof is a slight extension of the above theorem: suppose that $u_1, u_2 \in X_T$ solve the Navier-Stokes equation with the same initial data $u_0 \in L^\infty(\mathbb{R}^n)$. Then the above argument shows that there exists a sufficiently small $T_0 > 0$ such that $u_1 \equiv u_2$ in $\mathbb{R}^n \times [0, T_0]$. Then we can repeat the same argument to show that $u_1 \equiv u_2$ in $\mathbb{R}^n \times [T_0, 2T_0]$. After finite steps, it follows that $u_1 \equiv u_2$ in $\mathbb{R}^n \times [0, T]$.

## 7.3 Serrin's blow-up criterion

Consider $u_0 \in L^\infty \cap L^2$, and let $0 < T < +\infty$ be the maximal interval of existence of mild solutions or the Leray solution $u$. Then we have

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \to +\infty, \text{ as } t \uparrow T.$$

Let $1 << M_1 < M_2 < \cdots <\to +\infty$ and let $t_j \in (0, T)$ be the first time such that $t \to \|u(t)\|_{L^\infty(\mathbb{R}^n)}$ takes the value $M_j$. Let $x_j \in \mathbb{R}^n$ such that $|u(x_j, t_j)| \approx M_j$. Note that

$$|u(x,t)| \leq M_j, \ \ \forall x \in \mathbb{R}^n, \ 0 \leq t \leq t_j.$$

Define

$$v_j(y,s) = \frac{1}{M_j}u\left(x_j + \frac{y}{y_j}, t_j + \frac{t}{M_j^2}\right), \ y \in \mathbb{R}^n, \ -M_j^2 t_j \leq t \leq M_j^2(T - t_j).$$

By the scaling and translation invariance of the Navier-Stokes equation, we have that $v_j$ is a solution of the Navier-Stokes equation in $\mathbb{R}^n \times [-M_j^2 t_j, M_j^2(T-t_j)]$. Moreover,

$$|v_j(0,0)| = 1.$$

Hence by the Hölder continuity, there exists $\rho > 0$ such that

$$|v_j(x,t)| \geq \frac{1}{2}, \ \forall (x,t) \in B_\rho \times [-\rho^2, 0].$$

This implies, after rescaling, that

$$|u(x,t)| \geq \frac{M_j}{2}, \ (x,t) \in B_{\frac{\rho}{M_j}}(x_j) \times [t_j - \frac{\rho^2}{M_j^2}, t_j].$$

In other words, this indicates that $|u(x,t)|$ reach a "peak" at $z_j = (x_j, t_j)$, with hight $M_j$ and width in $x$-direction $\frac{\rho}{M_j}$ and in $t$-direction $\frac{\rho^2}{M_j^2}$. This implies that if $\frac{2}{p} + \frac{n}{q} = 1$, then

$$\left\| u \right\|_{L_t^p L_x^q (B_{\frac{\rho}{M_j}}(x_j) \times [t_j - \frac{\rho^2}{M_j^2}, t_j])} \geq c\rho^{\frac{2}{p}+\frac{n}{q}} M_j^{1-\frac{2}{p}+\frac{n}{q}} \geq c\rho.$$

This shows that the $L_t^p L_x^q$-norm of $u$ concentrates in infinitesimal region at time approaches $T$, and thus we have

**Theorem 7.4.** *Assume that a mild solution $u$ blows up at $0 < T < +\infty$. Let $q > n$ and $p \geq 2$ be such that $\frac{2}{p} + \frac{n}{q} = 1$. Then for any $\tau > 0$,*

$$\int_{T-\tau}^{T} \left( \int_{\mathbb{R}^n} |u(x,t)|^q \, dx \right)^{\frac{p}{q}} \, dt = +\infty. \tag{7.11}$$

Now we present Serrin's interior regularity theorem.

**Theorem 7.5.** *Let $u \in L^{\infty,2}(R) \cap L^2 H^1(R)$ be a weak solution of the Navier-Stokes equation. Suppose, in addition, that $u \in L_t^{s'} L_x^s(R)$ for a pair of exponents $s$ and $s'$ satisfying*

$$\frac{2}{s'} + \frac{n}{s} < 1,$$

*then $u$ is $C^\infty$ in the space variable. If $u$ is strongly differentiable with respect to $t$, then $u, \nabla_x^k u$ is absolutely continuous with respect to time $t$.*

*Proof.* The argument is based on the vorticity equation: $\omega = \nabla \times u$ satisfies

$$\omega_t - \Delta\omega = \operatorname{div}(\omega u - u\omega) \ \text{ in } \ R \tag{7.12}$$

Thus we can represent $\omega$ by

$$\omega(x,t) = \int\int k(x-y, t-s) g(y,s) \, dyds + B \ \text{ in } \ R, \tag{7.13}$$

where $B$ solves the heat equation on $R$, $k = \nabla K$ and

$$K = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4t}), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

is the heat kernel, and $g(y,s) = \eta^2(\omega u - u\omega)(y,s)$ with $\eta \in C_0^\infty(\mathbb{R}^{n+1})$ satisfying $0 \le \eta \le 1$, and $\eta \equiv 1$ in $R$.

For $\rho, \rho' \ge 1$, if $\omega \in L^{\rho',\rho}(R)$, then, since $u \in L^{s',s}(R)$, we have

$$g \in L^{q',q}(\mathbb{R}^{n+1})$$

with

$$\frac{1}{q'} = \frac{1}{s'} + \frac{1}{\rho'}, \ \frac{1}{q} = \frac{1}{s} + \frac{1}{\rho}.$$

Hence, by using the properties of the kernel $k$ and the equation (7.13), we have

$$\omega \in L^{r',r}(R),$$

where

$$r = \frac{\rho}{1-k\rho}, \ \ r' = \frac{\rho'}{1-k\rho'}, \ \ k = \frac{1-\frac{n}{s}-\frac{2}{s'}}{n+3} > 0.$$

Note that $r > \rho, r' > \rho'$, which shows that there is an improvement of the integrability of $\omega$. Starting with $(\rho,\rho') = (2,2)$, after a finite number of steps, we would obtain that $\omega \in L^\infty(R)$. Once we have that the vorticity $\omega$ is bounded, the higher order regularity follows from the standard theory, we leave the details to the interested readers. □

*Remark* 7.2. M. Struwe has extended Serrin's regularity theorem and showed that
i) if $u \in L^{p,q}(Q_T)$, with $\frac{2}{p} + \frac{n}{q} \le 1$ and $q > n$, or
ii) if $u \in L^{\infty,n}(Q_T)$ satisfies that for some absolutely constant $\epsilon$, there exists a $R > 0$ such that

$$\int_{B_R(x)\cap\Omega} |u(x,t)|^n \, dx \le \epsilon, \ \forall t \in [0,T],$$

then $u \in L^\infty(Q_T)$.

## 7.4 Sketch of Struwe's Proof

The idea is based on the Nash-Moser iterations method to the vorticity equation: For $\phi \in C_0^\infty(Q_T) \ge 0$ and $s \ge 1$, multiplying (7.12) by $\omega|\omega|^{2s-2}\phi^2$ and integrating over $Q_T$, we obtain

$$\begin{aligned} &\int \partial_t\left(\frac{|\omega|^{2s}\phi^2}{2s}\right) + |\nabla\omega|^2\omega^{2s-2}\phi^2 + \frac{1}{2}(s-1)|\nabla|\omega|^2|^2|\omega|^{2s-4}\phi^2 \\ &= \int \frac{|\omega|^{2s}}{s}\phi\partial_t\phi + \nabla|\omega|^2|\omega|^{2s-2}\phi\nabla\phi + (u\omega - \omega u)\nabla(\omega|\omega|^{2s-2}\phi^2). \end{aligned}$$

This implies

$$\begin{aligned} &\sup_{0\le t\le T}\int_\Omega (|\omega|^s\phi)^2 + \int_{Q_T} |\nabla(|\omega|^s\phi)|^2 \\ \le\ & C(\phi)\int_{Q_T} |\omega|^{2s} + C\int_{Q_T} [|u||\omega|^{2s}\phi|\nabla\phi| + ||u|^2|\omega|^{2s}\phi^2] \\ \le\ & C(\phi)\int_{Q_T} |\omega|^{2s} + C\int_{Q_T} |u|^2(|\omega|^s\phi)^2. \end{aligned} \tag{7.14}$$

The second term in the right hand side of the last inequality can be estimated by

$$\int_{Q_T} |u|^2(|\omega|^s\phi)^2 \leq \|u\|_{L^{p,q}(Q_T)}\||\omega|^s\phi\|_{L^{p*,q*}(Q_T)}, \tag{7.15}$$

where

$$\frac{1}{p^*} = \frac{1}{2} - \frac{1}{p}, \quad \frac{1}{q^*} = \frac{1}{2} - \frac{1}{q}.$$

Since

$$\frac{2}{p^*} + \frac{n}{q^*} = \frac{n}{2} + 1 - (\frac{2}{p} + \frac{n}{q}) \geq \frac{n}{2},$$

it follows from the Sobolev-interpolation inequality that

$$\||\omega|^2\phi\|^2_{L^{p*,q*}(Q_T)} \leq C(\phi)\|\omega\|^{2s}_{L^{2s}(Q_T)} + C\|u\|^2_{L^{p,q}(\text{supp}\phi)}\||\omega|^2\phi\|^2_{L^{p*,q*}(Q_T)}. \tag{7.16}$$

If $\|u\|^2_{L^{p,q}(\text{supp}\phi)} \leq \epsilon$, then we have

$$\||\omega|^2\phi\|^2_{L^{p*,q*}(Q_T)} \leq C(\phi)\|\omega\|^{2s}_{L^{2s}(Q_T)}. \tag{7.17}$$

In particular, we obtain that for any $\pi, \rho$ satisfying

$$\frac{2}{\pi} + \frac{n}{\rho} \geq \frac{n}{2},$$

then

$$\||\omega|^2\phi\|^2_{L^{\pi,\rho}(Q_T)} \leq C(\phi)\|\omega\|^{2s}_{L^{2s}(Q_T)}. \tag{7.18}$$

Thus we have that for $\beta = \frac{n+2}{n} > 1$, it holds

$$|\omega|^s\phi \in L^{2\beta}(Q_T).$$

Starting with $s_0 = 1$, $s_1 = \beta s_0 = \beta$, $s_{k+1} = \beta s_k$, and $Q_0 = Q_T$, $Q_{k+1} = \{(x,t) \mid \phi_k(x,t) \geq 1\}$ for $\phi_{k+1} \in C_0^\infty(Q_k)$. Then we obtain that $\omega \in L^\infty_{\text{loc}}(Q_T)$.

## 7.5 Beale-Kato-Majda criterion on finite time singularity

For $u_0 \in H^1(\mathbb{R}^n)$ $(s \geq n)$, thee exists $T_0 > 0$ depending only on $\|u_0\|_{H^s}$ so that the initial value problem of the Navier-Stokes equation has a unique solution $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$.

**Theorem 7.6.** (Beale-Kato-Majda) *Let* $0 < T_*$ *be the maximal time interval. If* $T_* < +\infty$, *then*

$$\int_0^{T_*} \|\nabla \times u(t)\|_{L^\infty(\mathbb{R}^n)}\, dt = +\infty. \tag{7.19}$$

*In particular,*

$$\limsup_{t \uparrow T_*} \|\nabla \times u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

*Proof.* First we observe that

$$T_* < +\infty \text{ iff } \limsup_{t\uparrow T_*} \|u(t)\|_{H^s(\mathbb{R}^n)} = +\infty.$$

We want to prove that if

$$\int_0^{T_*} \|\nabla \times u(t)\|_{L^\infty(\mathbb{R}^n)}\, dt < +\infty, \tag{7.20}$$

then

$$\|u(t)\|_{H^s(\mathbb{R}^n)} \le C_0,\ \forall\, 0 < t < T_*. \tag{7.21}$$

For simplicity, we present the argument for the Euler equation. In this case, the vorticity equation is

$$\omega_t + u\cdot\nabla\omega = \omega\cdot\nabla u.$$

Since $\nabla\times u = \omega$ and $\mathrm{div} u = 0$, we have

$$\|\nabla u\|_{L^2} \le C\|\omega\|_{L^2}.$$

Hence we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 &= -\|\nabla\omega\|_{L^2}^2 + (\omega\cdot\nabla u, \omega)_{L^2}\\
&\le -\|\nabla\omega\|_{L^2}^2 + C\|\omega\|_{L^\infty}\|\omega\|_{L^2}\|\nabla u\|_{L^2}\\
&\le -\|\nabla\omega\|_{L^2}^2 + C\|\omega\|_{L^\infty}\|\omega\|_{L^2}^2.
\end{aligned}$$

By Gronwall's inequality, we have

$$\|\omega(t)\|_{L^2} \le M_0\|\omega(0)\|_{L^2}\ \forall\, 0 \le t < T_*,\ M_0 = \exp(c\int_0^{T_*} |\omega(t)|_{L^\infty}\, dt). \tag{7.22}$$

For $|\alpha| \le s$, let $v = \nabla^\alpha u$. Then we have

$$v_t + u\cdot\nabla v + \nabla q = F := -\nabla^\alpha(u\cdot\nabla u) - v\cdot\nabla(\nabla^\alpha u).$$

By the Leibnitz rule and Sobolev's inequality, we have

$$\|\nabla^\alpha(fg) - f\nabla^\alpha g\|_{L^2} \le C(\|f\|_{H^s}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}}). \tag{7.23}$$

Applying (7.23) to $F$, we obtain

$$\|F\|_{L^2} \le C\|\nabla u\|_{L^\infty}\|u\|_{H^s}.$$

Thus we have

$$\frac{d}{dt}\|u(t)\|_{H^s}^2 \le C\|\nabla u(t)\|_{L^\infty}\|u(t)\|_{H^s}^2, \tag{7.24}$$

and

$$\|u(t)\|_{H^s} \le \|u(0)\|_{H^s}\exp(c\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau). \tag{7.25}$$

Now we need the following key inequality:

$$\|\nabla u(t)\|_{L^\infty} \le C\left(1 + (1 + \ln^+\|u(t)\|_{H^3})\|\omega(t)\|_{L^\infty} + \|\omega(t)\|_{L^2}\right). \tag{7.26}$$

Here

$$\ln^+ a = \begin{cases} \ln a & \text{if } a \geq 1 \\ 0 & \text{if } a < 1. \end{cases}$$

Assume (7.26) for the moment, we proceed as follows.

$$\|\nabla u(t)\|_{L^\infty} \leq C(1 + \ln(e + \|u\|_{H^3})\|\omega(t)\|_{L^\infty}).$$

Set $y(t) = e + \|u(t)\|_{H^s}$. Then we have

$$y(t) \leq y(0) \exp\left(c \int_0^t (1 + \|\omega(\tau)\|_{L^\infty} \ln y(\tau)\, d\tau\right).$$

Set $z(t) = \ln y(t)$. Then $z$ satisfies

$$z(t) \leq z(0) + c \int_0^t (1 + \|\omega(\tau)\|_{L^\infty} \ln y(\tau)\, d\tau.$$

This implies that $z(t)$ is bounded by $T_*$, $\|u_0\|_{H^s}$, and $M_1 = \int_0^{T_*} \|\omega(t)\|_{L^\infty}\, dt$. Hence $\|u(t)\|_{H^s}$ is uniformly bounded for $0 \leq t < T_*$. Thus $T_*$ is not the maximal time interval.

Now we return to the proof of (7.26). To do it, we first recall by the Biot-Savart law,

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)\, dy = \int_{\mathbb{R}^3} K(x-y)\omega(y)\, dy.$$

For $0 < \rho \leq 1$, let $\xi_\rho \in C_0^\infty(\mathbb{R}) \geq 0$ such that $\xi_\rho = \begin{cases} 1 & |x| < \rho \\ 0 & |x| \geq 2\rho \end{cases}$, and $|\nabla \xi_\rho| \leq \frac{2}{\rho}$. Then we can write $\nabla u(x) = \nabla u^1(x) + \nabla u^2(x)$, where

$$\nabla u^1(x) = \int \xi_\rho(x-y) K(x-y) \nabla\omega(y)\, dy,$$

and

$$\nabla u^2(x) = \int \nabla[K(x-y)(1 - \xi_\rho(x-y))]\omega(y)\, dy.$$

We estimate $\nabla u^1$ and $\nabla u^2$ separately as follows. Since $|K(x-y)| \lesssim |x-y|^{-2} \in L^p(B_{2\rho}(x))$ for any $p < \frac{3}{2}$, we have

$$|\nabla u^1(x)| \leq \|K\|_{L^{\frac{4}{3}}(B_{2\rho}(x))} \|\nabla\omega\|_{L^4(B_{2\rho}(x))} \leq C\rho^{\frac{1}{4}} \|\nabla\omega\|_{H^1} \leq C\rho^{\frac{1}{4}} \|u\|_{H^3}.$$

While we can split $\nabla u^2 = \nabla u^3 + \nabla u^4$, where

$$\nabla u^3(x) = \int_{\rho \leq |x-y| \leq 1} \nabla[K(x-y)(1 - \xi_\rho(x-y))]\omega(y)\, dy,$$

and

$$\nabla u^4(x) = \int_{|x-y| \geq 1} \nabla[K(x-y)(1 - \xi_\rho(x-y))]\omega(y)\, dy.$$

For $\nabla u^3$, we have

$$|\nabla u^3(x)| \lesssim [\int_\rho^1 r^{-3} r^2\, dr + \int_\rho^{2\rho} r^{-2}\rho^{-1} r^2\, dr]\|\omega\|_{L^\infty} \leq C(1 + \ln\frac{1}{\rho})\|\omega\|_{L^\infty}.$$

Since $\nabla K \in L^2(\mathbb{R}^3 \setminus B_1(x)$, we have

$$|\nabla u^4(x)| \le C\|\omega\|_{L^\infty}.$$

Putting these estimates together yields

$$|\nabla u\|_{L^\infty} \lesssim (\rho^{\frac{1}{4}}\|u\|_{H^3} + (1 - \ln\rho)\|\omega\|_{L^\infty} + \|\omega\|_{L^2}).$$

If we choose $\rho$ by

$$\rho = \begin{cases} 1 & \text{if } \|u\|_{H^3} \le 1 \\ \|u\|_{H^3}^{-4} & \text{if } \|u\|_{H^3} \ge 1. \end{cases}$$

Then (7.26) follows. The proof is now complete. □

# 8 Caffarelli-Kohn-Nirenberg's theorem on the incompressible Navier-Stokes equation

We consider the Cauchy problem for the incompressible Navier-Stokes equation in $\mathbb{R}^3 \times (0, \infty)$:

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = \Delta v, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \text{div v} = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ v(x,0) = v_0(x), & \text{in } \mathbb{R}^3, \end{cases} \tag{8.1}$$

where $v = v(x,t) \in \mathbb{R}^3$ is the velocity field, $p(x,t)$ is the scalar pressure function, and $v_0(x)$, with div $v_0 = 0$, is the initial velocity field.

The study of the incompressible Navier-Stokes equation in three space dimension has a long history. The existence of Leray-Hopf's solutions has been established by J. Leray in 1934 for $\Omega = \mathbb{R}^n$, and by E. Hopf in 1940 for $\Omega \subset \mathbb{R}^n$ being a bounded smooth domain.

- A typical property of Leray-Hopf's solutions is the weak energy inequality:

$$\|v(.,t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(.,s)\|_{L^2}^2 \, ds \le \|v_0\|_{L^2}^2, \quad t \ge 0. \tag{8.2}$$

- $v \in L^\infty(0,T; L^2(\mathbb{R}^n)) \cap L^2(0,T; H^1(\mathbb{R}^n)), \quad \forall T > 0.$
- $v$ is weakly continuous from $[0,T)$ to $L^2(\mathbb{R}^3)$.
- $v$ verifies (8.1) in the sense of distributions, i.e.,

$$\int_0^T \int_{\mathbb{R}^n} \Big(\frac{\partial \phi}{\partial t} + (v\cdot\nabla)\phi\Big) v \, dxdt + \int_{\mathbb{R}^n} v_0 \phi(x,0)\, dx = \int_0^T \int_{\mathbb{R}^n} \nabla v : \nabla \phi \, dxdt.$$

  for all $\phi \in C_0^\infty(\mathbb{R}^n \times [0,T))$ with div$\phi = 0$, and

$$\int_0^T \int_{\mathbb{R}^n} v \cdot \nabla \phi \, dxdt = 0.$$

  for all $\phi \in C_0^\infty(\mathbb{R}^n \times [0,T))$.

- If $v_0 \in C^\infty(\mathbb{R}^n)$, with div $v_0 = 0$, then there exist $T_0 = T_0(v_0) > 0$ and a unique smooth solution $v \in C^\infty(\mathbb{R}^n \times [0,T_0], \mathbb{R}^n)$ of (8.1).

**Suitable weak solutions and generalized energy inequalities**

A weak solution $(v,p)$ is called a suitable weak solution of (8.1) in $Q_T \equiv \Omega \times [0,T] \subset \mathbb{R}^3 \times (0,\infty)$, provided that the following properties hold:

- $p \in L^{\frac{3}{2}}(Q_T)$ and $L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$.
- $(v,p)$ satisfies (8.1) in the sense of distributions

$$\int\int_{Q_T} v\partial_t \varphi - p\nabla\cdot\varphi + \nabla v \cdot \nabla\varphi + v\cdot\nabla\varphi v \, dxdt = 0, \quad \forall \varphi \in C_c^\infty(Q_T).$$

- $(v, p)$ satisfies the generalized energy inequality:

$$2\int_0^T\int_\Omega |\nabla v|^2\varphi\,dxdt \le \int_0^T\int_\Omega |v|^2(\varphi_t+\Delta\varphi)\,dxdt + \int_0^T\int_\Omega (|v|^2+2p)v\cdot\nabla\varphi\,dxdt, \tag{8.3}$$

  holds for all $\varphi \in C_c^\infty(Q_T),\ \varphi \ge 0$.

**Lemma 8.1.** *If $(v, p)$ is smooth solution of (8.1), then the generalized energy inequality (8.3) must hold.*

*Proof.* . Multiplying (8.1) by $v\varphi$ and integrating over $Q_T$, we have

$$\int_0^T\int_\Omega v_t(v\varphi) + v\cdot\nabla v(v\varphi) + \nabla p\cdot(v\varphi)\,dxdt = -\int_0^T\int_\Omega \nabla v\cdot\nabla(v\varphi)\,dxdt.$$

$$\begin{aligned}
\text{RHS} &= -\int_0^T\int_\Omega \nabla v\cdot\nabla(v\varphi)\,dxdt\\
&= -\int_0^T\int_\Omega |\nabla v|^2\varphi\,dxdt - \int_0^T\int_\Omega \nabla v\cdot\nabla\varphi v\,dxdt\\
&= -\int_0^T\int_\Omega |\nabla v|^2\varphi\,dxdt - \int_0^T\int_\Omega \nabla\varphi\cdot\nabla(\frac{1}{2}|v|^2)\,dxdt\\
&= -\int_0^T\int_\Omega |\nabla v|^2\varphi\,dxdt + \int_0^T\int_\Omega \Delta\varphi(\frac{1}{2}|v|^2)\,dxdt.
\end{aligned}$$

For the terms in the left side, we estimate them one by one as follows:

$$\begin{aligned}
(\text{LHS})_1 &= \int_0^T\int_\Omega \frac{\partial}{\partial t}(\frac{1}{2}|v|^2\varphi)\,dxdt - \int_0^T\int_\Omega \frac{1}{2}|v|^2\partial_t\varphi\,dxdt\\
&= -\int_0^T\int_\Omega \frac{1}{2}|v|^2\partial_t\varphi\,dxdt.
\end{aligned}$$

By the divergence free condition of $v$, we can conclude that

$$\begin{aligned}
(\text{LHS})_2 &= \int_0^T\int_\Omega v\cdot\nabla(\frac{1}{2}|v|^2\varphi)\,dxdt - \int_0^T\int_\Omega \frac{1}{2}|v|^2 v\cdot\nabla\varphi\,dxdt\\
&= -\int_0^T\int_\Omega \mathrm{div}(v)(\frac{1}{2}v^2\varphi)\,dxdt - \int_0^T\int_\Omega \frac{1}{2}|v|^2 v\cdot\nabla\varphi\,dxdt\\
&= -\int_0^T\int_\Omega \frac{1}{2}|v|^2 v\cdot\nabla\varphi\,dxdt.
\end{aligned}$$

Finally we turn to the last term. By the divergence free condition of $v$, we have

$$\begin{aligned}
(\text{LHS})_3 &= \int_0^T\int_\Omega \nabla p(v\varphi)\,dxdt\\
&= -\int_0^T\int_\Omega p\,\mathrm{div}(v\varphi)\,dxdt\\
&= -\int_0^T\int_\Omega pv\cdot\nabla\varphi\,dxdt.
\end{aligned}$$

Putting all these estimates together, we obtain the generalized energy inequality. □

*Remark* 8.1. If $\varphi \in C_0^\infty(\Omega \times (0,t])$, $\varphi \geq 0$, then the generalized energy inequality (8.3) yields

$$\int_\Omega |v|^2 \varphi\, dx\Big|_t + 2\int_0^t \int_\Omega |\nabla v|^2 \varphi\, dxdt \leq \int_0^t \int_\Omega |v|^2 (\varphi_t + \Delta\varphi)\, dxdt + \int_0^t \int_\Omega (|v|^2 + 2p) v \cdot \nabla\varphi\, dxdt. \tag{8.4}$$

*Proof.* For $t_0 > 0$ and $0 < \epsilon < t_0$, let $\eta_\epsilon \in C_0^\infty(\mathbb{R})$ be a cut off function such that

$$\eta_\epsilon(s) = \begin{cases} 1, & 0 \leq s \leq t_0 - \epsilon, \\ \text{linear}, & \text{otherwise}, \\ 0, & s \geq t_0. \end{cases} \tag{8.5}$$

Then $\varphi(x,t)\eta_\epsilon(t) \in C_0^\infty(\Omega \times (0,t_0))$ and the previous energy inequality yields

$$\begin{aligned} 2\int_0^{t_0} \int_\Omega |\nabla v|^2 \varphi \eta_\epsilon\, dxdt \quad \leq \quad & \int_0^{t_0} \int_\Omega |v|^2 \Big[ (\varphi_t + \Delta\varphi)\eta_\epsilon + \varphi\eta_\epsilon' \Big]\, dxdt \\ & + \int_0^{t_0} \int_\Omega (|v|^2 + 2p) v \cdot (\nabla\varphi\eta_\epsilon)\, dxdt. \end{aligned}$$

Taking $\epsilon \downarrow 0$, we have

$$\begin{aligned} 2\int_0^{t_0} \int_\Omega |\nabla v|^2\, dxdt \leq & \int_0^{t_0} \int_\Omega |v|^2 \Big[ (\varphi_t + \Delta\varphi) \Big]\, dxdt + \int_0^{t_0} \int_\Omega (|v|^2 + 2p) v \cdot (\nabla\varphi)\, dxdt \\ & + \lim_{\epsilon \downarrow 0} \int_0^{t_0} \int_\Omega |v|^2 \varphi \eta_\epsilon'\, dxdt. \end{aligned}$$

Thanks to the definition of $\eta_\epsilon$, it is easy to show that

$$\lim_{\epsilon \downarrow 0} \int_0^{t_0} \int_\Omega |v|^2 \varphi \eta_\epsilon'\, dxdt = -\int_\Omega |v|^2 \varphi(x,t_0)\, dx.$$

Thus we can get

$$\begin{aligned} \int_\Omega |v|^2(x,t_0)\varphi(x,t_0)\, dx + 2\int_0^{t_0} \int_\Omega |\nabla v|^2 \varphi\, dxdt \leq & \int_0^{t_0} \int_\Omega |v|^2 (\varphi_t + \Delta\varphi)\, dxdt \\ & + \int_0^{t_0} \int_\Omega (|v|^2 + 2p) v \cdot \nabla\varphi\, dxdt. \end{aligned}$$

□

*Remark* 8.2. Now we make some comments:

- It is an open problem whether Leray-Hopf's weak solutions (e.g., constructed by Galerkin's method) are suitable weak solutions.
- However, Caffarelli-Kohn-Nirenberg did obtain the existence of suitable weak solutions by a different method.

**Scheffer's partial regularity**

(1) It is well-known that if $(u, p)$ solves the Navier-Stokes equation, then so does $(u_\lambda, p_\lambda)$ for all $\lambda > 0$ in $\mathbb{R}^n$, where

$$\begin{cases} u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \\ p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t). \end{cases}$$

(2) If $v \in L_t^\infty L_x^2(Q_T) \cap L_t^2 H_x^1(Q_T)$, then $v \in L^{\frac{10}{3}}(Q_T)$.

*Proof.* It is a direct consequence of interpolations. For the convenience, we present the details. For $2 \le p \le 2^*(=6)$, one has

$$\|v(t)\|_{L^p(\mathbb{R}^3)} \le \|v(t)\|_{L^2(\mathbb{R}^3)}^\theta \|v(t)\|_{L^{2^*}(\mathbb{R}^3)}^{1-\theta},$$

where

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2^*}. \tag{8.6}$$

Taking the $L^q$−norm with respect to time variable, we have

$$\begin{aligned} \Big(\int_0^T \|v(t)\|_{L^p}^q \, dt\Big)^{\frac{1}{q}} \le& \Big(\int_0^T \|v(t)\|_{L^2}^{q\theta} \|v(t)\|_{L^{2^*}}^{q(1-\theta)} \, dt\Big)^{\frac{1}{q}} \\ \le& \|v\|_{L_t^\infty L_x^2}^\theta \Big(\int_0^T \|\nabla v(t)\|_{L^2}^{q(1-\theta)} \, dt\Big)^{\frac{1}{q}}. \end{aligned}$$

Choose $q$ such that $q(1-\theta) = 2$. Together with (8.6), we can show that

$$\frac{1}{p} = \frac{1-\frac{2}{q}}{2} + \frac{1}{3q} = \frac{1}{2} - \frac{1}{q} + \frac{1}{3q} = \frac{1}{2} + \frac{2}{3q},$$

or equivalently,

$$\frac{3}{p} + \frac{2}{q} = \frac{3}{2}. \tag{8.7}$$

Thus we have

$$\|v(x,t)\|_{L_t^q L_x^p} \lesssim \|v(x,t)\|_{L_t^\infty L_x^2}^{1-\frac{2}{q}} \|v(x,t)\|_{L_t^2 L_x^2}^{\frac{2}{q}}.$$

Choose $p = q = \frac{10}{3}$. The proof is complete. □

(3) Leray-Hopf solutions satisfy the following estimates:

- $\int_0^T \int_\Omega (|v|^{\frac{10}{3}} + |p|^{\frac{5}{3}}) \, dxdt < \infty$,
- $\int_0^T \int_\Omega |\nabla v|^2 \, dxdt < \infty$.

**Theorem 8.2** ($\varepsilon_0$-regularity)**.** *Let $Q_r \triangleq \{(x,t) | \quad |x| \le r, \ -r^2 \le t \le 0\}$. There exists $\varepsilon_0 > 0$ such that if $(v, p)$ is a suitable weak solution of (8.1) in $Q_r$ and satisfies*

$$r^{-2} \int_{Q_r} (|v|^3 + |p|^{\frac{3}{2}}) \, dxdt \le \varepsilon_0,$$

*then $v \in C^\infty(Q_{\frac{r}{2}}, \mathbb{R}^3)$ and $\|v\|_{C^k(Q_{\frac{r}{2}})} \le C(\varepsilon_0, k, r)$.*

**Lemma 8.3** ($\varepsilon_0$−decay). *There exist $\varepsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if $(v, p)$ is a suitable weak solution of (8.1) in $Q_r$ satisfying*

$$r^{-2}\int_{Q_r}(|v|^3 + |p|^{\frac{3}{2}})\,dxdt \le \varepsilon_0,$$

*then*

$$(\theta_0 r)^{-2}\int_{Q_{\theta_0 r}}(|v|^3 + |p|^{\frac{3}{2}})\,dxdt \le \frac{1}{2}r^{-2}\int_{Q_r}(|v|^3 + |p|^{\frac{3}{2}})\,dxdt.$$

*Proof.* (By contradiction)
Firstly, by scalings, we may assume that $r = 1$. If the conclusion were false, then for any $\theta \in (0, \frac{1}{2})$, there would exist a sequence of suitable weak solutions $(v_i, p_i)$ of (1.1) that satisfying

$$\Big(\int_{Q_1}|v_i|^3\,dxdt\Big)^{\frac{1}{3}} + \Big(\int_{Q_1}|p_i|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}} = \varepsilon_i \to 0,$$

but

$$\Big(\theta^{-2}\int_{Q_\theta}|v_i|^3\,dxdt\Big)^{\frac{1}{3}} + \Big(\theta^{-2}\int_{Q_\theta}|p_i|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}} > \frac{1}{2}\varepsilon_i.$$

Next we define the blow-up sequence

$$u_i = \frac{v_i}{\varepsilon_i}, \qquad Q_i = \frac{p_i}{\varepsilon_i}.$$

Then one has

$$\Big(\int_{Q_1}|u_i|^3\,dxdt\Big)^{\frac{1}{3}} + \Big(\int_{Q_1}|Q_i|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}} = \frac{\Big(\int_{Q_1}|v_i|^3\,dxdt\Big)^{\frac{1}{3}} + \Big(\int_{Q_1}|p_i|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}}}{\varepsilon_i} = 1,$$

while

$$\begin{aligned}\Big(\theta^{-2}\int_{Q_\theta}|u_i|^3\,dxdt\Big)^{\frac{1}{3}} + \Big(\theta^{-2}\int_{Q_\theta}|Q_i|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}} &= \frac{1}{\varepsilon_i}\Big(\theta^{-2}\int_{Q_\theta}|v_i|^3\,dxdt\Big)^{\frac{1}{3}} \\ &+\Big(\theta^{-2}\int_{Q_\theta}|p_i|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}} \\ &> \frac{1}{2}.\end{aligned}$$

It is easy to show that $(u_i, Q_i)$ satisfies the following blow-up equations

$$\begin{cases}\partial_t u_i + \varepsilon_i u_i\cdot\nabla u_i + \nabla Q_i = \Delta u_i, & \text{in } \mathbb{R}^3\times(0,\infty),\\ \operatorname{div} u_i = 0, & \textit{in } \mathbb{R}^3\times(0,\infty).\end{cases} \tag{8.8}$$

We may assume that

$$u_i \rightharpoonup u \text{ weakly in } L^3(Q_1), \qquad Q_i \rightharpoonup Q \text{ weakly in } L^{\frac{3}{2}}(Q_1).$$

Then we can show $(u, Q)$ solves the linear Stokes equation

$$\begin{cases}\partial_t u + \nabla Q = \Delta u, & \text{in } \mathbb{R}^3\times(0,\infty),\\ \operatorname{div} u = 0, & \textit{in } \mathbb{R}^3\times(0,\infty).\end{cases} \tag{8.9}$$

and by the lower semicontinuity,

$$\Big(\int_{Q_1}|u|^3\,dxdt\Big)^{\frac13}+\Big(\int_{Q_1}|Q|^{\frac32}\,dxdt\Big)^{\frac23}\le 1,$$

By the regularity of the Stokes equation, we have $(u,Q)\in C^\infty(Q_{\frac12})$ and

$$\begin{aligned}&\Big(\theta^{-2}\int_{Q_\theta}|u|^3\,dxdt\Big)^{\frac13}+\Big(\theta^{-2}\int_{Q_\theta}|Q|^{\frac32}\,dxdt\Big)^{\frac23}\\ \le& C\theta\Big\{\Big(\int_{Q_1}|u|^3\,dxdt\Big)^{\frac13}+\Big(\int_{Q_1}|Q|^{\frac32}\,dxdt\Big)^{\frac23}\Big\}\\ \le& C\theta.\end{aligned}$$

Now we want to show that

$$\Big(\theta^{-2}\int_{Q_\theta}|u_i|^3\,dxdt\Big)^{\frac13}\approx\Big(\theta^{-2}\int_{Q_\theta}|u|^3\,dxdt\Big)^{\frac13}+o(\frac1i)$$

and

$$\Big(\theta^{-2}\int_{Q_\theta}|Q_i|^{\frac32}\,dxdt\Big)^{\frac23}\approx\Big(\theta^{-2}\int_{Q_\theta}|Q|^{\frac32}\,dxdt\Big)^{\frac23}+o(\frac1i).$$

Suppose that these were established. Then we reach the desired contradiction.

By the Aubin-Lions lemma, whose condition will be verified below, the generalized energy inequality for $(v_i,p_i)$: for $\forall-\frac14\le t\le 0$ and $\varphi\in C_c^\infty(B_1\times[-1,t])$ with $\phi\ge 0$,

$$\begin{aligned}&\int_{B_1}|v_i|^2(x,t)\varphi(x,t)\,dx+2\int_0^t\int_{B_1}|\nabla v_i|^2(x,t)\varphi(x,t)\,dxdt\\ &\le\int_0^t\int_{B_1}|v_i|^2(\varphi_t+\Delta\varphi)\,dxdt+\int_0^t\int_{B_1}(|v_i|^2+2p_i)v_i\cdot\nabla\varphi\,dxdt,\end{aligned}$$

yields that $(u_i,Q_i)$ satisfies

$$\begin{aligned}&\int_{B_1}|u_i|^2(x,t)\varphi(x,t)\,dx+2\int_0^t\int_{B_1}|\nabla u_i|^2(x,t)\varphi(x,t)\,dxdt\\ &\le\int_0^t\int_{B_1}|u_i|^2(\varphi_t+\Delta\varphi)\,dxdt+\int_0^t\int_{B_1}(\varepsilon_i|u_i|^2+2Q_i)u_i\cdot\nabla\varphi\,dxdt,\end{aligned}$$

Therefore, we can deduce that

$$\begin{aligned}&\sup_{-\frac14\le t\le 0}\int_{B_{\frac12}}|u_i|^2(x,t)\,dx+2\int_{-\frac14}^0\int_{B_{\frac12}}|\nabla u_i|^2(x,t)\,dxdt\\ \lesssim&\int_{P_1}(|u_i|^2+\varepsilon_i|u_i|^3+|Q_i||u_i|)\,dxdt,\\ \lesssim&\Big(\int_{P_1}|u_i|^3\,dxdt\Big)^{\frac23}+\varepsilon_i\int_{P_1}|u_i|^3\,dxdt+\Big(\int_{P_1}|Q_i|^{\frac32}\,dxdt\Big)^{\frac23}\Big(\int_{P_1}|u_i|^3\,dxdt\Big)^{\frac13}\\ \lesssim&1,\end{aligned}$$

where we have used the Hölder inequality.

Now we verify the condition of Aubin-Lions' lemma.

$$\partial_t u_i = -\varepsilon_i u_i \cdot \nabla u_i - \nabla Q_i + \Delta u_i,$$

It is not hard to see that

$$\Delta u_i \in H^{-1}(\mathbb{R}^3), \quad \nabla Q_i \in (W_0^{1,3})^* = W^{-1,\frac{3}{2}}.$$

and

$$\varepsilon_i u_i \cdot \nabla u_i \in L^{\frac{5}{4}}(\mathbb{R}^3),$$

because that $u \in L^{\frac{10}{3}}(Q_T)$ and $\nabla u \in L^2(Q_T)$.
Therefore

$$\partial_t u_i \in L_t^2 H_x^{-1} + L_t^{\frac{5}{4}} L_x^{\frac{5}{4}} + L_t^{\frac{3}{2}} W_x^{-1,\frac{3}{2}},$$

and

$$\left\|\partial_t u_i\right\|_{L_t^2 H_x^{-1} + L_t^{\frac{5}{4}} L_x^{\frac{5}{4}} + L_t^{\frac{3}{2}} W_x^{-1,\frac{3}{2}}(Q_{\frac{1}{2}})}$$

is bounded uniformly in $i$. By the Aubin-Lions Lemma, we conclude that $\{u_i\}_{i=1}^{\infty} \subset L^2(P_{\frac{1}{2}})$ is pre-compact. Thus, after taking a subsequence if necessary, we may assume that

$$u_i \to u \text{ strongly in } L^3(Q_{\frac{1}{2}})$$

which implies that

$$\begin{aligned}\Big(\theta^{-2}\int_{P_\theta} |u_i|^3\, dxdt\Big)^{\frac{1}{3}} =& \Big(\theta^{-2}\int_{P_\theta} |u|^3\, dxdt\Big)^{\frac{1}{3}} + o(\frac{1}{i})\\ \lesssim& C\theta + o(\frac{1}{i}).\end{aligned}$$

Since $Q_i$ satisfies the Poisson equation: for any $t \in [-1, 0]$,

$$-\Delta Q_i = \varepsilon_i \mathrm{div}(u_i \cdot \nabla u_i) = \varepsilon_i \mathrm{div}(\mathrm{div}(u_i \otimes u_i)) \quad in\ B_1.$$

Let $\widetilde{Q_i} : \mathbb{R}^3 \to \mathbb{R}$ satisfy

$$\widetilde{Q_i}(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} D^2_{\alpha,\beta}(\frac{1}{|x-y|^3}) : \chi_{B_1}(y)\varepsilon_i u_i^\alpha u_i^\beta(y,t)\, dy.$$

Then

$$-\Delta\widetilde{Q_i} = \varepsilon_i \mathrm{div}(\mathrm{div}(u_i \otimes u_i)) \quad \text{in } B_1.$$

Hence

$$-\Delta(Q_i - \widetilde{Q_i}) = 0 \quad in\ B_1.$$

One thus deduces from the boundedness of Calderon-Zygmund operators, we have

$$\begin{aligned}\|\widetilde{Q_i}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim& \varepsilon_i \|u_i \otimes u_i\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}^3)}\\ \lesssim& \varepsilon_i \|u_i\|^3_{L^3(\mathbb{R}^3)}.\end{aligned}$$

Thus we obtain

$$\int_{-1}^0\int_{\mathbb{R}^3} |\widetilde{Q_i}|^{\frac{3}{2}}\, dxdt \lesssim \varepsilon_i \int_{-1}^0\int_{B_1} |u_i|^3\, dxdt \leq C\varepsilon_i.$$

By the mean value property of harmonic functions, we have

$$\theta^{-3}\int_{B_\theta}|Q_i-\widetilde{Q}_i|^{\frac{3}{2}}\,dx \le \int_{B_1}|Q_i-\widetilde{Q}_i|^{\frac{3}{2}}\,dx.$$

Therefore

$$\theta^{-2}\int_{P_\theta}|Q_i-\widetilde{Q}_i|^{\frac{3}{2}}\,dx \le C\theta.$$

Thus we have

$$\begin{aligned}\Big(\theta^{-2}\int_{P_\theta}|Q_i|^{\frac{3}{2}}\,dx\Big)^{\frac{2}{3}} \lesssim&\Big(\theta^{-2}\int_{-1}^{0}\int_{\mathbb{R}^3}|\widetilde{Q}_i|^{\frac{3}{2}}\,dx\Big)^{\frac{2}{3}}+\Big(\theta^{-2}\int_{P_\theta}|Q_i-\widetilde{Q}_i|^{\frac{3}{2}}\,dx\Big)^{\frac{2}{3}}\\ \le& C\theta^{\frac{2}{3}}+(\varepsilon_i\theta^2)^{\frac{2}{3}}\\ \le&\frac{1}{8},\end{aligned}$$

provided that $i$ is chosen sufficiently large and $\theta$ is chosen sufficiently small. This contradicts the choices of $(v_i, p_i)$. □

**Lemma 8.4.** *There exist $\varepsilon_0 > 0$ and $\alpha_0 \in (0, \frac{1}{2})$ such that if $(v, p)$ is a suitable weak solution of (8.1) in $P_r$ satisfying*

$$\Big(r^{-2}\int_{P_r(x_0,t_0)}|v|^3\,dxdt\Big)^{\frac{1}{3}}+\Big(r^{-2}\int_{P_r(x_0,t_0)}|p|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}}\le\varepsilon_0,$$

*then for any $(x_1, t_1) \in P_{\frac{r}{2}}(x_0, t_0)$ and $0 < \tau \le \frac{r}{2}$*

$$\Big(\tau^{-2}\int_{P_\tau(x_1,t_1)}|v|^3\,dxdt\Big)^{\frac{1}{3}}+\Big(\tau^{-2}\int_{P_\tau(x_1,t_1)}|p|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}}\le C(\varepsilon_0)\tau^{\alpha_0}.$$

*Proof.* For simplicity, we assume $(x_0, t_0) = (0, 0)$ and $(x_1, t_1) = (0, 0)$. Iterating the above process $k$-times, we arrive at

$$\begin{aligned}&\Big((\theta^k r)^{-2}\int_{P_{\theta^k r}}|v|^3\,dxdt\Big)^{\frac{1}{3}}+\Big((\theta^k r)^{-2}\int_{P_{\theta^k r}}|p|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}}\\ \le&(\frac{1}{2})^k\Big\{\Big(r^{-2}\int_{P_r}|v|^3\,dxdt\Big)^{\frac{1}{3}}+\Big(r^{-2}\int_{P_r}|p|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}}\Big\}.\end{aligned}$$

For $0 < \tau \le \frac{r}{2}$, there exists $k \ge 1$ such that $\theta^{k+1}r \le \tau \le \theta^k r$.
Hence

$$\theta^k \approx \frac{\tau}{r} \Rightarrow k \approx \frac{\ln(\frac{\tau}{r})}{\ln\theta}.$$

Therefore

$$\begin{aligned}\Big(\tau^{-2}\int_{P_\tau(x_1,t_1)}|v|^3\,dxdt\Big)^{\frac{1}{3}}+&\Big(\tau^{-2}\int_{P_\tau(x_1,t_1)}|p|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}}\\ &\le(\frac{1}{2})^{\frac{\ln(\frac{\tau}{r})}{\ln\theta}}\varepsilon_0\\ &\le(\frac{\tau}{r})^{\alpha_0}\varepsilon_0,\end{aligned}$$

where $\alpha_0 = \frac{\ln\frac{1}{2}}{\ln\theta} \in (0, 1)$. □

**Riesz potential estimates between on Morrey spaces**

- Morrey spaces: For $1 \le p \le \infty$ and $0 \le \lambda \le 5$, define

$$M^{p,\lambda}(\mathbb{R}^3 \times \mathbb{R}) \equiv \Big\{ f \in L^p_{loc}(\mathbb{R}^3 \times \mathbb{R}) |\, \|f\|_{M^{p,\lambda}(\mathbb{R}^3\times\mathbb{R})} < \infty \Big\},$$

  where

$$\|f\|^p_{M^{p,\lambda}(\mathbb{R}^3\times\mathbb{R})} \equiv \Big\{ \sup_{z_0 \in \mathbb{R}^3\times\mathbb{R},\ 0<r<\infty} r^{\lambda-5} \int\int_{P_r(z_0)} |f|^p\, dydt \Big\}.$$

- Let $\eta \in C_0^\infty(P_r(0,0))$ such that

$$0 \le \eta \le 1,\ \phi \equiv 1\ in\ P_{\frac{r}{2}}(0,0),\ |\nabla^2\eta| + |\phi_t| + |\nabla\eta|^2 \lesssim \frac{1}{r^2}.$$

Define $v$ by

$$\begin{aligned} v^i(x,t) = &- \int_{\mathbb{R}^4} \nabla_j H(x-y, t-s)[\eta^2 u^i u^j](y,s)\, dyds \\ &- \int_{\mathbb{R}^4} \nabla_i H(x-y,t-s)\eta^2 p(y,s)\, dyds, \end{aligned}$$

where

$$H(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-(\frac{|x|^2}{4t})}$$

is the heat kernel in $\mathbb{R}^3$. Note that

$$|\nabla H(x,t)| \lesssim \frac{1}{\delta((x,t),(0,0))^{5-1}},$$

where

$$\delta((x,t),(0,0)) \triangleq \max\{|x|,\ \sqrt{|t|}\}$$

is the parabolic norm in the space $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$.

Define the parabolic Riesz potential of $\alpha$-order:

$$I_\alpha f(x,t) \equiv \int_{\mathbb{R}^4} \frac{|f(y,s)|}{\delta(x-y,t-s)^{5-\alpha}}\, dyds$$

for $0 \le \alpha \le 5$. Therefore we have

$$|v(x,t)| \lesssim I_1(\eta^2(|u|^2 + |p|))(x,t).$$

Note that

$$\eta^2(|u|^2 + |p|) \in M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)$$

and

$$\Big\|\eta^2(|u|^2 + |p|)\Big\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \le C\varepsilon_0.$$

**Lemma 8.5.** *For $1 < p < \lambda$ and $0 < \lambda < 5$, $I_1 : M^{p,\lambda}(\mathbb{R}^4) \hookrightarrow M^{\widetilde{p},\lambda}(\mathbb{R}^4)$, where $\widetilde{p} = \frac{\lambda p}{\lambda - p}$. Moreover*

$$\|I_1(f)\|_{M^{\widetilde{p},\lambda}(\mathbb{R}^4)} \lesssim \|f\|_{M^{p,\lambda}(\mathbb{R}^4)}.$$

*Proof.* The proof can be founded in Huang-Wang's paper. □

Now we continue the proof. By the lemma, we can obtain

$$\|v\|_{M^{\widetilde{p},3-3\alpha}(\mathbb{R}^4)} \lesssim \|\eta^2(|u|^2)+|p|)\|_{M^{\frac{3}{2},3-3\alpha}(\mathbb{R}^4)},$$

where

$$\widetilde{p} = \frac{\frac{3}{2}(3-3\alpha)}{\frac{3}{2}-(3-3\alpha)} = \frac{\frac{3}{2}(3-3\alpha)}{3\alpha-\frac{3}{2}} = \frac{3(1-\alpha)}{2\alpha-1} \to \infty \quad \text{as} \quad \alpha \downarrow \frac{1}{2}.$$

Hence we have

$$\|v\|_{L^q(P_r)} \le C(q,r)\Big\{\|u\|_{L^3(P_r)} + \|p\|_{L^{\frac{3}{2}}(P_r)}\Big\}$$

Since

$$\partial_t v - \Delta v = -(u\cdot\nabla u + \nabla p) \quad in \quad P_{\frac{r}{2}},$$

it follows

$$\partial_t(u-v) = 0 \quad in \quad P_{\frac{r}{2}}.$$

Thus

$$u - v \in L^\infty(P_{\frac{r}{4}}).$$

Therefore we obtain that for any $1<q<\infty$

$$u \in L^q(P_{\frac{r}{4}}).$$

Since

$$-\Delta p = \operatorname{div}(\operatorname{div}(u\otimes u)) \quad \text{in } B_1,$$

one also has that $p \in L^q(P_{\frac{r}{8}})$ and

$$\int_{P_{\frac{r}{8}}} |p|^q \lesssim (\int_{P_1} |p|^2)^{\frac{q}{2}} + \int_{P_1} |u|^{2q}.$$

Therefore we have that for any $1<q<\infty$, $(u,p)\in L^q(P_{\frac{r}{8}})$ Hence $v \in C^\infty(P_{\frac{r}{8}},\mathbb{R}^3)$ and $\|v\|_{C^k(P_{\frac{r}{8}})} \le C(\varepsilon_0,k,r)$.

**Strong version of $\varepsilon_0$-regularity**

**Theorem 8.6.** *There exists $\varepsilon_0>0$ if $(v,p)$ is a suitable weak solution satisfying*

$$\overline{\lim_{r\to 0}}\, r^{-1}\int\int_{P_r} |\nabla v|\,dxdt \le \varepsilon_0, \tag{8.10}$$

*then $\exists\ \theta_0 \in (0,1)$ and $r_0\in(0,1)$ such that either*

$$A^{\frac{3}{2}}(\theta_0 r) + D^2(\theta_0 r) \le \frac{1}{2}(A^{\frac{3}{2}}(r)+D^2(r)), \tag{8.11}$$

*or*

$$(A^{\frac{3}{2}}(r)+D^2(r)) \le \varepsilon_1 \ll 1 \qquad \textit{where } 0<r<r_0. \tag{8.12}$$

*Here*

$$A(r) \equiv \sup_{-r^2\le t\le 0} r^{-1}\int_{B_r} |v|^2(x,t)\,dx, \qquad D(r) \equiv r^{-2}\int\int_{P_r} |p|^{\frac{3}{2}}(x,t)\,dxdt.$$

**Preparation of the proof.**

I) Some interpolation inequalities: For $B_r \subset \mathbb{R}^3$ (the ball of radial $r$) and for every $2 \le q \le 6$, and $a = \frac{3}{2}(1-\frac{q}{6})$, we have

$$\int_{B_r} |v|^q(x,t)\,dx \lesssim \Big(\int_{B_r} |\nabla v|^2(x,t)\,dx\Big)^{\frac{q}{2}-a}\Big(\int_{B_r} |v|^2(x,t)\,dx\Big)^a + r^{3(1-\frac{q}{2})}\Big(\int_{B_r} |v|^2(x,t)\,dx\Big)^{\frac{q}{2}}. \tag{8.13}$$

**Proof of inequality (8.13)**. First, we have

$$\begin{aligned}\Big(\int_{B_r} |v|^q\,dx\Big)^{\frac{1}{q}} &\lesssim \Big(\int_{B_r} |v-v_r|^q\,dx\Big)^{\frac{1}{q}} + r^{\frac{3}{q}}\frac{1}{|B_r|}\int_{B_r}|v|\,dx\\ &\lesssim \Big(\int_{B_r} |v-v_r|^2\,dx\Big)^{\frac{\theta}{2}}\Big(\int_{B_r} |v-v_r|^6\,dx\Big)^{\frac{1-\theta}{6}} + r^{\frac{3}{q}}\frac{1}{|B_r|}\Big(\int_{B_r}|v|^2\,dx\Big)^{\frac{1}{2}}|B_r|^{\frac{1}{2}}\end{aligned}$$

where $\theta \in (0,1)$ satisfies

$$\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{6} \tag{8.14}$$

$$\lesssim \Big(\int_{B_r} |v|^2\,dx\Big)^{\frac{\theta}{2}}\Big(\int_{B_r} |\nabla v|^2\,dx\Big)^{\frac{1-\theta}{2}} + r^{\frac{3}{q}-\frac{3}{2}}\Big(\int_{B_r}|v|^2\,dx\Big)^{\frac{1}{2}}.$$

Thus we can get the following inequality

$$\int_{B_r} |v|^q\,dx \lesssim \Big(\int_{B_r} |v|^2\,dx\Big)^{\frac{q\theta}{2}}\Big(\int_{B_r} |\nabla v|^2\,dx\Big)^{\frac{q(1-\theta)}{2}} + r^{3-\frac{3q}{2}}\Big(\int_{B_r}|v|^2\,dx\Big)^{\frac{q}{2}}.$$

Now we set $a = \frac{q\theta}{2}$ and we have from (8.14) that $\theta = (\frac{1}{q}-\frac{1}{6})\times 3 = \frac{3}{q}-\frac{1}{2}$. Hence

$$a = \frac{q}{2}(\frac{3}{q}-\frac{1}{2}) = \frac{3}{2}(1-\frac{q}{6}) \in (0,\frac{3}{2}).$$

II) Next we define some quantities which are useful as follows

$$A(r) \equiv \sup_{-r^2\le t\le 0} \frac{1}{r}\int_{B_r} |v|^2(x,t)\,dx,$$

$$B(r) \equiv \frac{1}{r}\int_{P_r} |\nabla v|^2(x,t)\,dxdt,$$

$$C(r) \equiv \frac{1}{r^2}\int_{P_r} |v|^3(x,t)\,dx,$$

and

$$P_r \equiv B_r \times [-r^2, 0],$$

**Lemma 8.7.** *For any $v \in L^\infty([-r^2,0];L^2)\cap L^2([-r^2,0];H^1)$ it holds for any $0 < r \le \rho$*

$$C(r) \lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho). \tag{8.15}$$

*Proof.* With the help of (8.13), we obtain

$$\int_{B_r}|v|^3(x,t)\,dx \lesssim \Big(\int_{B_r}|\nabla v|^2(x,t)\,dx\Big)^{\frac34}\Big(\int_{B_r}|v|^2(x,t)\,dx\Big)^{\frac34}+r^{-\frac32}\Big(\int_{B_r}|v|^2(x,t)\,dx\Big)^{\frac32}. \quad (8.16)$$

Some computations show that

$$\begin{aligned}\int_{B_r}|v|^2\,dx &\lesssim \int_{B_r}\Big||v|^2-(|v|^2)_{B_\rho}\Big|\,dx+\Big(\frac{r}{\rho}\Big)^3\int_{B_\rho}|v|^2\,dx\\ &\lesssim \rho\int_{B_\rho}|v||\nabla v|\,dx+\Big(\frac{r}{\rho}\Big)^3\int_{B_\rho}|v|^2\,dx\\ &\lesssim \rho^{\frac32}\Big(\rho^{-1}\int_{B_\rho}|v|^2\,dx\Big)^{\frac12}\Big(\int_{B_\rho}|\nabla v|^2\,dx\Big)^{\frac12}+\Big(\frac{r}{\rho}\Big)^3\int_{B_\rho}|v|^2\,dx\\ &\lesssim \rho^{\frac32}A^{\frac12}(\rho)\Big(\int_{B_\rho}|\nabla v|^2\,dx\Big)^{\frac12}+\Big(\frac{r}{\rho}\Big)^3\rho A(\rho). \quad (8.17)\end{aligned}$$

Substituting the estimate (1.8) into the second term of the right hand side of (8.16), we can conclude that

$$\begin{aligned}\int_{B_r}|v|^3(x,t)\,dx &\lesssim \rho^{\frac34}\Big(\rho^{-1}\int_{B_r}|\nabla v|^2(x,t)\,dx\Big)^{\frac34}\Big(\int_{B_r}|v|^2(x,t)\,dx\Big)^{\frac34}+r^{-\frac32}\Big(\int_{B_r}|v|^2(x,t)\,dx\Big)^{\frac32}\\ &\lesssim \rho^{\frac34}A^{\frac34}(\rho)\Big(\int_{B_r}|\nabla v|^2(x,t)\,dx\Big)^{\frac34}+r^{-\frac32}\Big(\int_{B_r}|v|^2(x,t)\,dx\Big)^{\frac32}\\ &\lesssim \Big\{\rho^{\frac34}+\frac{\rho^{\frac94}}{r^{\frac32}}\Big\}\Big(\int_{B_r}|\nabla v|^2(x,t)\,dx\Big)^{\frac34}A^{\frac34}(\rho)+\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho).\end{aligned}$$

Integrating the resulting inequality over $[-r^2,0]$ together with Hölder's inequality yields

$$\begin{aligned}\frac{1}{r^2}\int_{P_r}|v|^3(x,t)\,dx &\lesssim\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho)+\Big\{\rho^{\frac34}+\frac{\rho^{\frac94}}{r^{\frac32}}\Big\}\int_{-r^2}^{0}\Big(\int_{B_r}|\nabla v|^2(x,t)\,dx\Big)^{\frac34}dtA^{\frac34}(\rho)\\ &\lesssim\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho)+r^{-\frac32}\Big\{\rho^{\frac34}+\frac{\rho^{\frac94}}{r^{\frac32}}\Big\}A^{\frac34}(\rho)B^{\frac34}(\rho)\rho^{\frac34}\\ &\lesssim\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho)+\Big\{(\frac{\rho}{r})^{\frac32}+(\frac{\rho}{r})^3\Big\}A^{\frac34}(\rho)B^{\frac34}(\rho)\\ &\lesssim\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho)+(\frac{\rho}{r})^3A^{\frac34}(\rho)B^{\frac34}(\rho).\end{aligned}$$

Thus we get

$$C(r)\lesssim\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho)+\Big(\frac{\rho}{r}\Big)^3A^{\frac34}(\rho)B^{\frac34}(\rho).$$

This completes the proof. □

**Lemma 8.8** (pressure estimate)**.** *Let $(v,p)$ be a weak solution of (8.1) in $P_1$. Then for any $0<r\le 1$ and $0<\tau\le\frac{r}{2}$, it holds*

$$\frac{1}{\tau^2}\int_{P_\tau}|p|^{\frac32}(x,t)\,dxdt\lesssim\Big(\frac{r}{\tau}\Big)^2\frac{1}{r^2}\int_{P_r}|v-v_r(t)|^3(x,t)\,dxdt+\frac{\tau}{r}\frac{1}{r^2}\int_{P_r}|p|^{\frac32}(x,t)\,dxdt. \quad (8.18)$$

*Proof.* Since all the quantities are scaling invariant , we only consider the case $r = 1$. Taking use of the divergence-free condition of $v$, we deduce from (8.1) that

$$-\Delta p = \operatorname{div}(v\cdot\nabla v) = \operatorname{div}(\operatorname{div}(v\otimes v)) = \operatorname{div}(\operatorname{div}((v-v_1)\otimes(v-v_1))).$$

Here $v_1$ is the average of $v$ over $P_1$. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut off function of $B_{\frac{1}{2}}$ such that

$$\begin{cases} \eta \equiv 1, & in\ B_{\frac{1}{2}}, \\ \eta \equiv 0, & in\ \mathbb{R}^n \setminus B_1, \\ 0 \le \eta \le 1, & |\nabla\eta| \le 8. \end{cases} \tag{8.19}$$

Now we define an axillary function

$$\widetilde{p}(x,t) = -\int_{\mathbb{R}^3} \nabla_y^2 G(x-y) : \eta^2(y)(v-v_1)\otimes(v-v_1)(y,t)\,dy.$$

By an easy calculation, we have that

$$-\Delta\widetilde{p} = \operatorname{div}(\operatorname{div}((v-v_1)\otimes(v-v_1))) \quad in\ B_{\frac{1}{2}},$$

$$-\Delta(p-\widetilde{p}) = 0 \quad in\ B_{\frac{1}{2}}.$$

One thus deduces from the boundedness of Calderon-Zygmund operators shows that

$$\|\widetilde{p}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim \|\eta^2(v-v_1)^2\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim \int_{B_1} |v-v_1|^3\,dx.$$

Together with the change of variable, we have

$$\frac{1}{\tau^2}\|p-\widetilde{p}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_\tau)} \lesssim \tau\|p-\widetilde{p}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1)} \lesssim \tau(\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1)} + \|\widetilde{p}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1)}).$$

Integrating above inequality over $[-r^2, 0]$, we get

$$\frac{1}{\tau^2}\int_{P_\tau} |p|^{\frac{3}{2}}(x,t)\,dxdt \lesssim \tau\Big(\int_{P_1} |p|^{\frac{3}{2}}(x,t)\,dxdt + \int_{P_1} |v-v_1|^3\,dxdt\Big).$$

Thus

$$\frac{1}{\tau^2}\int_{P_\tau} |p|^{\frac{3}{2}}(x,t)\,dxdt \lesssim \tau\int_{P_1} |p|^{\frac{3}{2}}(x,t)\,dxdt + \frac{1}{\tau^2}\int_{P_1} |v-v_1|^3\,dxdt.$$

Together with the following interpolation inequality

$$\frac{1}{\rho^2}\int_{P_\rho} |v-v_\rho|^3\,dxdt \lesssim \sup_{-\rho^2\le t\le 0}\Big(\rho^{-1}\int_{B_\rho} |v|^2(x,t)\,dx\Big)^{\frac{3}{4}}\Big(\rho^{-1}\int_{P_\rho} |\nabla v|^2(x,t)\,dxdt\Big)^{\frac{3}{4}},$$

the following holds

$$D(r) \le C\Big\{\frac{r}{\rho}D(\rho) + \Big(\frac{\rho}{r}\Big)^2 A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho)\Big\}.$$

Now we employ the local energy inequality as follows. Let $\phi \in C_0^\infty(P_\rho)$ be a function such that $\phi \equiv 1$ *in* $P_r$ and $|\nabla\phi| \lesssim \frac{1}{\rho}$, $|\nabla^2\phi| + |\phi_t| \lesssim \frac{1}{\rho^2}$.
Then we have that

$$\begin{aligned}
&\sup_{-r^2\le t\le 0} r^{-1}\int_{P_r} |\nabla v|^2(x,t)\,dxdt + r^{-1}\int_{B_r} |v|^2(x,t)\,dx \\
&\lesssim \int_{P_\rho} |v|^2(|\phi_t| + |\Delta\phi|)\,dxdt + \int_{P_\rho} (|v|^2 + 2p)v\cdot\nabla\phi\,dxdt \\
&\lesssim \frac{1}{\rho^2}\int_{P_\rho} |v|^2\,dxdt + \int_{P_\rho} (|v|^2 - |v|_\rho^2)v\cdot\nabla\phi\,dxdt + \int_{P_\rho} 2pv\cdot\nabla\phi\,dxdt \\
&\lesssim \frac{1}{\rho^2}\int_{P_\rho} |v|^2\,dxdt + \frac{1}{\rho}\int_{P_\rho} (||v|^2 - |v|_\rho^2|)|v|\,dxdt + \frac{1}{\rho}\int_{P_\rho} |p||v|\,dxdt.
\end{aligned}$$

Putting all these estimates together, we have

$$A(r) + B(r) \lesssim \frac{\rho}{r}C^{\frac{2}{3}}(\rho) + \frac{\rho}{r}A^{\frac{1}{2}}(\rho)B^{\frac{1}{2}}(\rho)C^{\frac{1}{3}}(\rho) + \frac{\rho}{r}C^{\frac{1}{3}}(\rho)D^{\frac{2}{3}}(\rho),$$

$$D(r) \lesssim \frac{r}{\rho}D(\rho) + \Big(\frac{\rho}{r}\Big)^2 A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho),$$

$$C(r) \lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho).$$

Therefore we can deduce that

$$A(\theta_0 r) + B(\theta_0 r) \lesssim \theta_0^{-1}\Big\{C^{\frac{2}{3}}(r) + A^{\frac{1}{2}}(r)B^{\frac{1}{2}}(\rho)C^{\frac{1}{3}}(\rho) + C^{\frac{1}{3}}(\rho)D^{\frac{2}{3}}(r)\Big\},$$

$$D^2(\theta_0 r) \lesssim \theta_0^2\Big(D^2(r) + \theta_0^{-6}A^{\frac{3}{2}}(r)B^{\frac{3}{2}}(r)\Big),$$

$$C(\theta_0 r) \lesssim \theta_0^3 A^{\frac{3}{2}}(r) + \theta_0^{-3}A^{\frac{3}{4}}(r)B^{\frac{3}{4}}(r).$$

$$\begin{aligned}
A(\theta_0^2 r) \lesssim& \theta_0^{-1}C^{\frac{2}{3}}(\theta_0 r) + \theta_0^{-1}A^{\frac{1}{2}}(\theta_0 r)B^{\frac{1}{2}}(\theta_0 r)C^{\frac{1}{3}}(\theta_0 r) + \theta_0^{-1}C^{\frac{1}{3}}(\theta_0 r)D^{\frac{2}{3}}(\theta_0 r) \\
\lesssim& \theta_0^{-1}\Big(\theta_0^2 A(r) + \theta_0^{-2}A^{\frac{1}{2}}(r)B^{\frac{1}{2}}(r)\Big) + \theta_0^{-\frac{11}{6}}\Big\{C^{\frac{2}{3}}(r) + A^{\frac{1}{2}}(r)B^{\frac{1}{2}}(\rho)C^{\frac{1}{3}}(\rho) + C^{\frac{1}{3}}(\rho)D^{\frac{2}{3}}(r)\Big\}^{\frac{5}{6}} \\
&\times\Big\{\theta_0^3 A^{\frac{3}{2}}(r) + \theta_0^{-3}A^{\frac{3}{4}}(r)B^{\frac{3}{4}}(r)\Big\}^{\frac{1}{3}} + \theta_0^{-1}\Big\{\theta_0^3 A^{\frac{3}{2}}(r) + \theta_0^{-3}A^{\frac{3}{4}}(r)B^{\frac{3}{4}}(r)\Big\}^{\frac{1}{3}} \\
&\times\Big\{\theta_0^2\Big(D^2(r) + \theta_0^{-6}A^{\frac{3}{2}}(r)B^{\frac{3}{2}}(r)\Big)\Big\}^{\frac{2}{3}}.
\end{aligned}$$

Therefore we can deduce that

$$A(\theta_0^2 r)^{\frac{3}{2}} + D^2(\theta_0 r) \lesssim C\theta_0\Big(A(r)^{\frac{3}{2}} + D^2(r)\Big) + \epsilon_1$$

where

$$\epsilon_1 \approx \theta_0^{-N}B(r).$$

If we choose $r_0$ sufficiently small, then we can guarantee that for $0 < r \le r_0$ there exists $\epsilon_1 \ll 1$ such that
If $A(r)^{\frac{3}{2}} + D^2(r) \le 8\epsilon_1$, then the $\varepsilon_0-$regularity theorem implies $(0,0)$ is a smooth point.

For otherwise, $A(r)^{\frac{3}{2}} + D^2(r) > 8\epsilon_1$, for any for $0 < r \leq r_0$.
Hence,

$$\begin{aligned} A(\theta_0^2 r)^{\frac{3}{2}} + D^2(\theta_0 r) \lesssim & C\theta_0\Big(A(r)^{\frac{3}{2}} + D^2(r)\Big) + \frac{1}{8}\Big(A(r)^{\frac{3}{2}} + D^2(r)\Big) \\ \leq & (C\theta_0 + \frac{1}{8})\Big(A(r)^{\frac{3}{2}} + D^2(r)\Big) \\ \leq & \frac{1}{2}\Big(A(r)^{\frac{3}{2}} + D^2(r)\Big). \end{aligned}$$

After iterating finitely many times, it reduce to the former case. □

**Theorem 8.9** (Compactness of suitable weak solutions)**.** *Let* $(v_n, p_n)$ *be a sequence of suitable weak solution of (8.1) in* $P_1$ *such that*

$$\sup_{-1\leq t\leq 0} \int_{B_1} |v_n|^2(x,t)\, dx \leq C_1,$$

$$\int_{P_1} |\nabla v_n|^2(x,t)\, dxdt \leq C_2,$$

$$\int_{P_1} |p|^{\frac{3}{2}}(x,t)\, dxdt \leq C_3.$$

*Suppose*

$$v_n \rightharpoonup v \text{ weakly in } L_t^\infty L_x^2 \cap L_t^2 H_x^1$$

$$p_n \rightharpoonup p \text{ weakly in } L_x^3.$$

*Then* $(v, p)$ *is also a suitable weak solution of (8.1).*

*Proof.* It is sufficient to show that $v_n \rightharpoonup v$ strongly in $L^a$ for $1 \leq a < \frac{10}{3}$. Assume that this is true for the moment. Then by the local energy inequality for $(v_n, p_n)$, we have

$$2\int_{P_1} |\nabla v_n|^2\phi\, dxdt \lesssim \int_{P_1} |v_n|^2(|\phi_t| + |\Delta\phi|)\, dxdt + \int_{P_1} (|v_n|^2 + 2p_n)v_n \cdot \nabla\phi\, dxdt.$$

Thus we take the limit,

$$2\lim_n \int_{P_1} |\nabla v_n|^2\phi\, dxdt \leq \int_{P_1} |v|^2(|\phi_t| + |\Delta\phi|)\, dxdt + \int_{P_1} (|v|^2 + 2p)v \cdot \nabla\phi\, dxdt.$$

By the lower semicontinity, we have

$$\int_{P_1} |\nabla v|^2\phi\, dxdt \leq \lim_n \int_{P_1} |\nabla v_n|^2\phi\, dxdt.$$

Let

$$Z = H^{-2}(B_1) = (H_0^2(B_1))^*.$$

Since $\partial_t v_n = -(v_n \cdot \nabla v_n + \nabla P_n - \Delta v_n)$, we have

$$\|\partial_t v_n\|^{\frac{3}{2}}_{L^{\frac{3}{2}}([-1,0];Z)} \leq C_0,$$

where $C_0$ depends only on $C_1, C_2, C_3$.

Thus

$$v_n \in C([-1,0];Z),\ \forall n.$$

Applying the well-known Aubin-Lions Lemma, we have that $v_n \rightharpoonup v$ strongly in $L^2$. Therefore, by the interpolation inequalities, we also have that $v_n \rightharpoonup v$ strongly in $L^a$ for $1 \le a < \frac{10}{3}$. □

**Theorem 8.10.** *Let $(v,p)$ be a suitable weak solution of (8.1), then $\mathcal{P}^1(\mathrm{sing}(v)) = 0$, where sing(v) denotes the discontinuous set of v. Here $\mathcal{P}^1$ is the 1-dimensional Hausdorff measure in $\mathbb{R}^4$ with respect to the parabolic norm $\delta$:*

$$\mathcal{P}^1(E) \equiv \lim_{\delta\downarrow 0} \mathcal{P}^1_\delta(E),$$

*and*

$$\mathcal{P}^1_\delta(E) \equiv \inf\Big\{ \sum_{i=1}^\infty r_i : \bigcup_{i=1}^\infty \mathcal{P}_{r_i}(x_i,t_i) \supset E,\ r_i \le \delta \Big\}.$$

*Proof.*

$$(x,t) \in \mathrm{sing}(v) \Longleftrightarrow \overline{\lim_{r\to 0}} r^{-1} \int_{P_r(x,t)} |\nabla v|^2\,dxdt \ge \epsilon_1.$$

Let $V$ be a neighborhood of sing $(v)$ and $\delta > 0$ such that for all $(x,t) \in \mathrm{sing}(v)$ and $\forall r < \delta$ such that

$$r^{-1} \int_{P_r(x,t)} |\nabla v|^2\,dxdt \ge \epsilon_1, \quad P_r(x,t) \subset V.$$

By Vitali's five times covering Lemma, $\exists (x_i,t_i) \in V,\ 0 < r_i < \delta$ such that $\{P_{r_i}(x_i,t_i)\}_{i=1}^\infty$ are mutually disjoint and $\bigcup_{i=1}^\infty P_{5r_i}(x_i,t_i) \supset \mathrm{sing}(v)$. Therefore we can obtain

$$\begin{aligned}\sum_i r_i \le& \frac{1}{\epsilon_1} \sum_i \int_{P_{r_i}(x_i,t_i)} |\nabla v|^2\,dxdt \\ \le& \frac{1}{\epsilon_1} \int_{\bigcup_i P_{r_i}(x_i,t_i)} |\nabla v|^2\,dxdt \\ \le& \frac{1}{\epsilon_1} \int_{\bigcup_i P_{r_i}(x_i,t_i)} |\nabla v|^2\,dxdt \\ \le& \frac{1}{\epsilon_1} \int_V |\nabla v|^2\,dxdt.\end{aligned}$$

Now we can get

$$\mathcal{P}^1_{5\delta}(\mathrm{sing}(v)) \le \sum_i 5r_i \le \frac{5}{\epsilon_1} \int\int_V |\nabla v|^2\,dxdt < +\infty.$$

Therefore sing$(v)$ has zero Lesbegue measure so that $|V|$ can be arbitrarily small. By the absolute continuity , we have

$$\int_V |\nabla v|^2\,dxdt \to 0$$

as $|V| \to 0$. Hence

$$\lim_{\delta\to 0} \mathcal{P}^1_{5\delta}(\mathrm{sing}(v)) = 0.$$

Thus $\mathcal{P}^1(\mathrm{sing}(v)) = 0$. □

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