Landau-Lifshitz-Maxwell equation in dimension three

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Abstract

In dimension three, we establish the existence of weak solutions $\{u, H, E\}$ to the Landau-Lifshitz equation (1.1) coupled with the time-dependent Maxwell equation (1.2)-(1.3) such that u is Hölder continuous away from a closed set Σ, which has locally finite 3-dimensional parabolic Hausdorff measure. For two reduced Maxwell equations (1.17) and (1.18), Hölder continuity of ∇u away from Σ is also established.

1 Introduction

For a bounded, smooth domain $\Omega \subseteq \mathbb{R}^3$, we consider the Landau-Lifshitz-Maxwell equation:

$$\frac{\partial u}{\partial t} = \beta_1 u \times (\Delta u + H) - \beta_2 u \times (u \times (\Delta u + H)) \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$\nabla \times H = \epsilon_0 \frac{\partial E}{\partial t} + \sigma E \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1.2}$$

$$\nabla \times E = -\frac{\partial}{\partial t}(H + \beta \bar{u}) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1.3}$$

where $u : \Omega \times \mathbb{R}_+ \to S^2$ is the magnetization field, $H : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ is the magnetic field, $E : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ is the electric field, $H^e \equiv \triangle Z + H$ is the effective magnetic field, and β_1 is the gynomagnetisic coefficients and $\beta_2 \geq 0$ is the Gilbert damping coefficient and $\epsilon_0 \geq 0$ and $\sigma \geq 0$ is the conductivity constant and β is the magnetic permeability of free space, and $\bar{u}$ is an extension of u such that $\bar{u} = 0$ outside Ω. The system (1.1)-(1.3) was originally proposed by Landau and Lifshitz [23] in 1935 to model the dynamics of magnetization, magnetic field, electric field for the ferromagnetic materials.

The coupled Maxwell equation (1.2) and (1.3) can be written as

$$\frac{\partial B}{\partial t} = -\nabla \times E \quad \text{and} \quad \frac{\partial D}{\partial t} + \sigma E = \nabla \times H \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1.4}$$

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where D and B are the electric and magnetic displacements given by

$$D = \epsilon_0 E, \quad B = H + \beta \bar{u} \ \text{ in } \mathbb{R}^3 \times \mathbb{R}_+. \tag{1.5}$$

Note that when $H = E = 0$ and $\beta = 0$, the system (1.1)-(1.3) reduces to the Landau-Lifshitz-Gilbert equation for $Z : \Omega \times \mathbb{R}_+ \to S^2$:

$$\frac{\partial Z}{\partial t} = \beta_1 Z \times \triangle Z - \beta_2 Z \times (Z \times \triangle Z) \tag{1.6}$$

It is well-known that the equation (1.6) is the hybrid between the Schördinger flow into S^2 (i.e., $\frac{\partial u}{\partial t} = u \times \Delta u$ for $\beta_2 = 0$) and the heat flow of harmonic map into S^2 (i.e., $\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u$ for $\beta_1 = 0$). There have been many works on both the existence and regularity of weak solutions to equation (1.6) in recent years. Zhou-Guo [30] proved the existence of global weak solutions of (1.6) under suitable initial-boundary conditions. The unique smooth solution of (1.6) in dimension one was established by Zhou-Guo-Tan [31]. F. Alouges and A. Soyeur [1] proved that if $0 < \beta_2$, and the initial data $u_0 : \mathbb{R}^3 \to S^2$ with $\nabla u_0 \in L^2(\mathbb{R}^3)$, then there exists a global weak solution of (1.6) in $\mathbb{R}^3$. Moreover, if $u_0 \in H^1(\Omega)$ and $\beta_2 > 0$, then the Neumann boundary value problem of (1.6) in a bounded domain $\Omega \subset \mathbb{R}^3$ may admit infinitely many weak solutions. For regularity of weak solutions to the equation (1.6), Guo-Hong [18] established the existence of a global, weak solution with finitely many singular points in dimension two, and Chen-Ding-Guo [4] proved the uniqueness of weak solutions whose energies are non-increasing in time at dimension two. In dimension three, Melcher [26] proved the existence of global weak solutions to the equation (1.6) for $\Omega = \mathbb{R}^3$, which are smooth away from a closed set of locally finite 3-dimensional parabolic Hausdorff measure. Later, Wang [29] established the existence of partially smooth weak solutions to the equation (1.6) in any bounded domain Ω of dimensions ≤ 4. It is unknown whether the results by [26] and [29] can be extended to dimensions at least 5. It is also an interesting question to study regularity of *suitable* weak solutions to (1.6). Moser [27] proved, in dimensions $n \leq 4$, a partial regularity theorem of weak solutions of the equation (1.6) that are *stationary*, a notion analogous to that of heat flow of harmonic maps introduced by [14], [6], and [8] (see also some related works by Liu [25]). More recently, Ding-Wang [12] proved that the short time, smooth solution to the equation (1.6) may develop finite time singularity in dimensions 3 and 4 for suitable initial-boundary data.

Motivated by these studies on the equation (1.6), we are interested in the Landau-Lifshitz system coupled with time-dependent Maxwell equations (1.1)-(1.3).

There were some previous works on the system (1.1)-(1.3). Guo-Su [20] used the Galerkin's method to establish the existence of global, weak solutions with periodic initial conditions in dimension three. Carbou-Fabrie [3] used the Ginzburg-Landau approximation scheme to show the existence of global, weak solutions to the system (1.1)-(1.3) under the Neumann boundary condition in dimension three, and studied the long time behavior of the weak solution by the method of time average. See also Joly-Komech-Vacus [22] and Ding-Guo-Lin-Zeng [11] for related results.

The regularity issue of the system (1.1)-(1.3) is a challenging problem. There are very few results in the literature. Ding-Guo [9] proved a partial regularity theorem for stationary solutions to the Landau-Lifshitz equation (1.1) coupled with the quasi-stationary Maxwell equation:

$$\operatorname{div}(H+\beta\bar{u})=0 \ \text{ and } \nabla\times H=0 \ \text{ in } \mathcal{D}'(\mathbb{R}^3). \tag{1.7}$$

By modifying the techniques by [29], Ding-Guo [10] proved the existence of partially smooth weak solutions to (1.1) and (1.7) in dimension three.

We remark that there is an essential difference between (1.7) and (1.2)-(1.3): (1.7) is elliptic and $H\in\cap_{p>1}L^p(\mathbb{R}^3,\mathbb{R}^3)$; while (1.2)-(1.3) is a hyperbolic system and the regularity for $H(\cdot,t)$ and $E(\cdot,t)$ are no better than that of $H(\cdot,0)$ and $E(\cdot,0)$. The hyperbolicity of (1.2) and (1.3) imposes serious difficulties to study the regularity of (1.1).

In this paper, we attempt to establish the existence of partially regular, weak solutions of the Landau-Lifshitz-Maxwell system (1.1)-(1.3) with respect to the following initial-boundary conditions:

$$\frac{\partial u}{\partial\nu} = 0 \ \text{ on } \ \Omega\times\mathbb{R}_+, \tag{1.8}$$

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{1.9}$$

$$H(x,0) = H_0(x) \text{ in } \mathbb{R}^3, \tag{1.10}$$

$$E(x,0) = E_0(x) \text{ in } \mathbb{R}^3. \tag{1.11}$$

We assume throughout the paper

$$|u_0|=1 \ \text{ a.e. in } \Omega,\ H_0\in L^2(\mathbb{R}^3,\mathbb{R}^3),\ E_0\in L^2(\mathbb{R}^3,\mathbb{R}^3). \tag{1.12}$$

For the convenience, we study an equivalent form of the Landau-Lifshitz-Maxwell equation (1.1) (see [18, 19]):

$$\alpha_1\frac{\partial u}{\partial t}+\alpha_2 u\times\frac{\partial u}{\partial t}=\left(\Delta u+|\nabla u|^2u\right)+\left(H-\langle H,u\rangle\, u\right) \ \text{ in } \Omega\times\mathbb{R}_+. \tag{1.13}$$

where α_1 and $\alpha_2\in\mathbb{R}$ is a suitable normalization of β_2 and β_1 respectively such that

$$0<\alpha_1<1,\ \alpha_1^2+\alpha_2^2=1.$$

Now we recall the definition of weak solutions to (1.13), (1.2) and (1.3) along with the initial-boundary conditions (1.8)-(1.11).

Definition 1.1 $\{u,H,E\}$ is a weak solution of (1.13), (1.2), (1.3), and (1.8)-(1.11), if
(i) $u\in L^\infty_{\text{loc}}\left(\mathbb{R}_+,H^1(\Omega,S^2)\right)$, $\frac{\partial u}{\partial t}\in L^2_{\text{loc}}\left(\Omega\times\mathbb{R}_+\right)$, H and $E\in L^\infty_{\text{loc}}\left(\mathbb{R}_+,L^2(\mathbb{R}^3)\right)$.
(ii) u satisfies the equation (1.13) in the distribution sense, i.e., for any $\Phi\in C^\infty(\Omega\times\mathbb{R}_+,\mathbb{R}^3)$ with $\Phi(\cdot,0)=\Phi(\cdot,+\infty)=0$,

$$\int_{\Omega\times\mathbb{R}_+}\left(\alpha_1\frac{\partial u}{\partial t}+\alpha_2u\times\frac{\partial u}{\partial t}\right)\cdot\Phi = \int_{\Omega\times\mathbb{R}_+}\left(-\nabla u\cdot\nabla\Phi+|\nabla u|^2u\cdot\Phi\right) + \int_{\Omega\times\mathbb{R}_+}\left(H-\langle H,u\rangle\, u\right)\cdot\Phi, \tag{1.14}$$

and $u(\cdot,0)=u_0$ in the sense of trace.
(iii) For any $\Phi\in C^\infty(\mathbb{R}^3\times\mathbb{R}_+,\mathbb{R}^3)$ with $\Phi(\cdot,+\infty)=0$,

$$-\int_{\mathbb{R}^3\times\mathbb{R}_+}\left(\epsilon_0 E\cdot\frac{\partial\Phi}{\partial t}+H\cdot\nabla\times\Phi\right)+\sigma\int_{\mathbb{R}^3\times\mathbb{R}_+}E\cdot\Phi=\epsilon_0\int_{\mathbb{R}^3}E_0(x)\cdot\Phi(x,0). \quad (1.15)$$

(iv) For any $\Phi\in C^\infty(\mathbb{R}^3\times\mathbb{R}_+,\mathbb{R}^3)$ with $\Phi(\cdot,+\infty)=0$,

$$\begin{aligned}&-\int_{\mathbb{R}^3\times\mathbb{R}_+}(H+\beta\bar{u})\cdot\frac{\partial\Phi}{\partial t}+\int_{\mathbb{R}^3\times\mathbb{R}_+}E\cdot\nabla\times\Phi\\&=\beta\int_\Omega u_0(x)\cdot\Phi(x,0)+\int_{\mathbb{R}^3}H_0(x)\cdot\Phi(x,0). \quad (1.16)\end{aligned}$$

To state our results, we also need some notations. For $z_0=(x_0,t_0)\in\mathbb{R}^3\times\mathbb{R}$ and $r>0$, denote

$$B_r(x_0)=\left\{x\in\mathbb{R}^3:|x-x_0|<r_0\right\},\quad\text{and}\quad P_r(z_0)=B_r(x_0)\times(t_0-r^2,t_0).$$

For any subset $D\subset\mathbb{R}^4$, the three dimensional parabolic Hausdorff measure, $\mathcal{P}^3(D)$, is defined by

$$\mathcal{P}^3(D)=\lim_{\delta\downarrow 0}\left(\inf\left\{\sum_{i=1}^\infty r_i^3:\ D\subset\cup_{i=1}^\infty P_{r_i}(z_i),\ 0<r_i\le\delta\right\}\right).$$

We say a subset $D\subset\mathbb{R}^4$ has locally finite 3-dimensional parabolic Hausdorff measure, if

$$\mathcal{P}^3\left(D\cap P_R(0)\right)<+\infty,\ \forall R>0.$$

Our first theorem is

Theorem 1.2 *For any $u_0\in H^1(\Omega,S^2)$, $H_0\in L^2(\mathbb{R}^3,\mathbb{R}^3)$ and $E_0\in L^2(\mathbb{R}^3,\mathbb{R}^3)$, there exists a global weak solution $\{u,H,E\}$ to the Landau-Lifshitz-Maxwell system (1.13), (1.2) and (1.3) under the initial-boundary conditions (1.8)-(1.11) such that there exists a closed subset $\Sigma\subset\Omega\times\mathbb{R}_+$, which has locally finite 3-dimensional parabolic Hausdorff measure, so that $u\in C^{\frac{1}{2}}(\Omega\times\mathbb{R}_+\setminus\Sigma,S^2)$.*

To study the higher order regularity of weak solutions to (1.13) and (1.2)-(1.3), obtained by theorem 1.2, we restrict to the two special cases:
(i) The constant $\epsilon_0=0$ in (1.2), and (1.2) and (1.3) become

$$\nabla\times(\nabla\times H)=-\sigma\frac{\partial}{\partial t}(H+\beta\bar{u})\ \text{ in }\mathbb{R}^3. \quad (1.17)$$

(ii) The constant $\beta=0$ in (1.3), and (1.2) and (1.3) become

$$\nabla\times H=\epsilon_0\frac{\partial E}{\partial t}+\sigma E,\quad \nabla\times E=-\frac{\partial H}{\partial t}\ \text{ in }\mathbb{R}^3. \quad (1.18)$$

Our second theorem is

Theorem 1.3 *For any $u_0 \in H^1(\Omega, S^2)$ and $H_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $\nabla \cdot (H_0 + \beta\overline{u_0}) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$, there exists a weak solution $\{u, H\}$ of the Landau-Lifshitz system (1.13) coupled with (1.17) under the initial-boundary condition (1.8), (1.9) and (1.10) such that $H \in \cap_{T>0} H^1(\mathbb{R}^3 \times [0,T], \mathbb{R}^3)$ and there exists a closed subset $\Sigma \subset \Omega \times \mathbb{R}_+$, which has locally finite 3-dimensional parabolic Hausdorff measure, so that $\nabla u \in C^\alpha(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $0 < \alpha < 1$, and $\nabla^2 u, \frac{\partial u}{\partial t} \in L^6_{loc}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$.*

Our third theorem is

Theorem 1.4 *For any $u_0 \in H^1(\Omega, S^2)$, and $H_0, E_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $\nabla \cdot H_0 = \nabla \cdot E_0 = 0$ in $\mathcal{D}'(\mathbb{R}^3)$, there exists a weak solution $\{u, H, E\}$ of the Landau-Lifshitz system (1.13) coupled with (1.18) under the initial-boundary condition (1.8), (1.9), (1.10) and (1.11) such that $\frac{\partial H}{\partial t}, \frac{\partial E}{\partial t} \in L^\infty_{loc}(\mathbb{R}_+, L^2(\mathbb{R}^3))$, and there exists a closed subset $\Sigma \subset \Omega \times \mathbb{R}_+$, which has locally finite 3-dimensional parabolic Hausdorff measure, so that $\nabla u \in C^\alpha(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $0 < \alpha < 1$, and $\nabla^2 u, \frac{\partial u}{\partial t} \in L^6_{loc}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$.*

The ideas to approach these theorems are based on analysis of the Ginzburg-Landau approximate equation: for $\epsilon > 0$,

$$\alpha_1 \frac{\partial u^\epsilon}{\partial t} + \alpha_2 u^\epsilon \times \frac{\partial u^\epsilon}{\partial t} = \Delta u^\epsilon + \frac{1}{\epsilon^2}(1 - |u^\epsilon|^2)u^\epsilon + u^\epsilon \times (H^\epsilon \times u^\epsilon) \quad \text{in } \Omega \times \mathbb{R}_+. \tag{1.19}$$

We would like to remark that by adopting our argument in this paper, similar to [12], it is not hard to see that the corresponding partial regularity property at the boundary also holds for the weak solution obtained in theorems 1.2, 1.3, and 1.4. For example, theorem 1.2 can be extended so that there exists a closed subset $\Sigma_1 \subseteq \partial\Omega$, with $\mathcal{P}^3(\Sigma_1) < +\infty$, such that $u \in C^{\frac{1}{2}}(\overline{\Omega} \setminus (\Sigma \cup \Sigma_1), S^2)$.

The paper is written as follows. In §2, we establish a uniform energy estimate of the equation (1.19). In §3, we sketch the time slice monotonicity. In §4 we establish a lower bound estimate of solutions to (1.19). In §5, we obtain the decay estimate of solutions to (1.19) under the smallness condition and prove theorem 1.2. In §6, we establish a partial C^α-regularity of ∇u and prove both theorem 1.3 and 1.4.

2 Energy estimate of the equation (1.19)

In this section, we sketch the existence of global weak solutions to (1.19) (1.2), (1.3), associated with (1.8)-(1.11) by Galerkin's method and their corresponding energy estimates. Here we modify the argument by Carbou-Fabrie [3] to handle the equation (1.19). We would like to point out the difference between (1.19) and the approximate equation employed by Carbou-Fabrie [3]: we approximate the term $H - \langle H, u\rangle u$ in (1.13) by $u^\epsilon \times (H^\epsilon \times u^\epsilon)$ in (1.19), while Carbou-Fabrie [3] approximated the term $H - \langle H, u\rangle u$ in (1.13) by H^ϵ. An advantage of our approximation is that we have the upper bound $|u^\epsilon| \leq 1$, which plays a crucial role to establish apriori continuity estimates of u^ϵ and hence the existence of partially smooth solutions; while the one by [3] yields an optimal energy inequality (cf. [3] page 387, (2.12)), which is important in their study of long time behaviors by the method of time average.

We begin with a general L^∞-estimate of weak solutions u^ϵ to (1.19).

Lemma 2.1 *For $\epsilon > 0$, assume $u_0 \in H^1(\Omega, S^2)$, $H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $E_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. Let $\{u^\epsilon, H^\epsilon, E^\epsilon\}$ be any weak solution of (1.19), (1.2)-(1.3), and (1.8)-(1.11). Then $|u^\epsilon|(x,t) \le 1$ for any $(x,t) \in \Omega \times \mathbb{R}_+$.*

Proof. Multiplying (1.19) by u^ϵ and using the fact that $u^\epsilon \cdot u^\epsilon \times \frac{\partial u^\epsilon}{\partial t} = 0$ and $u^\epsilon \cdot u^\epsilon \times (H^\epsilon \times u^\epsilon) = 0$, we have

$$\left(\alpha_1 \frac{\partial}{\partial t} - \Delta\right)\left(|u^\epsilon|^2 - 1\right) = -2\left(|\nabla u^\epsilon|^2 + \frac{1}{\epsilon^2}(|u^\epsilon|^2 - 1)|u^\epsilon|^2\right), \tag{2.1}$$

hence

$$\left(\alpha_1 \frac{\partial}{\partial t} - \Delta\right)\left(|u^\epsilon|^2 - 1\right)_+ \le 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

where $\left(|u^\epsilon|^2 - 1\right)_+$ is the positive part of $(|u^\epsilon|^2 - 1)$. The conclusion now follows from the weak maximum principle of the heat equation (cf. Liberman [24]). □

Now we sketch the existence of weak solutions to (1.19) that enjoy energy estimates by Galerkin's method. To do it, we first recall some notations (see [3] page 388-395). Let $\{\phi_k\}_k \subseteq H^2(\Omega)$ be eigenfunctions of Δ, with zero Neumann boundary condition, that form an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^1(\Omega)$ and $H^2(\Omega)$. For $1 \le N < +\infty$, set $V_N = \text{span}\{\phi_1, \cdots, \phi_N\}$. Define

$$\mathbb{H}_{\text{curl}}(\mathbb{R}^3) = \left\{\psi \in L^2(\mathbb{R}^3, \mathbb{R}^3),\ \nabla \times \psi \in L^2(\mathbb{R}^3, \mathbb{R}^3)\right\}.$$

Let $\{\psi_k\}_k$ be an orthogonal basis of $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$ that is orthonormal in $L^2(\mathbb{R}^3)$ and $W_N = \text{span}\{\psi_1, \cdots, \psi_N\}$. Denote by $\Pi_{V_N} : L^2(\Omega) \to V_N$, and $\Pi_{W_N} : L^2(\Omega) \to W_N$ the orthogonal projections. Define the retraction map $\Pi : \mathbb{R}^3 \to B_1$ by letting

$$\begin{aligned} \Pi(p) &= p \quad \text{if } |p| \le 1; \\ &= \frac{p}{|p|} \quad \text{if } |p| > 1. \end{aligned}$$

Now we define $(u_N, H_N, E_N) \in V_N \times W_N \times W_N$ by

$$u_N(x,t) = \sum_{k=1}^N v_k(t)\phi_k(x),\ H_N(x,t) = \sum_{k=1}^N h_k(t)\psi_k(x),\ E_N(x,t) = \sum_{k=1}^N e_k(t)\psi_k(x), \tag{2.2}$$

which solves

$$\begin{aligned} \int_\Omega \left(\alpha_1 \frac{\partial u_N}{\partial t} + \alpha_2 u_N \times \frac{\partial u_N}{\partial t}\right)\cdot \Phi &= \int_\Omega [-\nabla u_N \cdot \nabla \Phi + \frac{1}{\epsilon^2}(1 - |u_N|^2)u_N \cdot \Phi] \quad (2.3) \\ &\quad + \int_\Omega \Pi(u_N) \times (H_N \times \Pi(u_N)) \cdot \Phi,\ \forall \Phi \in V_N \\ \int_{\mathbb{R}^3} \left(\epsilon_0 \frac{\partial E_N}{\partial t} + \sigma E_N\right)\cdot \Psi &= \int_{\mathbb{R}^3} H_N \cdot (\nabla \times \Psi),\ \forall \Psi \in W_N \quad (2.4) \\ \int_{\mathbb{R}^3} \frac{\partial}{\partial t}(H_N + \beta \overline{u_N}) \cdot \Psi &= -\int_{\mathbb{R}^3} E_N \cdot (\nabla \times \Psi),\ \forall \Psi \in W_N, \quad (2.5) \end{aligned}$$

under the initial condition:

$$u_N\big|_{t=0} = \Pi_{V_N}(u_0),\ H_N\big|_{t=0} = \Pi_{W_N}(H_0),\ E_N\big|_{t=0} = \Pi_{W_N}(E_0). \tag{2.6}$$

Throughout this section, we will use the following fact:

$$\lim_{N\to\infty}\int_\Omega e_\epsilon(u_N(0)) = \int_\Omega \frac{1}{2}|\nabla u_0|^2. \tag{2.7}$$

Note that (2.3)-(2.6) reduces to a system of first order ODEs for $(v_k, h_k, e_k)_k$. Moreover, since $P(u_N)(v) = \alpha_1 v + \alpha_2 u_N \times v : \mathbb{R}^3 \to \mathbb{R}^3$ is one to one, we can solve (2.3) for the derivative in time. Hence it is well known that there exists a local solution (u_N, H_N, E_N) of (2.3)-(2.6). The following uniform estimate shows that (u_N, H_N, E_N) is also global in time and converges to a global weak solution of (1.19), (1.2)-(1.3).

Lemma 2.2 *For $\epsilon > 0$, assume $u_0 \in H^1(\Omega, S^2)$ and $H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $E_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. Then there exists a global weak solution $\{u^\epsilon, H^\epsilon, E^\epsilon\}$ to (1.19), (1.2)-(1.3), with (1.8)-(1.11), such that for any $0 < T < +\infty$, it holds*

$$\sigma\int_0^T\int_{\mathbb{R}^3}|E^\epsilon|^2 + \alpha_1\int_0^T\int_\Omega\left|\frac{\partial u^\epsilon}{\partial t}\right|^2 + \mathcal{E}_\epsilon(T) \le e^{CT}\mathcal{E}_0, \tag{2.8}$$

where

$$\mathcal{E}_\epsilon(t) = \int_\Omega e_\epsilon(u^\epsilon(t)) + \int_{\mathbb{R}^3}\left(\frac{\epsilon_0}{2}|E^\epsilon(t)|^2 + \frac{1}{2}|H^\epsilon(t)|^2\right), \tag{2.9}$$

$$e_\epsilon(u^\epsilon(t)) = \left(\frac{1}{2}|\nabla u^\epsilon(t)|^2 + \frac{1}{4\epsilon^2}(1-|u^\epsilon(t)|^2)^2\right),$$

$C = C(\beta, \alpha_1) > 0$ depends only on β and α_1, and

$$\mathcal{E}_0 = \int_\Omega \frac{1}{2}|\nabla u_0|^2 + \int_{\mathbb{R}^3}\left(\frac{\epsilon_0}{2}|E_0|^2 + \frac{1}{2}|H_0|^2\right).$$

Proof. We first establish the estimate (2.8) for Galerkin's approximate solutions $\{u_N, H_N, E_N\}$. Then we employ this estimate to extract a subsequence that converges to a global weak solution $(u^\epsilon, H^\epsilon, E^\epsilon)$ to (1.19), (1.2), and (1.3).

Testing (2.3) with $\Phi = \frac{\partial u_N}{\partial t}$ and integrating over Ω gives

$$\begin{aligned}\int_\Omega \alpha_1|\frac{\partial u_N}{\partial t}|^2 + \frac{d}{dt}\int_\Omega e_\epsilon(u_N) &= \int_\Omega \Pi(u_N)\times(H_N\times\Pi(u_N))\cdot\frac{\partial u_N}{\partial t}\\ &\le \int_\Omega |H_N||\frac{\partial u_N}{\partial t}|,\end{aligned} \tag{2.10}$$

where we use the fact that $|\Pi(u_N)| \le 1$ and $|\Pi(u_N)\times(H_N\times\Pi(u_N))| \le |H_N|$.

Testing (2.4) with $\Psi = E_N$ and integrating over $\mathbb{R}^3$ gives

$$\int_{\mathbb{R}^3}\nabla\times H_N\cdot E_N = \frac{d}{dt}\int_{\mathbb{R}^3}\frac{\epsilon_0}{2}|E_N|^2 + \int_{\mathbb{R}^3}\sigma|E_N|^2. \tag{2.11}$$

Testing (2.5) with $\Psi = H_N$ and integrating over $\mathbb{R}^3$ gives

$$-\int_{\mathbb{R}^3} \nabla \times E_N \cdot H_N = \frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}|H_N|^2 + \beta \int_\Omega H_N \cdot \frac{\partial u_N}{\partial t}. \tag{2.12}$$

Adding together (2.11) and (2.12), and using the identity

$$\int_{\mathbb{R}^3} (\nabla \times H_N \cdot E_N - \nabla \times E_N \cdot H_N) = 0,$$

we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3} (\frac{\epsilon_0}{2}|E_N|^2 + \frac{1}{2}|H_N|^2) + \sigma \int_{\mathbb{R}^3} |E_N|^2 &= -\beta \int_\Omega H_N \cdot \frac{\partial u_N}{\partial t} \\ &\le \beta \int_\Omega |H_N||\frac{\partial u_N}{\partial t}|. \end{aligned} \tag{2.13}$$

Adding (2.10) and (2.13) together gives

$$\begin{aligned} &\sigma \int_{\mathbb{R}^3} |E_N|^2 + \alpha_1 \int_\Omega |\frac{\partial u_N}{\partial t}|^2 + \frac{d}{dt}\left[\int_\Omega e_\epsilon(u_N) + \int_{\mathbb{R}^3} (\frac{\epsilon_0}{2}|E_N|^2 + \frac{1}{2}|H_N|^2)\right] \\ &\le (1+\beta)\int_\Omega |H_N||\frac{\partial u_N}{\partial t}| \\ &\le \frac{\alpha_1}{4}\int_\Omega |\frac{\partial u_N}{\partial t}|^2 + \frac{(1+\beta)^2}{\alpha_1}\int_{\mathbb{R}^3} |H_N|^2, \end{aligned} \tag{2.14}$$

where we have used the Cauchy-Schwarz inequality in the last step. Applying the Gronwall's inequality to (2.14) and integrating from $t = 0$ to $t = T$ gives

$$\begin{aligned} &\sigma \int_0^T \int_{\mathbb{R}^3} |E_N|^2 + \alpha_1 \int_0^T \int_\Omega |\frac{\partial u_N}{\partial t}|^2 \\ &+ \left[\int_\Omega e_\epsilon(u_N) + \int_{\mathbb{R}^3} (\frac{\epsilon_0}{2}|E_N|^2 + \frac{1}{2}|H_N|^2)\right](T) \\ &\le e^{CT}\left[\int_\Omega e_\epsilon(u_N) + \int_{\mathbb{R}^3} (\frac{\epsilon_0}{2}|E_N|^2 + \frac{1}{2}|H_N|^2)\right](0) \\ &\le e^{CT}\left[o(1) + \int_\Omega e_\epsilon(u_0) + \int_{\mathbb{R}^3} (\frac{\epsilon_0}{2}|E_0|^2 + \frac{1}{2}|H_0|^2)\right] \\ = \ & e^{CT}(\mathcal{E}_0 + o(1)). \end{aligned} \tag{2.15}$$

Here we have used (2.7) and $o(1)$ denotes the quantity such that $\lim_{N\to\infty} o(1) = 0$. It follows from the bound (2.15) that there exists a subsequence of (u_N, H_N, E_N), still denoted as itself, such that for any $0 < T < +\infty$,

$$u_N \rightharpoonup u^\epsilon \text{ weak}^* \text{ in } L^\infty([0,T], H^1(\Omega)),\ \frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\epsilon}{\partial t} \text{ in } L^2(\Omega \times [0,T]);$$

$$E_N \rightharpoonup E^\epsilon \text{ weak}^* \text{ in } L^\infty([0,T], L^2(\mathbb{R}^3)),\ H_N \rightharpoonup H^\epsilon \text{ weak}^* \text{ in } L^\infty([0,T], L^2(\mathbb{R}^3)).$$

Furthermore, by Aubin's lemma, we have

$$u_N \to u^\epsilon \ \text{ strongly in } L^4(\Omega \times [0,T]).$$

Since $|\Pi(u_N)| \le |u_N|$ and $\int_\Omega |\nabla(\Pi(u_N))|^2 \le \int_\Omega |\nabla u_N|^2$, we also have

$$\Pi(u_N) \to \Pi(u^\epsilon) \ \text{ strongly in } L^4(\Omega \times [0,T]).$$

It is readily seen that (2.15) yields that $(u^\epsilon, H^\epsilon, E^\epsilon)$ satisfies (2.8) and the initial condition (1.8)-(1.11). It is also not hard to see that (H^ϵ, E^ϵ) are weak solutions to the equations (1.2) and (1.3). Similar to [3] page 392, we can check that

$$\alpha_1 \frac{\partial u^\epsilon}{\partial t} + \alpha_2 u^\epsilon \times \frac{\partial u^\epsilon}{\partial t} = \Delta u^\epsilon + \frac{1}{\epsilon^2}(1 - |u^\epsilon|^2) u^\epsilon + \Pi(u^\epsilon) \times (H^\epsilon \times \Pi(u^\epsilon)) . \qquad (2.16)$$

Multiplying (2.16) by u^ϵ and observing that $\Pi(u^\epsilon) \times (H^\epsilon \times \Pi(u^\epsilon)) \cdot u^\epsilon = 0$, we have that u^ϵ satisfies (2.1). Hence lemma 2.1 implies that $|u^\epsilon| \le 1$. Thus $\Pi(u^\epsilon) = u^\epsilon$ and (2.16) yields (1.19). The proof is complete. $\square$

In order to establish a partial C^α-regularity of ∇u for weak solutions u to (1.13) coupled with the Maxwell equations (1.17) or (1.18), we need uniform estimates of H^ϵ, E^ϵ in $H^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}_+)$. More precisely, we have

Lemma 2.3 *For any $u_0 \in H^1(\Omega, S^2)$ and $H_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $\nabla \cdot (H_0 + \beta \overline{u_0}) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. Then there exists a global weak solution $\{u^\epsilon, H^\epsilon\}$ to (1.19) and (1.17), under the initial-boundary conditions (1.8)-(1.10) such that for any $0 < T < +\infty$,*

$$\begin{aligned} &\alpha_1 \int_0^T \int_\Omega \left| \frac{\partial u^\epsilon}{\partial t} \right|^2 + \int_0^T \int_{\mathbb{R}^3} \left(\left| \frac{\partial H^\epsilon}{\partial t} \right|^2 + |\nabla H^\epsilon|^2 \right) \\ &+ \left[\int_\Omega e_\epsilon(u^\epsilon) + \int_{\mathbb{R}^3} \left(\frac{\sigma}{2} |H^\epsilon|^2 + |\nabla H^\epsilon|^2 \right) \right](T) \\ \le\ & \int_{\mathbb{R}^3} |\nabla H_0|^2 + e^{CT} \left(\int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} |H_0|^2 \right), \end{aligned} \qquad (2.17)$$

for some $C = C(\beta, \alpha_1) > 0$.

Proof. For $N \ge 1$, let $(u_N, H_N) \in V_N \times W_N$ be given by (2.2) such that u_N solves (2.3) and H_N solves

$$\int_{\mathbb{R}^3} (\nabla \times H_N) \cdot (\nabla \times \Psi) = -\sigma \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (H_N + \beta \overline{u_N}) \cdot \Psi, \ \forall \Psi \in W_N \qquad (2.18)$$

subject to the initial condition $(u_N, H_N)|_{t=0} = (\Pi_{V_N}(u_0), \Pi_{W_N}(H_0))$.

Testing (2.18) with $\Psi = H_N$ and integrating over $\mathbb{R}^3$ gives

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{\sigma}{2} |H_N|^2 + \int_{\mathbb{R}^3} |\nabla \times H_N|^2 = -\beta\sigma \int_\Omega H_N \cdot \frac{\partial u_N}{\partial t} \le \beta\sigma \int_\Omega |H_N| \left| \frac{\partial u_N}{\partial t} \right| . \qquad (2.19)$$

Combining (2.19) with (2.10) and applying Cauchy-Schwarz inequality yields

$$\begin{aligned} &\int_\Omega \alpha_1 \left|\frac{\partial u_N}{\partial t}\right|^2 + \frac{d}{dt} \left(\int_\Omega e_\epsilon(u_N) + \int_{\mathbb{R}^3} \frac{\sigma}{2} |H_N|^2 \right) + \int_{\mathbb{R}^3} |\nabla \times H_N|^2 \\ \le\ & C(\alpha_1, \beta) \int_{\mathbb{R}^3} |H_N|^2. \end{aligned} \qquad (2.20)$$

This, combined with the Gronwall's inequality, yields that for any $0 < T < +\infty$,

$$\begin{aligned}
& \alpha_1 \int_0^T \int_\Omega |\frac{\partial u_N}{\partial t}|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla \times H_N|^2 + \left(\int_\Omega e_\epsilon(u_N) + \int_{\mathbb{R}^3} \frac{\sigma}{2} |H_N|^2 \right)(T) \\
\leq \ & e^{CT} \left(o(1) + \int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} \frac{\sigma}{2} |H_0|^2 \right) \qquad (2.21)
\end{aligned}$$

for some $C = C(\beta, \alpha_1) > 0$, here we have used (2.7).

Now test (2.18) with $\Psi = \frac{\partial H_N}{\partial t}$ and integrate over $\mathbb{R}^3$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \times H_N|^2 + \sigma \int_{\mathbb{R}^3} |\frac{\partial H_N}{\partial t}|^2 = -\beta\sigma \int_\Omega \frac{\partial H_N}{\partial t} \cdot \frac{\partial u_N}{\partial t}. \qquad (2.22)$$

Thus by the Cauchy-Schwarz inequality, this implies

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \times H_N|^2 + \sigma \int_{\mathbb{R}^3} \left| \frac{\partial H_N}{\partial t} \right|^2 \leq 16\beta^2 \sigma \int_{\mathbb{R}^3} \left| \frac{\partial u_N}{\partial t} \right|^2. \qquad (2.23)$$

Integrating for $0 \leq t \leq T$ and applying (2.21), this implies

$$\begin{aligned}
& \int_{\mathbb{R}^3} |\nabla \times H_N|^2(T) + \sigma \int_0^T \int_{\mathbb{R}^3} \left| \frac{\partial H_N}{\partial t} \right|^2 \\
\leq \ & \int_{\mathbb{R}^3} |\nabla H_0|^2 + 16\beta^2 \sigma \int_0^T \int_\Omega \left| \frac{\partial u_N}{\partial t} \right|^2 \\
\leq \ & \int_{\mathbb{R}^3} |\nabla H_0|^2 + e^{CT}(o(1) + \int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} |H_0|^2). \qquad (2.24)
\end{aligned}$$

Adding (2.21) and (2.24) together, we obtain

$$\begin{aligned}
& \alpha_1 \int_0^T \int_\Omega |\frac{\partial u_N}{\partial t}|^2 + \int_0^T \int_{\mathbb{R}^3} (|\frac{\partial H_N}{\partial t}|^2 + |\nabla \times H_N|^2) \\
& + \left(\int_\Omega e_\epsilon(u_N) + \int_{\mathbb{R}^3} [\frac{\sigma}{2} |H_N|^2 + |\nabla \times H_N|^2] \right)(T) \\
\leq \ & \int_{\mathbb{R}^3} |\nabla H_0|^2 + e^{CT} \left(o(1) + \int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} \frac{\sigma}{2} |H_0|^2 \right) \qquad (2.25)
\end{aligned}$$

It follows from (2.25) that we may assume, after taking subsequences, that for any $0 < T < +\infty$,

$$u_N \rightharpoonup u^\epsilon \text{ weak}^* \text{ in } L^\infty([0,T], H^1(\Omega)), \ \frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\epsilon}{\partial t} \text{ in } L^2(\Omega \times [0,T]);$$

$$H_N \rightharpoonup H^\epsilon, \ \frac{\partial H_N}{\partial t} \rightharpoonup \frac{\partial H^\epsilon}{\partial t}, \ \nabla \times H_N \rightharpoonup \nabla \times H^\epsilon \text{ in } L^2(\mathbb{R}^3 \times [0,T]).$$

As in Lemma 2.2, we can show that (u^ϵ, H^ϵ) are weak solutions to the equations (1.19) and (1.17), and the initial condition (1.8)-(1.10). By the lower semicontinuity, we also have that (2.25) holds with (u_N, H_N) replaced by (u^ϵ, H^ϵ). In order to obtain the bound of L^2-norm for ∇H, we need to use the condition $\nabla\cdot(H_0+\beta\overline{u_0})=0$ in $\mathcal{D}'(\mathbb{R}^3)$. Note that

$$\int_{\mathbb{R}^3}(\nabla\times H^\epsilon)\cdot(\nabla\times\Psi)=-\sigma\int_{\mathbb{R}^3}\frac{\partial}{\partial t}(H^\epsilon+\beta\overline{u^\epsilon})\cdot\Psi,\forall\Psi\in H^1(\mathbb{R}^3).$$

By choosing $\Psi=\nabla\psi$ for $\psi\in C_0^\infty(\mathbb{R}^3)$ and observing $\nabla\times(\nabla\psi)=0$ in $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3}\frac{\partial}{\partial t}(H^\epsilon+\beta\overline{u^\epsilon})\cdot\nabla\psi=0$$

so that for a.e. $t>0$,

$$\int_{\mathbb{R}^3}\nabla\cdot(H^\epsilon+\beta\overline{u^\epsilon})\psi=\int_{\mathbb{R}^3}\nabla\cdot(H_0+\beta\overline{u_0})\psi=0.$$

Thus

$$\nabla\cdot(H^\epsilon+\beta\overline{u^\epsilon})=0\ \text{ in } \mathcal{D}'(\mathbb{R}^3)\ \text{ for a.e. } t>0.$$

This, combined with the inequality:

$$\int_{\mathbb{R}^3}|\nabla H^\epsilon|^2\le C\int_{\mathbb{R}^3}(|\nabla\times H^\epsilon|^2+|\nabla\cdot H^\epsilon|^2)\le C(\beta)[\int_{\mathbb{R}^3}|\nabla\times H^\epsilon|^2+\int_\Omega|\nabla u^\epsilon|^2],$$

and (2.25) with $(u_N,H_N)=(u^\epsilon,H^\epsilon)$, yields (2.17). Hence the proof is complete. □

For the equations (1.19) and (1.18), we have

Lemma 2.4 *For any $u_0\in H^1(\Omega,S^2)$, $H_0\in H^1(\mathbb{R}^3,\mathbb{R}^3)$ and $E_0\in H^1(\mathbb{R}^3,\mathbb{R}^3)$ with $\nabla\cdot E_0=\nabla\cdot H_0=0$ in $\mathcal{D}'(\mathbb{R}^3)$, there exists a global weak solution $\{u^\epsilon,H^\epsilon,E^\epsilon\}$ to (1.19) and (1.18) under the initial-boundary conditions (1.8), (1.9, (1.10) and (1.11) such that for any $0<T<+\infty$, the following holds:*

$$\begin{aligned}&\int_0^T\int_\Omega|\frac{\partial u^\epsilon}{\partial t}|^2+\mathcal{E}_\epsilon(T)\\&+\int_{\mathbb{R}^3}\left[|H^\epsilon|^2+|E^\epsilon|^2+|\frac{\partial H^\epsilon}{\partial t}|^2+|\frac{\partial E^\epsilon}{\partial t}|^2+|\nabla H^\epsilon|^2+|\nabla E^\epsilon|^2\right](T)\\\le\ & C(\epsilon_0,\sigma,T)\left[\int_\Omega|\nabla u_0|^2+\int_{\mathbb{R}^3}(|H_0|^2+|E_0|^2+|\nabla H_0|^2+|\nabla E_0|^2)\right].\end{aligned}\tag{2.26}$$

Proof. For $N\ge1$, let $(u_N,H_N,E_N)\in V_N\times W_N\times W_N$ be given by (2.2) and solve (2.3), (2.4), (2.5), and (2.6). Since $\beta=0$ in this case, testing (2.4) with $\Psi=E_N$ and (2.5) with $\Psi=H_N$ and adding the resulting identities together gives

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left(|H_N|^2+\epsilon_0|E_N|^2\right)+2\sigma\int_{\mathbb{R}^3}|E_N|^2=0.\tag{2.27}$$

Differentiating both equations (2.4) and (2.5) with respect to t and testing the resulting equations with Ψ being $\frac{\partial E_N}{\partial t}$ and $\frac{\partial H_N}{\partial t}$ respectively, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\frac{\partial E_N}{\partial t}\cdot\nabla\times\frac{\partial H_N}{\partial t}-\frac{\partial H_N}{\partial t}\cdot\nabla\times\frac{\partial E_N}{\partial t}\right)\\=\ &\int_{\mathbb{R}^3}\left[\frac{\partial E_N}{\partial t}\cdot(\epsilon_0\frac{\partial^2E_N}{\partial t^2}+\sigma\frac{\partial E_N}{\partial t})+\frac{\partial H_N}{\partial t}\cdot\frac{\partial^2H_N}{\partial t^2}\right].\end{aligned}$$

Since

$$\int_{\mathbb{R}^3} \left(\frac{\partial E_N}{\partial t} \cdot \nabla \times \frac{\partial H_N}{\partial t} - \frac{\partial H_N}{\partial t} \cdot \nabla \times \frac{\partial E_N}{\partial t} \right) = 0,$$

we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} (\epsilon_0 |\frac{\partial E_N}{\partial t}|^2 + |\frac{\partial H_N}{\partial t}|^2) + 2\sigma \int_{\mathbb{R}^3} |\frac{\partial E_N}{\partial t}|^2 = 0. \tag{2.28}$$

Combining (2.27) with (2.28), we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \left[|H_N|^2 + \epsilon_0 (|E_N|^2 + |\frac{\partial E_N}{\partial t}|^2) + |\frac{\partial H_N}{\partial t}|^2 \right] \\ = & \ -2\sigma \int_{\mathbb{R}^3} (|E_N|^2 + \left| \frac{\partial E_N}{\partial t} \right|^2). \end{aligned} \tag{2.29}$$

Since

$$\frac{\partial H_N}{\partial t}|_{t=0} = -\nabla \times (\Pi_{W_N}(E_0)),\ \epsilon_0 \frac{\partial E}{\partial t}|_{t=0} = \nabla \times (\Pi_{W_N}(H_0)) - \sigma \Pi_{W_N}(E_0),$$

integrating (2.29) for $0 \leq t \leq T$ yields

$$\begin{aligned} & \int_{\mathbb{R}^3} \left[|H_N|^2 + \epsilon_0 (|E_N|^2 + |\frac{\partial E_N}{\partial t}|^2) + |\frac{\partial H_N}{\partial t}|^2 \right] (T) + 2\sigma \int_0^T \int_{\mathbb{R}^3} (|E_N|^2 + |\frac{\partial E_N}{\partial t}|^2) \\ & \leq \int_{\mathbb{R}^3} [|\Pi_{W_N}(H_0)|^2 + \epsilon_0 |\Pi_{W_N}(E_0)|^2 + |\nabla \times (\Pi_{W_N}(E_0))|^2 \\ & \quad + \epsilon_0^{-1} |\nabla \times (\Pi_{W_N}(H_0)) - \sigma \Pi_{W_N}(E_0)|^2] \\ & \leq C(\epsilon_0, \sigma) \int_{\mathbb{R}^3} \left[|H_0|^2 + |E_0|^2 + |\nabla H_0|^2 + |\nabla E_0|^2 \right] \end{aligned} \tag{2.30}$$

For u_N, by testing (2.3) with $\Phi = \frac{\partial u_N}{\partial t}$ as in (2.10) of Lemma 2.2, we have

$$\alpha_1 \int_\Omega |\frac{\partial u_N}{\partial t}|^2 + \frac{d}{dt} \int_\Omega e_\epsilon(u_N) \leq C \int_{\mathbb{R}^3} |H_N|^2. \tag{2.31}$$

This, with the help of (2.30), implies that for any $0 < T < +\infty$,

$$\begin{aligned} \alpha_1 \int_0^T \int_\Omega |\frac{\partial u_N}{\partial t}|^2 + \int_\Omega e_\epsilon(u_N(T)) \ \leq \ & CT \int_{\mathbb{R}^3} \left[|H_0|^2 + |E_0|^2 + |\nabla H_0|^2 + |\nabla E_0|^2 \right] \\ & + \int_\Omega |\nabla u_0|^2 + o(1). \end{aligned} \tag{2.32}$$

Here we have used (2.7) in the last step.

It follows from (2.30), (2.4), and (2.5) with $\beta = 0$ that

$$\begin{aligned} & \int_{\mathbb{R}^3} [|\nabla \times H_N|^2 + |\nabla \times E_N|^2](T) \\ \leq \ & C \int_{\mathbb{R}^3} [|\frac{\partial E_N}{\partial t}|^2 + |E_N|^2 + |\frac{\partial H_N}{\partial t}|^2](T) \\ \leq \ & C \int_{\mathbb{R}^3} [|H_0|^2 + |E_0|^2 + |\nabla H_0|^2 + |\nabla E_0|^2]. \end{aligned} \tag{2.33}$$

It follows from (2.30), (2.32), and (2.33) that we may assume, after taking subsequences, that for any $0 < T < +\infty$,

$$u_N \rightharpoonup u^\epsilon \text{ weak}^* \text{ in } L^\infty([0,T], H^1(\Omega)), \frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\epsilon}{\partial t} \text{ in } L^2(\Omega \times [0,T]);$$

$$H_N \rightharpoonup H^\epsilon, \ \frac{\partial H_N}{\partial t} \rightharpoonup \frac{\partial H^\epsilon}{\partial t}, \ \nabla \times H_N \rightharpoonup \nabla \times H^\epsilon \text{ in } L^2(\mathbb{R}^3 \times [0,T]);$$

$$E_N \rightharpoonup E^\epsilon, \ \frac{\partial E_N}{\partial t} \rightharpoonup \frac{\partial E^\epsilon}{\partial t}, \ \nabla \times E_N \rightharpoonup \nabla \times E^\epsilon \text{ in } L^2(\mathbb{R}^3 \times [0,T]).$$

As in the previous lemmas, it is standard to check that $(u^\epsilon, H^\epsilon, E^\epsilon)$ solves (1.19), (1.18), and the initial-boundary conditions (1.8), (1.9, (1.10) and (1.11). Moreover, by the lower semicontinuity, we have that for $0 < T < +\infty$,

$$\begin{aligned}
&\int_0^T \int_\Omega |\frac{\partial u^\epsilon}{\partial t}|^2 + \mathcal{E}_\epsilon(T) \\
&+ \int_{\mathbb{R}^3} \left[|H^\epsilon|^2 + |E^\epsilon|^2 + |\frac{\partial H^\epsilon}{\partial t}|^2 + |\frac{\partial E^\epsilon}{\partial t}|^2 + |\nabla \times H^\epsilon|^2 + |\nabla \times E^\epsilon|^2 \right] \\
\leq\ & C(\epsilon_0, \sigma, T) \left(\int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} [|H_0|^2 + |E_0|^2 + |\nabla H_0|^2 + |\nabla E_0|^2] \right). \quad (2.34)
\end{aligned}$$

As in the previous lemma, we can check that $\nabla \cdot H_0 = \nabla \cdot E_0 = 0$ is preserved under the equation (1.18), i.e.,

$$\nabla \cdot H^\epsilon(t) = \nabla \cdot E^\epsilon(t) = 0 \ \text{ a.e. } t > 0. \quad (2.35)$$

Finally, it is not hard to see that (2.34) and (2.35) yield (2.26). Hence the proof is complete. □

Remark 2.5 It follows from lemma 2.3 and lemma 2.4 that for any $0 < T < +\infty$, H^ϵ is uniformly bounded in $L^\infty([0,T], H^1(\mathbb{R}^3))$. Hence by the Sobolev embedding inequality that H^ϵ is uniformly bounded in $L^\infty([0,T], L^6(\mathbb{R}^3))$. This property plays an important role in the proof of C^α-regularity of ∇u claimed in both theorems 1.3 and 1.4.

We end this section with a local energy inequality.

Lemma 2.6 *There exists $C > 0$ such that for any $\epsilon > 0$, $u_0 \in H^1(\Omega, S^2)$, $H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $E_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, let $\{u^\epsilon, H^\epsilon, E^\epsilon\}$ be the global weak solution of (1.19), (1.2)-(1.3), with (1.8), (1.9)-(1.11) obtained in Lemma 2.2. Then for any $x_0 \in \Omega$, $t_0 > 0$, and $0 < r < \min\{dist(x_0, \partial\Omega), \frac{\sqrt{t_0}}{2}\}$,*

$$\begin{aligned}
&r^{-1} \int_{P_{\frac{r}{2}}(z_0)} \left| \frac{\partial u^\epsilon}{\partial t} \right|^2 + r^{-1} \max_{t \in [t_0 - \frac{r^2}{4}, t_0]} \int_{B_{\frac{r}{2}}(x_0)} e_\epsilon(u^\epsilon) \\
\leq\ & C r^{-3} \int_{P_r(z_0)} e_\epsilon(u^\epsilon) + C r^{-1} \int_{P_r(z_0)} |H^\epsilon|^2. \quad (2.36)
\end{aligned}$$

Proof. Write (u, H) for (u^ϵ, H^ϵ). For $x_0 \in \Omega$ and $0 < r < \min\{\text{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$, by Fubini's theorem there is $\alpha \in (\frac{1}{2}, \frac{7}{8})$ such that

$$\int_{B_r(x_0)} e_\epsilon(u)(t_0 - \alpha^2 r^2) \leq 8r^{-2} \int_{P_r(z_0)} e_\epsilon(u). \tag{2.37}$$

Let $\phi(x) \in C_0^\infty(B_r(x_0))$ be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{\frac{r}{2}}(x_0)$. Multiplying (1.19) by $\phi^2 \frac{\partial u}{\partial t}$ and integrating over $B_r(x_0)$, we get

$$\begin{aligned} & \alpha_1 \int_{B_r(x_0)} \left|\frac{\partial u}{\partial t}\right|^2 \phi^2 + \frac{d}{dt} \int_{B_r(x_0)} e_\epsilon(u)\phi^2 \\ = & -2 \int_{B_r(x_0)} \phi \nabla\phi \nabla u \cdot \frac{\partial u}{\partial t} - \int_{B_r(x_0)} \phi^2 u \times (H \times u) \cdot \frac{\partial u}{\partial t} \\ \leq & \frac{\alpha_1}{2} \int_{B_r(x_0)} |\frac{\partial u}{\partial t}|^2 \phi^2 + C(\alpha_1) \int_{B_r(x_0)} \left(|\nabla\phi|^2 |\nabla u|^2 + \phi^2 |H|^2\right). \end{aligned} \tag{2.38}$$

Integrating (2.38) from $t_0 - \alpha^2 r^2$ to $t \in [t_0 - \frac{r^2}{4}, t_0]$ and applying (2.37), we can obtain (2.36). □

3 Energy monotonicity on time slices

An energy monotonicity analogous to that of Struwe [28] (see also [7] and [5]) is unknown for Landau-Lifshitz type equations. In order to derive an prior estimate for $\{u^\epsilon, E^\epsilon, H^\epsilon\}$ under the small energy condition, we need an energy monotonicity of u^ϵ on time slices, which can be derived by the Pohozaev type argument as in [29].

Lemma 3.1 *For $\epsilon > 0$, let $\{u^\epsilon, H^\epsilon\}$ be a weak solution to the equation (1.19). Then for a.e. $t > 0$, any $x_0 \in \Omega$, and $0 < r \leq R < \min\{1, dist(x_0, \partial\Omega)\}$, there holds*

$$r^{-1} E_\epsilon(u^\epsilon, B_r(x_0)) \leq 2R^{-1} E_\epsilon(u^\epsilon, B_R(x_0)) + C_0 R \int_{B_R(x_0)} \left(|\frac{\partial u^\epsilon}{\partial t}|^2 + |H^\epsilon|^2\right), \tag{3.1}$$

and

$$\begin{aligned} \int_{B_R} |x - x_0|^{-1} \frac{(1 - |u^\epsilon|^2)^2}{\epsilon^2} \leq \; & 2R^{-1} E_\epsilon(u^\epsilon, B_R(x_0)) \\ & + C_0 R \int_{B_R(x_0)} \left(|\frac{\partial u^\epsilon}{\partial t}|^2 + |H^\epsilon|^2\right) \end{aligned} \tag{3.2}$$

for some $C_0 = C_0(\alpha_1) > 0$, where

$$E_\epsilon(u^\epsilon, A) = \int_A \left(\frac{1}{2}|\nabla u^\epsilon|^2 + \frac{(1 - |u^\epsilon|^2)^2}{2\epsilon^2}\right), \; A \subseteq \mathbb{R}^3.$$

Proof. The proof is a modification of [29] (see also [26] and [9]. We sketch it here. First observe that for a.e. $t > 0$, $\Delta u \in L^2(\Omega)$ and hence $\nabla^2 u \in L^2(\Omega)$. For $p \in \mathbb{R}^3$, define $R(p) : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$R(p)(v) = \alpha_1 v - \alpha_2 p \times v, \; \forall v \in \mathbb{R}^3.$$

Assume $x_0 = 0 \in \Omega$. Write $(u, H) = (u^\epsilon, H^\epsilon)$ and $B_r = B_r(0)$. Multiplying (1.19) by $x \cdot \nabla u$ and integrating over B_r yields

$$\begin{aligned}
&\int_{B_r} \langle R(u)(\frac{\partial u}{\partial t}) + u \times (u \times H),\ x \cdot \nabla u\rangle \\
= &\int_{B_r} \langle \Delta u + \frac{1}{\epsilon^2}(1 - |u|^2)u,\ x \cdot \nabla u\rangle \\
= &\ r\int_{\partial B_r} \left[|\frac{\partial u}{\partial r}|^2 - \frac{1}{2}|\nabla u|^2 - \frac{(1-|u|^2)^2}{4\epsilon^2}\right] + \int_{B_r}\left[\frac{1}{2}|\nabla u|^2 + \frac{3(1-|u|^2)^2}{4\epsilon^2}\right] \\
\geq &\ r\int_{\partial B_r} \left[|\frac{\partial u}{\partial r}|^2 - \frac{1}{2}|\nabla u|^2 - \frac{(1-|u|^2)^2}{4\epsilon^2}\right] + E_\epsilon(u, B_r). \qquad (3.3)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{d}{dr}\left(r^{-1}E_\epsilon(u, B_r) - r^{-1}\int_{B_r} \langle R(u)(\frac{\partial u}{\partial t}) + u \times (u \times H),\ x \cdot \nabla u\rangle\right) \qquad (3.4) \\
\geq &\ r^{-1}\int_{\partial B_r}[|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}] - r^{-1}\int_{\partial B_r}\langle R(u)(\frac{\partial u}{\partial t}) + u \times (u \times H),\ x \cdot \nabla u\rangle.
\end{aligned}$$

Since $|u| \leq 1$, we have $|u \times (u \times H)| \leq |H|$ and $\left|R(u)(\frac{\partial u}{\partial t})\right| \leq \left|\frac{\partial u}{\partial t}\right|$. The second term of the right hand side of (3.4) can be estimated by

$$\begin{aligned}
&-r^{-1}\int_{\partial B_r}\langle R(u)(\frac{\partial u}{\partial t}) + u \times (u \times H),\ x \cdot \nabla u\rangle \\
\geq &\ -\frac{1}{4}r^{-1}\int_{\partial B_r}|\frac{\partial u}{\partial r}|^2 - r\int_{\partial B_r}(|\frac{\partial u}{\partial t}|^2 + |H|^2). \qquad (3.5)
\end{aligned}$$

The second term of the left hand side of (3.4) can be estimated by

$$\begin{aligned}
&\left|r^{-1}\int_{B_r}\langle R(u)(\frac{\partial u}{\partial t}) + u \times (u \times H),\ x \cdot \nabla u\rangle\right| \\
\leq &\ \frac{1}{4}E_\epsilon(u, B_r) + r\int_{B_r}(|\frac{\partial u}{\partial t}|^2 + |H|^2). \qquad (3.6)
\end{aligned}$$

Putting (3.5) and (3.6) into (3.4) and integrating from r to R gives

$$\begin{aligned}
&2R^{-1}E_\epsilon(u, B_R) + R\int_{B_R}(|\frac{\partial u}{\partial t}|^2 + |H|^2) \\
&\geq \frac{1}{2}r^{-1}E_\epsilon(u, B_r) - r\int_{B_r}(|\frac{\partial u}{\partial t}|^2 + |H|^2) + \int_{B_R \setminus B_r}\frac{1}{|x|}\left[|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}\right] \\
&-\int_0^R s\int_{\partial B_s}(|\frac{\partial u}{\partial t}|^2 + |H|^2). \qquad (3.7)
\end{aligned}$$

Since

$$r\int_{B_r}(|\frac{\partial u}{\partial t}|^2 + |H|^2) \leq R\int_{B_R}(|\frac{\partial u}{\partial t}|^2 + |H|^2),$$

and

$$\int_0^R s\int_{\partial B_s}(|\frac{\partial u}{\partial t}|^2 + |H|^2) \leq R\int_{\partial B_R}(|\frac{\partial u}{\partial t}|^2 + |H|^2),$$

(3.7) clearly implies both (3.1) and (3.2). Hence the lemma is proved. □

4 On the lower bound of $|u^\epsilon|$

In this section, we will establish a lower bound estimate of $|u^\epsilon|$ on generic time slices, under the smallness condition of $r^{-3}\int_{P_r} e_\epsilon(u^\epsilon)$. First we define good time slices.

Definition 4.1 For any $\epsilon \in (0, \frac{1}{2})$, $x_0 \in \Omega$, $t_0 > 0$, $0 < r < \min\{\mathrm{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$, and $\Lambda > 0$, we define the set of good time slices by

$$G^\Lambda_{z_0,r} = \left\{ t \in [t_0 - r^2, t_0) : \int_{B_r(x_0)} |\frac{\partial u^\epsilon}{\partial t}|^2 \leq \frac{\Lambda^2}{r^2} \int_{P_r(z_0)} |\frac{\partial u^\epsilon}{\partial t}|^2 \right\}, \tag{4.1}$$

and the set of bad time slices

$$B^\Lambda_{z_0,r} = [t_0 - r^2, t_0) \setminus G^\Lambda_{z_0,r}. \tag{4.2}$$

By Fubini's theorem, we have

$$\left|B^\Lambda_{z_0,r}\right| \leq \frac{r^2}{\Lambda^2}. \tag{4.3}$$

Similar to [29] and [26], we have

Lemma 4.2 *For any $\epsilon > 0$, let $\{u^\epsilon, H^\epsilon\}$ be the weak solution of (1.19) obtained in Lemma 2.2. Denote $\|H^\epsilon\|_{L^\infty_t L^2_x(\mathbb{R}^3\times[0,t_0])} = C_0$. Then for any $\Lambda > 0$, there exist $\eta_0 > 0$ and $r_0 > 0$ depending on Λ and C_0 such that for any $z_0 = (x_0, t_0) \in \Omega \times (0, +\infty)$, and $0 < r < \min\{dist(x_0, \partial\Omega), \sqrt{t_0}, r_0\}$ if*

$$r^{-3} \int_{P_r(z_0)} e_\epsilon(u^\epsilon) \leq \eta_0^2, \tag{4.4}$$

then

$$|u^\epsilon|(x,t) \geq \frac{1}{2}, \ \forall x \in B_{\frac{r}{4}}(x_0) \quad and\ t \in G^\Lambda_{z_0,\frac{r}{2}}. \tag{4.5}$$

Proof. It is a modification of [26] and [29]. We prove a $C^{\frac{1}{2}}$-estimate of $u^\epsilon(\cdot, s)$ for $s \in G^\Lambda_{z_0,\frac{r}{2}}$ (see also [26] page 577, Lemma 5). Define $v^\epsilon(x,t) = u^\epsilon(x_0 + \epsilon x, s + \epsilon^2 t) : B_2 \times [-4, 4] \to \mathbb{R}^3$. Then $w^\epsilon(x) \equiv v^\epsilon(x, 0)$ satisfies

$$\Delta w^\epsilon = R(w^\epsilon)\left(\frac{\partial v^\epsilon}{\partial t}(0)\right) - (1 - |w^\epsilon|^2)w^\epsilon - w^\epsilon \times (\tilde{H}^\epsilon \times w^\epsilon), \tag{4.6}$$

where $\tilde{H}^\epsilon(x) = \epsilon^2 H^\epsilon(\epsilon x, s)$. By the standard $W^{2,2}$ estimate, we have

$$\begin{aligned}
\|\nabla^2 w^\epsilon\|^2_{L^2(B_1)} &\leq C\left[1 + \|\frac{\partial w^\epsilon}{\partial t}\|^2_{L^2(B_2)} + \|\tilde{H}^\epsilon\|^2_{L^2(B_2)}\right] \\
&\leq C\left[1 + \epsilon\int_{B_{2\epsilon}(x_0)} (|\frac{\partial u^\epsilon}{\partial t}|^2 + |H^\epsilon|^2)(s)\right] \\
&\leq C\left[1 + C_0^2 + r\int_{B_{\frac{r}{2}}(x_0)} |\frac{\partial u^\epsilon}{\partial t}|^2(s)\right] \\
&\leq C\left(1 + C_0^2 + \Lambda^2\epsilon_0^2\right),
\end{aligned} \tag{4.7}$$

where we have used both (4.1) and lemma 2.6 in the last step. Therefore, by the Sobolev embedding theorem, $w^\epsilon \in C^{\frac{1}{2}}(B_1)$. Moreover, by rescaling and (4.7), we have

$$[u^\epsilon(s)]_{C^{\frac{1}{2}}(B_{\frac{r}{2}}(x_0))} \leq C(\Lambda, \eta_0, C_0)\epsilon^{-\frac{1}{2}}, \ \forall s \in G^\Lambda_{z_0, \frac{r}{2}}. \tag{4.8}$$

Suppose that (4.5) were false. Then there exists $z_1 = (x_1, t_1) \in B_{\frac{r}{4}}(x_0) \times G^\Lambda_{z_0, \frac{r}{2}}$ such that $|u^\epsilon(z_1)| < \frac{1}{2}$. Hence for sufficiently small $\theta_0 > 0$, if $y \in B_{\theta_0^2\epsilon}(x_1)$, then

$$\begin{aligned} |u^\epsilon|(y, t_1) &\leq |u^\epsilon|(x_1, t_1) + [u^\epsilon(t_1)]_{C^{\frac{1}{2}}} |y - x_1|^{\frac{1}{2}} \\ &\leq \frac{1}{2} + C(\Lambda, \eta_0, C_0)\theta_0 \leq \frac{3}{4} \end{aligned}$$

so that

$$\int_{B_{\theta_0^2\epsilon}(x_1)} |x - x_1|^{-1} \frac{(1 - |u^\epsilon|^2)^2(x, t_1)}{\epsilon^2} \geq C_1. \tag{4.9}$$

On the other hand, (4.4) gives

$$\sup_{x \in B_{\frac{r}{2}}(x_0)} (\frac{r}{2})^{-3} \int_{P_{\frac{r}{2}}(x, t_0)} e_\epsilon(u^\epsilon) \leq 8\eta_0^2. \tag{4.10}$$

This, combined with lemma 2.6, implies

$$\sup_{t \in [t_0 - \frac{r^2}{16},\ t_0)} \sup_{x \in B_{\frac{r}{4}}(x_0)} (\frac{r}{4})^{-1} \int_{B_{\frac{r}{4}}(x)} e_\epsilon(u^\epsilon) \leq C(\eta_0^2 + C_0^2 r). \tag{4.11}$$

By the definition of $G^\Lambda_{z_0,\ \frac{r}{2}}$ and lemma 2.6, we have

$$\begin{aligned} \sup_{t \in G^\Lambda_{z_0,\ \frac{r}{2}}} \sup_{x \in B_{\frac{r}{4}}(x_0)} r \int_{B_{\frac{r}{4}}(x)} |\frac{\partial u^\epsilon}{\partial t}|^2(t) &\leq \sup_{t \in G^\Lambda_{z_0,\ \frac{r}{2}}} r \int_{B_{\frac{r}{2}}(x_0)} |\frac{\partial u^\epsilon}{\partial t}|^2(t) \\ &\leq C\left[\frac{\Lambda^2}{r^3} \int_{P_r(z_0)} e_\epsilon(u^\epsilon) + \Lambda^2 r \|H^\epsilon\|^2_{L^\infty_t L^2_x}\right] \\ &\leq C\Lambda^2(\eta_0^2 + C_0^2 r). \end{aligned} \tag{4.12}$$

With (4.11), (4.12), and the monotonicity inequality (3.2), we obtain

$$\int_{B_{\theta_0^2\epsilon}(x_1)} |x - x_1|^{-1} \frac{(1 - |u^\epsilon|^2)^2}{\epsilon_0^2}(t_1) \tag{4.13}$$

$$\begin{aligned} &\leq C\left[r^{-1} \int_{B_{\frac{r}{4}}(x_1)} e_\epsilon(u^\epsilon)(t_1) + r \int_{B_{\frac{r}{4}}(x_1)} (|\frac{\partial u^\epsilon}{\partial t}|^2 + |H^\epsilon|^2)(t_1)\right] \\ &\leq C(\Lambda^2\eta_0^2 + C_0^2 r_0). \end{aligned} \tag{4.14}$$

This contradicts (4.9) provide $r_0 > 0$ and $\eta_0 > 0$ are chosen sufficiently small. Hence the proof is complete. □

5 Energy decay estimates and proof of theorem 1.2

In this section , we first establish the decay estimate of the normalized energy $r^{-3}\int_{P_r(z)} e_\epsilon(u^\epsilon)$, provided that it is sufficiently small. Then we give a proof of theorem 1.2. The techniques employed in the proof are suitable modifications of that by Hélein [21] and Evans [13] in the context of harmonic maps. We begin with

Lemma 5.1 *For any given $L > 0$ and $\delta > 0$, there exist $C(\delta) > 0$, $\eta(\delta) > 0$, and $\epsilon_1(\delta) > 0$, such that if $\{u^\epsilon, H^\epsilon\}$ is the weak solution of (1.19) obtained by Lemma 2.2 and for $z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}^+$, $0 < r < \min\{dist(x_0, \partial\Omega), \sqrt{t_0}, \frac{\epsilon_1^2(\delta)}{L^2}\}$, and $0 < \epsilon \le \eta(\delta) r$, there holds*

$$\|H^\epsilon\|_{L_t^\infty L_x^2(P_r(z_0))} \le L,\ r^{-3}\int_{P_r(z_0)} e_\epsilon(u^\epsilon) \le \epsilon_1^2(\delta), \tag{5.1}$$

then we have

$$\begin{aligned}(\frac{r}{8})^{-3}\int_{P_{\frac{r}{8}}(z_0)} e_\epsilon(u^\epsilon) &\le \delta\left[r^{-3}\int_{P_r(z_0)} e_\epsilon(u^\epsilon) + r\,\|H^\epsilon\|^2_{L_t^\infty L_x^2(P_r(z_0))}\right] \\ &\quad + \frac{C(\delta)}{\delta} r^{-5}\int_{P_r(z_0)} \left|u^\epsilon - u^\epsilon_{P_r(z_0)}\right|^2, \end{aligned} \tag{5.2}$$

where $u^\epsilon_{P_r(z_0)} = \frac{1}{|P_r(z_0)|}\int_{P_r(z_0)} u^\epsilon$, $r > 0$, is the average of u^ϵ over $P_r(z_0)$.

Proof. We follow [29] page 1631, proposition 5.1 with suitable modifications, and outline the key steps here. For simplicity, write $(u, H) = (u^\epsilon, H^\epsilon)$ and assume $z_0 = (x_0, t_0) = (0, 1) \in \Omega \times \mathbb{R}_+$. For $r > 0$, let $u_r(x, t) = u(rx, 1 + r^2 t)$ and $H_r(x, t) = r^2 H(rx, 1 + r^2 t)$ for $(x, t) \in P_1$. Then it follows from (1.19) that (u_r, H_r) satisfies:

$$R(u_r)(\frac{\partial u_r}{\partial t}) = \Delta u_r + \frac{(1 - |u_r|^2)}{\hat\epsilon^2} u_r + u_r \times (H_r \times u_r) \text{ in } P_1,$$

where $\hat\epsilon = r^{-1}\epsilon$. Moreover,

$$\int_{P_1} e_{\hat\epsilon}(u_r) = r^{-3}\int_{P_r(0,1)} e_\epsilon(u) \le \epsilon_1^2(\delta),$$

and

$$\|H_r\|^2_{L_t^\infty L_x^2(P_1)} = r\|H\|^2_{L_t^\infty L_x^2(P_r(0,1))} \le L^2 r \le \epsilon_1^2(\delta),$$

as $r \le \frac{\epsilon_1^2(\delta)}{L^2}$. From this scaling argument, we may further assume that $r = 1$ and

$$\|H\|_{L_t^\infty L_x^2(P_1(0,1))} \le \epsilon_1(\delta). \tag{5.3}$$

Observe that

$$\begin{aligned}\int_{P_{\frac{1}{8}}(0,1)} e_\epsilon(u) &= \int_{(1-(\frac{1}{8})^2,1)\cap G^\Lambda_{(0,1),\frac{1}{2}}}\int_{B_{\frac{1}{8}}} e_\epsilon(u) + \int_{(1-(\frac{1}{8})^2,1)\cap B^\Lambda_{(0,1),\frac{1}{2}}}\int_{B_{\frac{1}{8}}} e_\epsilon(u) \\ &= I + II. \end{aligned} \tag{5.4}$$

By (4.3) and lemma 2.6, we can estimate

$$\begin{aligned} II &\le \left|B^{\Lambda}_{(0,1),\frac{1}{2}}\right| \sup_{t\in B^{\Lambda}_{(0,1),\frac{1}{2}}\cap[1-(\frac{1}{8})^2,1]} \int_{B_{\frac{1}{8}}} e_\epsilon(u) \\ &\le \frac{1}{\Lambda^2}\int_{P_1(0,1)} \left[e_\epsilon(u)+|H|^2\right]. \end{aligned} \tag{5.5}$$

To estimate I, observe that (5.3) and lemma 4.1 imply that

$$|u|(x,t)\ge \frac{1}{2},\ \forall x\in B_{\frac{1}{4}} \text{ and } t\in G^{\Lambda}_{(0,1),\frac{1}{2}}. \tag{5.6}$$

This, combined with the fact $|u|\le 1$ in $\Omega\times\mathbb{R}_+$, implies

$$|\nabla u|^2 \le 4|u|^2|\nabla u|^2 = 4|\nabla u\times u|^2 + |\nabla|u|^2|^2 \le 4(|\nabla u\times u|^2+|\nabla|u||^2).$$

Therefore for $t\in G^{\Lambda}_{(0,1),\frac{1}{2}}$,

$$\begin{aligned} \int_{B_{\frac{1}{8}}} e_\epsilon(u) &\le 2\int_{B_{\frac{1}{8}}} |\nabla u\times u|^2 + \int_{B_{\frac{1}{8}}}\left(2|\nabla|u||^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}\right) \\ &= III+IV. \end{aligned} \tag{5.7}$$

By the definition of $G^{\Lambda}_{(0,1),\frac{1}{2}}$ and lemma 2.6, we have

$$\begin{aligned} \int_{B_{\frac{1}{2}}} e_\epsilon(u) + \int_{B_{\frac{1}{2}}} |\frac{\partial u}{\partial t}|^2 &\le C\Lambda^2\left(\int_{P_1(0,1)} e_\epsilon(u) + \int_{P_1(0,1)} |H|^2\right) \\ &\le C\Lambda^2\left(\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right). \end{aligned} \tag{5.8}$$

Hence, for $t\in G^{\Lambda}_{(0,1),\frac{1}{2}}$, there holds

$$\begin{aligned} \sup_{x\in B_{\frac{1}{4}}}\left\{\int_{B_{\frac{1}{4}}(x)} e_\epsilon(u) + \int_{B_{\frac{1}{4}}(x)} |\frac{\partial u}{\partial t}|^2\right\} &\le C\Lambda^2\int_{P_1(0,1)} e_\epsilon(u) \\ &\quad + C\Lambda^2\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}. \end{aligned} \tag{5.9}$$

It follows from (5.9) and lemma 3.1 that

$$\begin{aligned} \sup\left\{s^{-1}\int_{B_s(x)} |\nabla u|^2 :\ x\in B_{\frac{1}{4}},\ 0<s<\frac{1}{4}\right\} &\le C\Lambda^2\int_{P_1(0,1)} e_\epsilon(u) \\ &\quad + C\Lambda^2\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}. \end{aligned} \tag{5.10}$$

To estimate III, let $\phi \in C_0^\infty(B_{\frac{1}{4}})$ be such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{\frac{1}{8}}$, and $|\nabla\phi| \le 128$. Then we have, by integration by parts,

$$
\begin{aligned}
\int_{B_{\frac{1}{8}}} |\nabla u \times u|^2 &\le \int_{\mathbb{R}^3} \phi^2 |\nabla u \times u|^2 \\
&= \int_{\mathbb{R}^3} \phi^2 (\nabla u \times u) \cdot (\nabla u \times u) \\
&= \int_{\mathbb{R}^3} \phi^2 (\nabla u \times u) \cdot \Big(\nabla (u - c_{\frac{1}{4}}(t)) \times u\Big) \\
&= \int_{\mathbb{R}^3} \big(\phi^2 (\nabla u \times u) \times \nabla u\big) \cdot \Big(u - c_{\frac{1}{4}}(t)\Big) \\
&\quad - \int_{\mathbb{R}^3} \nabla \cdot \big(\phi^2 (\nabla u \times u)\big) \cdot \Big((u - c_{\frac{1}{4}}(t)) \times u\Big) \\
&= \int_{\mathbb{R}^3} \phi^2 \left[(\nabla u \times u) \times \nabla u - \lambda\right] \cdot (u - c_{\frac{1}{4}}(t)) \\
&\quad + \lambda \int_{\mathbb{R}^3} \phi^2 (u - c_{\frac{1}{4}}(t)) \\
&\quad - \int_{\mathbb{R}^3} \nabla \cdot \big(\phi^2 (\nabla u \times u)\big) \cdot \Big((u - c_{\frac{1}{4}}(t)) \times u\Big) \\
&= III_1 + III_2 + III_3, \qquad (5.11)
\end{aligned}
$$

where

$$
\lambda = \frac{\int_{\mathbb{R}^3} \phi^2 (\nabla u \times u) \times \nabla u}{\int_{\mathbb{R}^3} \phi^2}, \ c_r(t) = \frac{1}{|B_r|} \int_{B_r} u(t) \ \text{ for } r > 0.
$$

It follows from lemma 2.6 that

$$
|\lambda| \le C \int_{B_{\frac{1}{4}}} |\nabla u|^2 \le C \left[\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L_t^\infty L_x^2(P_1(0,1))}\right] \qquad (5.12)
$$

so that by Hölder inequality and Poincaré inequality, we have

$$
\begin{aligned}
|III_2| &\le |\lambda| \left\| u - c_{\frac{1}{4}}(t) \right\|_{L^2(B_{\frac{1}{4}})} \\
&\le C \left[\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L_t^\infty L_x^2(P_1(0,1))}\right] \|\nabla u\|_{L^2(B_{\frac{1}{4}})} \\
&\le C \left[\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L_t^\infty L_x^2(P_1(0,1))}\right]^{\frac{3}{2}}. \qquad (5.13)
\end{aligned}
$$

To estimate III_3, we first note that (1.19) is equivalent to

$$
\nabla \cdot (\nabla u \times u) = \left[R(u)(\frac{\partial u}{\partial t}) + u \times (u \times H)\right] \times u. \qquad (5.14)
$$

Hence, by using (5.14), (5.10), and lemma 2.6, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \left|\nabla\cdot(\phi^2\nabla u\times u)\right|^2 &\le \int_{\mathbb{R}^3}\left[|\nabla\phi|^2|\nabla u|^2+\phi^2\left|\nabla\cdot(\nabla u\times u)\right|^2\right]\\ &\le C\int_{B_{\frac{1}{4}}}|\nabla u|^2+C\int_{B_{\frac{1}{4}}}\left[|\frac{\partial u}{\partial t}|^2+|H|^2\right]\\ &\le C\Lambda^2\left[\int_{P_1(0,1)} e_\epsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right].\end{aligned}$$

Therefore, by Hölder inequality we have that for any $\delta>0$,

$$\begin{aligned}|III_3| &\le \left\|\nabla\cdot(\phi^2\nabla u\times u)\right\|_{L^2(\mathbb{R}^3)}\left\|u-c_{\frac{1}{4}}(t)\right\|_{L^2(B_{\frac{1}{4}})}\\ &\le \frac{\delta}{4}\left[\int_{P_1(0,1)} e_\epsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right]\\ &\quad +C\frac{\Lambda^2}{\delta}\|u-c_{\frac{1}{4}}(t)\|^2_{L^2(B_{\frac{1}{4}})}.\end{aligned}\tag{5.15}$$

To estimate III_1, we utilize the duality between Hardy and BMO spaces (see also [29], [21], and [13]). First, by the definition of BMO norm, Poincaré inequality, and (5.10), we have

$$\begin{aligned}\left[u-c_{\frac{1}{4}}(t)\right]^2_{\mathrm{BMO}(B_{\frac{1}{4}})} &\le \sup\left\{s^{-1}\int_{B_s(x)}|\nabla u|^2:\ x\in B_{\frac{1}{4}},\ 0<s<\frac{1}{4}\right\}\\ &\le C\Lambda^2\left[\int_{P_1(0,1)} e_\epsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right].\end{aligned}\tag{5.16}$$

Therefore it follows from (5.15), (5.16), and [29] proposition 5.6, proposition 5.7 and proposition 5.8 that

$$\begin{aligned}|III_1| &= \left|\int_{\mathbb{R}^3}\phi^2\left((\nabla u\times u)\times\nabla u-\lambda\right)\cdot\left(u-c_{\frac{1}{4}}(t)\right)\right|\\ &\le C\left\|\phi^2((\nabla u\times u)\times\nabla u-\lambda)\right\|_{\mathcal{H}^1(\mathbb{R}^3)}\left[u-c_{\frac{1}{4}}(t)\right]_{\mathrm{BMO}(B_{\frac{1}{4}})}\\ &\le C\left\|\phi^2(\nabla u\times u)\times\nabla u\right\|_{\mathcal{H}^1(B_{\frac{1}{4}},B_{\frac{1}{2}})}\left[u-c_{\frac{1}{4}}(t)\right]_{\mathrm{BMO}(B_{\frac{1}{4}})}\\ &\le C\left[u-c_{\frac{1}{4}}(t)\right]_{\mathrm{BMO}(B_{\frac{1}{4}})}\left[\|\nabla u\|^2_{L^2(B_{\frac{1}{2}})}+\|\nabla\cdot(\nabla u\times u)\|^2_{L^2(B_{\frac{1}{2}})}\right]\\ &\le C\Lambda^3\left\{\int_{P_1(0,1)} e_\epsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right\}^{\frac{3}{2}}.\end{aligned}\tag{5.17}$$

Putting all the estimates (5.3), (5.13), (5.15) and (5.17) together, we get

$$\begin{aligned}\int_{B_{\frac{1}{8}}} |\nabla u \times u|^2 &\leq \left(C\Lambda^3 \epsilon_1(\delta) + \frac{\delta}{4}\right) \left[\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L_t^\infty L_x^2(P_1(0,1))}\right] \\ &\quad + C\frac{\Lambda^2}{\delta} \int_{B_{\frac{1}{4}}} |u - c_{\frac{1}{4}}(t)|^2. \end{aligned} \tag{5.18}$$

Now we estimate IV as follows. It follows from (5.6) that we can write $u = \rho\omega$, with $\rho = |u| \geq \frac{1}{2}$ and $\omega = \frac{u}{|u|}$. Then ρ satisfies

$$\Delta\rho - \rho|\nabla\omega|^2 + \frac{(1-\rho^2)\rho}{\epsilon^2} = R(u)(\frac{\partial u}{\partial t}) \cdot \omega, \text{ in } B_{\frac{1}{4}}. \tag{5.19}$$

Multiplying (5.19) by $\phi^2(1-\rho)$ for $\phi \in C_0^\infty(B_{\frac{1}{4}})$ and integrating over $B_{\frac{1}{4}}$, we get

$$\begin{aligned} &\int_{B_{\frac{1}{4}}} \phi^2 \left[|\nabla\rho|^2 + \frac{(1-\rho)^2}{\epsilon^2}\rho(1+\rho)\right] \\ &= \int_{B_{\frac{1}{4}}} (1-\rho)\nabla\rho \cdot \nabla\phi^2 + \int_{B_{\frac{1}{4}}} \phi^2(1-\rho)R(u)(\frac{\partial u}{\partial t}) \cdot \omega \\ &+ \int_{B_{\frac{1}{4}}} \phi^2\rho(1-\rho)|\nabla\omega|^2 \\ &= IV_1 + IV_2 + IV_3. \end{aligned} \tag{5.20}$$

Since $|\nabla\rho| \leq |\nabla u|$, we have from lemma 2.6 that

$$\begin{aligned} |IV_1| &\leq \int_{B_{\frac{1}{4}}} |\nabla u|(1-|\rho|^2) \leq \epsilon \left(\int_{B_{\frac{1}{4}}} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{B_{\frac{1}{4}}} \frac{(1-|u|^2)^2}{\epsilon^2}\right)^{\frac{1}{2}} \\ &\leq C\Lambda^2\epsilon \left(\int_{P_1(0,1)} e_\epsilon(u) + \|H\|_{L_t^\infty L_x^2(P_1(0,1))}\right). \end{aligned} \tag{5.21}$$

For IV_2, we have

$$\begin{aligned} |IV_2| &\leq \int_{B_{\frac{1}{4}}} \left|\frac{\partial u}{\partial t}\right| (1-|\rho|^2) \leq \epsilon \left(\int_{B_{\frac{1}{4}}} |\frac{\partial u}{\partial t}|^2\right)^{\frac{1}{2}} \left(\int_{B_{\frac{1}{4}}} \frac{(1-|u|^2)^2}{\epsilon^2}\right)^{\frac{1}{2}} \\ &\leq C\Lambda^2\epsilon \left(\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L_t^\infty L_x^2(P_1^+(0,1))}\right). \end{aligned} \tag{5.22}$$

Since $|\omega| = 1$ and $\rho \geq \frac{1}{2}$, we have $|\nabla\omega|^2 \leq 14|\nabla u \times u|^2$. Hence we have

$$|IV_3| \leq C \int_{B_{\frac{1}{4}}} |\nabla u \times u|^2. \tag{5.23}$$

Therefore, for $t \in G^{\Lambda}_{(0,1),\frac{1}{2}}$, we get

$$|IV| \le C\Lambda^2\epsilon\left(\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right) + C\int_{B_{\frac{1}{4}}} |\nabla u \times u|^2. \tag{5.24}$$

Putting the estimates for III and IV together, we obtain for any $t \in G^{\Lambda}_{(0,1),\frac{1}{2}}$,

$$\begin{aligned}\int_{B_{\frac{1}{8}}} e_\epsilon(u) \quad &\le \quad \left[C\Lambda^2(\epsilon + \Lambda\epsilon_1(\delta)) + \frac{\delta}{4}\right]\left[\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right] \\ &\quad + C\frac{\Lambda^2}{\delta}\int_{B_1} |u - c_1(t)|^2. \end{aligned} \tag{5.25}$$

Integrating (5.25) over $t \in G^{\Lambda}_{(0,1),\frac{1}{2}}$ and adding (5.5), we obtain

$$\begin{aligned} &(\frac{1}{8})^{-3}\int_{P_{\frac{1}{8}}(0,1)} e_\epsilon(u) \\ \le \quad &\left[C\Lambda^2(\epsilon + \Lambda\epsilon_1(\delta)) + \frac{\delta}{4} + \frac{1}{\Lambda^2}\right]\left[\int_{P_1(0,1)} e_\epsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right] \\ &+ \frac{C\Lambda^2}{\delta}\int_{P_1(0,1)} |u - c_1(t)|^2. \end{aligned} \tag{5.26}$$

Lemma 5.1 is proved if we choose, for any fixed small $\delta > 0$, sufficiently large $\Lambda = \frac{2}{\sqrt{\delta}} > 0$, sufficiently small $\epsilon = \frac{\delta}{16C}$ and $\epsilon_1(\delta) = \frac{\delta^{\frac{5}{2}}}{32C}$. Here we have also used in the last step the fact that

$$\int_{P_1(0,1)} |u - c_1(t)|^2 \le 2\int_{P_1(0,1)} |u - u_{P_1(0,1)}|^2.$$

□

Next we need

Lemma 5.2 *There exists a constant $C_0 > 0$ such that for any $L > 0$, $\theta \in (0, \frac{1}{4})$ there are $\epsilon(\theta) > 0$ and $\epsilon_1(\theta) > 0$ such that if (u^ϵ, H^ϵ) is the weak solution of (1.19) by Lemma 2.2 and for $z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}_+$, $0 < r < \min\{dist(x_0, \partial\Omega), \sqrt{t_0}, \frac{\epsilon_1^2(\theta)}{L^2}\}$, and $\epsilon < \epsilon(\theta)r$, there holds*

$$\|H^\epsilon\|_{L^\infty_t L^2_x(P_r(z_0))} \le L, \quad \int_{P_r(z_0)} e_\epsilon(u^\epsilon) \le \epsilon_1^2(\theta),$$

then

$$\frac{1}{(\theta r)^5}\int_{P_{\theta r}(z_0)} \left|u^\epsilon - u^\epsilon_{P_{\theta r}(z_0)}\right|^2 \le C_0\theta^2 \max\left\{r^{-3}\int_{P_r(z_0)} e_\epsilon(u^\epsilon), r\|H^\epsilon\|^2_{L^\infty_t L^2_x(P_r(z_0))}\right\} \tag{5.27}$$

where $u^\epsilon_{P_{\theta r}(z_0)}$ is the average of u^ϵ over $P_{\theta r}(z_0)$.

Proof. Write (u, H) for (u^ϵ, H^ϵ). Assume that $z_0 = (0, 1)$, $r = 1$, and

$$\|H\|_{L_t^\infty L_x^2(P_1(0,1))} \le \epsilon_1(\theta).$$

Now we argue by contradiction. Suppose that lemma 5.2 were false. Then there are $\theta_0 \in (0, \frac{1}{4})$, $\epsilon_k \downarrow 0$, and a sequence of weak solutions (u^k, H^k) of (1.19) corresponding to $\epsilon = \epsilon_k$ such that

$$\int_{P_1(0,1)} e_{\epsilon_k}(u^k) = \delta_k^2 \downarrow 0, \quad \|H^k\|^2_{L_t^\infty L^2(P_1(0,1))} \le \delta_k^2, \tag{5.28}$$

but

$$\theta_0^{-5} \int_{P_{\theta_0}(0,1)} |u^k - u^k_{P_{\theta_0}(0,1)}|^2 \ge k\theta_0^2 \max\left\{\int_{P_1(0,1)} e_{\epsilon_k}(u^k),\ \|H^k\|^2_{L_t^\infty L^2(P_1(0,1))}\right\}. \tag{5.29}$$

Define $v^k = \frac{u^k - u^k_{P_1(0)}}{\delta_k}$. Then by lemma 2.6 $\{v^k\}$ is uniformly bounded in $H^1(P_{\frac{1}{2}}(0,1))$ and $(v^k)_{P_1(0,1)} = 0$. Assume that $v^k \to v$ weakly in $H^1(P_{\frac{1}{2}}(0,1), \mathbb{R}^3)$, strongly in $L^2(P_{\frac{1}{2}}(0,1), \mathbb{R}^3)$, and $u^k \to p$ for some $p \in S^2$. It is not hard to show that $v \in T_pS^2$ and hence we have $R(p)(\frac{\partial v}{\partial t}) - \Delta v \in T_pS^2$. Observe that

$$\left[R(u^k)(\frac{\partial v^k}{\partial t}) - \Delta v^k - \delta_k^{-1}(u^k \times (H^k \times u^k))\right] \times u^k = 0,$$

and (5.29) implies

$$|\delta_k^{-1}(u^k \times (H^k \times u^k))] \times u^k| \le \frac{|H^k|}{\delta_k} \to 0 \text{ in } L^2(P_1(0,1)) \ \text{ as } k \to \infty.$$

By sending k to ∞, v solves

$$\left(R(p)(\frac{\partial v}{\partial t}) - \Delta v\right) \times p = 0.$$

Therefore

$$R(p)(\frac{\partial v}{\partial t}) - \Delta v = 0 \ \text{ in } P_{\frac{1}{2}}(0,1). \tag{5.30}$$

The standard parabolic theory (cf. [24]) implies

$$\theta_0^{-5} \int_{P_{\theta_0}(0,1)} |v|^2 \le C\theta_0^2 \int_{P_1} |\nabla v|^2,$$

which contradicts (5.29). The proof is complete. □

Combining lemma 5.1 and lemma 5.2, we can prove

Lemma 5.3 *For any $\gamma \in (0, 1)$, there are $\theta \in (0, \frac{1}{4})$, $C_1 > 0$, $k_0 > 0$, $\epsilon_2 > 0$ such that if (u^ϵ, H^ϵ) is the weak solution of (1.19) obtained by Lemma 2.2 and, for any*

$z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}_+$, $L > 0$, $0 < r < \min\{dist(x_0, \partial\Omega), \sqrt{t_0}, \frac{\epsilon_2^2}{L^2}\}$, *and* $0 < \epsilon \le k_0 r$, *satisfies*

$$\|H^\epsilon\|_{L_t^\infty L_x^2(P_r(z_0))} \le L, \quad r^{-3} \int_{P_r(z_0)} e_\epsilon(u^\epsilon) \le \epsilon_2^2, \tag{5.31}$$

then

$$(\theta r)^{-3} \int_{P_{\theta r}(z_0)} e_\epsilon(u^\epsilon) \le C_1 \left[\theta^{2\gamma} r^{-3} \int_{P_r(z_0)} e_\epsilon(u^\epsilon) + \theta r \|H^\epsilon\|^2_{L_t^\infty L_x^2(P_r(z_0))} \right]. \tag{5.32}$$

Proof. The ideas here are similar to [29] and [26]. To simplify the notations, write (u, H) for (u^ϵ, H^ϵ). As in the proof of lemma 5.1 and 5.2, we may assume that $z_0 = (0, 1)$, $r = 1$, and

$$\|H\|_{L_t^\infty L_x^2(P_1(0,1))} \le \epsilon_2. \tag{5.33}$$

Let $\delta = 8^{-3}$, $\theta = \theta(\gamma) \le \left(\frac{\delta^2}{2C_0 C(\delta)}\right)^{\frac{1}{2-2\gamma}}$, here $C_0 > 0$ and $C(\delta) > 0$ are given by lemma 5.2 and lemma 5.1 respectively, and $k \ge 1$ be such that $8^k \theta = 1$. For $0 < \rho < 1$, set

$$E(u, \rho) = \rho^{-3} \int_{P_\rho(0,1)} e_\epsilon(u), \;\; F(H, \rho) = \rho \|H\|^2_{L_t^\infty L_x^2(P_\rho(0,1))}.$$

For $0 \le i \le k-1$, if $E(u, 8^{i+1}\theta) \le \epsilon_1^2(\delta)$ and $E(u, 1) \le \epsilon_1^2(8^{i+1}\theta)$, then lemma 5.1 and lemma 5.2 would imply

$$\begin{aligned} E(u, 8^i\theta) \;\; &\le \;\; \delta \max\left\{E(u, 8^{i+1}\theta), F(H, 8^{i+1}\theta)\right\} \\ &\quad + \frac{C_0 C(\delta)}{\delta} \max\left\{E(u, 1), F(H, 1)\right\} \end{aligned} \tag{5.34}$$

Now we choose

$$\epsilon_2 \equiv \frac{\delta}{2C_0 C(\delta)} \min\left\{\epsilon_1(8\theta), \cdots, \epsilon_1(8^k\theta), \epsilon_1(\delta)\right\}.$$

Since

$$F(H, \rho) \le \rho F(H, 1) \le F(H, 1) \le \epsilon_2^2,$$

(5.34) implies that

$$E(u, 8^i\theta) \le \min\left\{\epsilon_1^2(8\theta), \cdots, \epsilon_1^2(8^k\theta), \epsilon_1^2(\delta)\right\}, \;\; \forall 0 \le i \le k.$$

Hence by iteration, (5.34) implies

$$\begin{aligned} E(u, \theta) \;\; &\le \;\; \delta^k E(u, 1) + \left(\sum_{i=1}^k (8\theta\delta)^i\right) F(H, 1) \\ &\quad + \frac{C_0 C(\delta)}{1 - 64\delta} \left(\frac{\theta}{\delta}\right)^2 \max\left\{E(u, 1), F(H, 1)\right\} \\ &\le \;\; \delta^k E(u, 1) + \left(\frac{8\delta}{1 - 8\delta\theta}\right) \theta F(H, 1) \\ &\quad + \frac{C_0 C(\delta)}{1 - 64\delta} \left(\frac{\theta}{\delta}\right)^2 \max\left\{E(u, 1), F(H, 1)\right\}. \end{aligned} \tag{5.35}$$

According to the definition, we have $\delta^k = \theta^3$ and $\frac{2C_0 C(\delta)}{\delta^2} \le \theta^{2-2\gamma}$. Therefore (5.35) implies

$$E(u,\theta) \le \max\left\{C_1\theta^{2\gamma}E(u,1), C_1\theta F(H,1)\right\}.$$

This clearly implies (5.32). The proof is complete. □

The following proposition plays a crucial role in the proof of theorem 1.2

Proposition 5.4 *For any given $u_0 \in H^1(\Omega, S^2)$, $H_0 \in L^2(\mathbb{R}^3,\mathbb{R}^3)$, $E_0 \in L^2(\mathbb{R}^3,\mathbb{R}^3)$, $\epsilon > 0$ and $0 < T < +\infty$, let $\{u^\epsilon, H^\epsilon, E^\epsilon\} \in H^1(\Omega\times[0,T],\mathbb{R}^3)\times L^2(\mathbb{R}^3\times[0,T],\mathbb{R}^3)\times L^2(\mathbb{R}^3\times[0,T],\mathbb{R}^3)$ be the weak solution of (1.19), (1.2)-(1.3), and (1.8)-(1.11) obtained by Lemma 2.2. Then there exist universal constants $k_0 > 0$, $\epsilon_3 > 0$, $C_2 > 0$, such that for any $z_0 = (x_0,t_0) \in \Omega\times\mathbb{R}_+$, $0 < r < \min\{dist(x_0,\partial\Omega), \sqrt{t_0}, \frac{\epsilon_3^2}{C_2}\}$, if*

$$\mathcal{E}(u^\epsilon, z_0, r) \equiv r^{-3}\int_{P_r(z_0)} e_\epsilon(u^\epsilon) \le \epsilon_3^2, \tag{5.36}$$

then for any $z \in P_{\frac{r}{2}}(z_0)$, $\frac{\epsilon}{k_0} \le \rho \le \frac{r}{4}$, we have

$$\rho^{-3}\int_{P_\rho(z)}\left[e_\epsilon(u^\epsilon) + \rho^2\left|\frac{\partial u^\epsilon}{\partial t}\right|^2\right] \le C_2\frac{\rho}{r}\max\left\{\mathcal{E}(u^\epsilon, z_0, r), r\|H^\epsilon\|^2_{L^\infty_t L^2_x(P_r(z_0))}\right\} \tag{5.37}$$

Proof. By (2.8) of lemma 2.2, we have that $H^\epsilon \in L^\infty([0,T], L^2(\mathbb{R}^3))$ and

$$\|H^\epsilon\|_{L^\infty_t L^2_x(\mathbb{R}^3\times[0,T])} \le e^{CT}\left[\int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3}(\epsilon_0|E_0|^2 + |H_0|^2)\right] \equiv C_2. \tag{5.38}$$

This implies that for any $0 < \rho \le r$ and $z \in P_{\frac{r}{2}}(z_0)$

$$\rho\|H^\epsilon\|_{L^\infty_t L^2_x(P_\rho(z))} \le r\|H^\epsilon\|_{L^\infty_t L^2_x(\mathbb{R}^3\times[0,T])} \le rC_2 \le \epsilon_3^2.$$

Choose $\epsilon_3 \le \epsilon_2$, where ϵ_2 is given by lemma 5.3. Then the condition (5.31) of lemma 5.3 is satisfied for $P_{\frac{r}{2}}(z)$ with $z \in P_{\frac{r}{2}}(z_0)$. Hence we can repeatedly apply lemma 5.3 with $\gamma = \frac{1}{2}$ to obtain that for $0 < \rho < \frac{r}{4}$, $\epsilon \le k_0\rho$,

$$\mathcal{E}(u^\epsilon, z, \rho) \le C_1\frac{\rho}{r}\max\left\{\mathcal{E}(u^\epsilon, z_0, r), r\|H^\epsilon\|^2_{L^\infty_t L^2_x(P_r(z_0))}\right\}. \tag{5.39}$$

This, combined with lemma 2.6, implies (5.37). The proof is complete. □

Proof of Theorem 1.2:

For $\epsilon > 0$, let $\{u^\epsilon, H^\epsilon, E^\epsilon\}$ be the weak solution of (1.19), (1.2)-(1.3), with (1.8)-(1.11) obtained by Lemma 2.2. It follows from (2.8) that we may assume that $u^\epsilon \to u$ weakly in $H^1_{\text{loc}}(\Omega\times\mathbb{R}_+,\mathbb{R}^3)$, $(H^\epsilon, E^\epsilon) \to (H, E)$ weakly in $L^2_{\text{loc}}(\mathbb{R}^3\times\mathbb{R}_+,\mathbb{R}^3)$. By the same argument as [3], we know that $\{u, H, E\}$ is a weak solution of the Landau-Lifshitz-Maxwell system (1.13), (1.2) and (1.3) under the initial-boundary conditions (1.8)-(1.11).

Now we want to show a partial regularity of u as follows. Let ϵ_3 be given by proposition 5.4, and define the concentrate set of $\{u^\epsilon\}$ by

$$\Sigma = \bigcap_{r>0} \left\{ z \in \Omega \times \mathbb{R}^+ : \liminf_{\epsilon \to 0} r^{-3} \int_{P_r(z)} e_\epsilon(u^\epsilon) \geq \epsilon_3^2 \right\}. \tag{5.40}$$

Then a standard covering argument (see [7]) shows that $\mathcal{P}^3(\Sigma \cap K) < \infty$ for any compact subset of $\Omega \times \mathbb{R}^+$. Since u is a weak limit in $H^1_{\text{loc}}(\Omega \times \mathbb{R}_+, \mathbb{R}^3)$ of u_ϵ as $\epsilon \downarrow 0$, we have that for any $z_0 \in \Omega \times \mathbb{R}_+ \setminus \Sigma$, the lower semicontinuity, (5.40), and proposition 5.4 imply that there exists $r_0 > 0$ such that for any $z \in P_{\frac{r}{2}}(z_0)$ and $0 < \rho \leq \frac{r}{4}$,

$$\rho^{-3} \int_{P_\rho(z)} \left(|\nabla u|^2 + \rho^2 \left| \frac{\partial u}{\partial t} \right|^2 \right) \leq C_3 \frac{\rho}{r} \tag{5.41}$$

for some universal constant $C_3 > 0$. This implies that $u \in C^{\frac{1}{2}}(\Omega \times \mathbb{R}_+ \setminus \Sigma, S^2)$, by the parabolic version of Morrey's Lemma (cf. [6]). This completes the proof of theorem 1.2. □

6 C^α-regularity of ∇u, proof of theorems 1.3 and 1.4

This section is devoted to the discussion of partial C^α-regularity of ∇u, when $\{u, H, E\}$ is a weak solution of (1.13), (1.2), and (1.3) obtained as in theorem 1.2 in two special cases that (i) either $\epsilon_0 = 0$ in (1.2) or (ii) $\beta = 0$ in (1.3). For the case (i), we assume that the initial data $(u_0, H_0) \in H^1(\Omega, S^2) \times H^1(\mathbb{R}^3, \mathbb{R}^3)$ and H_0 satisfies $\nabla \cdot (H_0 + \beta u_0) = 0$. For the case (ii), we assume that the initial data $(u_0, H_0, E_0) \in H^1(\Omega, S^2) \times H^1(\mathbb{R}^3, \mathbb{R}^3) \times H^1(\mathbb{R}^3, \mathbb{R}^3)$ and H_0, E_0 satisfy $\nabla \cdot H_0 = \nabla \cdot E_0 = 0$.

There are two steps to prove the C^α-regularity of ∇u in $\Omega \times \mathbb{R}_+ \setminus \Sigma$, where Σ is the concentration set defined by (5.40). The first step is to utilize $H \in L^\infty_t L^6_x(\mathbb{R}^3 \times [0,T])$ for any $0 < T < +\infty$ to show that $u \in C^\gamma(\Omega \times \mathbb{R}_+ \setminus \Sigma, S^2)$ for any $\gamma \in (0,1)$. The second step is to employ the parabolic hole filling technique similar to Giaquinta-Hildebrandt [16] and Giaquinta-Struwe [17] to show that for $z \in \Omega \times \mathbb{R}_+ \setminus \Sigma$,

$$\rho^{-5} \int_{P_\rho(z)} |\nabla u - (\nabla u)_{P_\rho(z)}|^2 \leq C \rho^{2\alpha}$$

for some $\alpha \in (0,1)$.

It can be summarized into the following lemma.

Lemma 6.1 *For any $u_0 \in H^1(\Omega, S^2)$, $H_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, and $0 < T < +\infty$, let $(u, H) \in H^1(\Omega \times [0,T], S^2) \times L^\infty_t L^2_x(\mathbb{R}^3 \times [0,T], \mathbb{R}^3)$ be a weak solution to (1.13) coupled with (1.17) under the initial-boundary condition (1.8), (1.9) and (1.10) obtained as the weak limit of (u^ϵ, H^ϵ) given by Lemma 2.3. Let $\Sigma \subset \Omega \times \mathbb{R}_+$ be defined by (5.40). Then for any $z_0 \in \Omega \times \mathbb{R}_+ \setminus \Sigma$, there exists $r_0 > 0$ such that $\nabla u \in C^\alpha(P_{r_0}(z_0))$ for some $\alpha \in (0,1)$.*

Proof. By (2.17) of lemma 2.3, we have that

$$\sup_{\epsilon>0}\left(\|H^\epsilon\|_{L_t^\infty L_x^2(\mathbb{R}^3\times[0,T])}+\|\nabla H^\epsilon\|_{L_t^\infty L_x^2(\mathbb{R}^3\times[0,T])}\right)$$
$$\leq e^{CT}\left(\|\nabla u_0\|^2_{L^2(\Omega)}+\|H_0\|^2_{L^2(\mathbb{R}^3)}+\|\nabla H_0\|^2_{L^2(\mathbb{R}^3)}\right). \tag{6.1}$$

By the Sobolev embedding theorem, (6.1) implies that $H^\epsilon \in L_t^\infty L_x^6(\mathbb{R}^3\times[0,T])$ and

$$\sup_{\epsilon>0}\|H^\epsilon\|_{L_t^\infty L_x^6(\mathbb{R}^3\times[0,T])}\leq C_3\equiv C\left(T,\|u_0\|_{H^1(\Omega)},\|H_0\|_{H^1(\mathbb{R}^3)}\right). \tag{6.2}$$

Since $z_0\in\Omega\times\mathbb{R}_+\setminus\Sigma$, it follows from (5.40) that there exists $0<r_0\leq\frac{\epsilon_3^2}{C_3^2}$ such that

$$\mathcal{E}(u^\epsilon,z_0,r_0)\equiv r_0^{-3}\int_{P_{r_0}(z_0)}e_\epsilon(u^\epsilon)\leq\epsilon_3^2, \tag{6.3}$$

and

$$\mathcal{F}(H^\epsilon,z_0,r_0)\equiv r_0\|H^\epsilon\|^2_{L_t^\infty L_x^2(P_{r_0}(z_0))}\leq\epsilon_3^2. \tag{6.4}$$

Hence we can apply lemma 5.3 to conclude that for any $\theta\in(0,\frac{1}{2})$, $\gamma\in(0,1)$, $z\in P_{\frac{r_0}{2}}(z_0)$ and $0<r<\frac{r_0}{2}$, there is $C_4>0$ such that

$$\mathcal{E}(u^\epsilon,z,\theta r)\leq C_4\theta^{2\gamma}\mathcal{E}(u^\epsilon,z,r)+C_4\theta\mathcal{F}(H^\epsilon,z,r). \tag{6.5}$$

By Hölder inequality we have

$$\mathcal{F}(H^\epsilon,z,r)\leq r^3\|H^\epsilon\|_{L_t^\infty L_x^2(P_r(z))}\leq C_3r^3,\ \forall 0<r\leq r_0.$$

Therefore (6.5) yields that for $z\in P_{\frac{r_0}{2}}(z_0)$ and $0<r<\frac{r_0}{2}$,

$$\mathcal{E}(u^\epsilon,z,\theta r)\leq C_5\left(\theta^{2\gamma}\mathcal{E}(u^\epsilon,z,r)+\theta r^3\right). \tag{6.6}$$

Iterating (6.6) for k-times, we would have

$$\begin{aligned}\mathcal{E}(u^\epsilon,z,\theta^k r) &\leq \left(C_5\theta^{2\gamma}\right)^k\mathcal{E}(u^\epsilon,z,r)+\left(\sum_{i=0}^{k-1}(C_5\theta^{2\gamma})^{k-1-i}(\theta^3)^i\right)r^3\\ &\leq \left(C_5\theta^{2\gamma}\right)^k\left[\mathcal{E}(u^\epsilon,z,r)+\frac{r^3}{C_5\theta^{2\gamma}-\theta^3}\right].\end{aligned} \tag{6.7}$$

In particular, we have

$$\mathcal{E}(u^\epsilon,z,s)\leq(\frac{s}{r_0})^{2\gamma}\left(\mathcal{E}(u^\epsilon,z,\frac{r_0}{2})+C_6r_0^3\right),\ \forall z\in P_{\frac{r_0}{2}}(z_0),0<s\leq\frac{r_0}{2}. \tag{6.8}$$

Applying lemma 2.6 and taking $\epsilon\downarrow 0$, (6.8) implies that for $z\in P_{\frac{r_0}{2}}(z_0)$ and $0<s\leq\frac{r_0}{2}$,

$$s^{-3}\int_{P_s(z)}\left(|\nabla u|^2+s^2\left|\frac{\partial u}{\partial t}\right|^2\right)\leq(\frac{s}{r_0})^{2\gamma}\left(\epsilon_3^2+C_6r_0^3\right). \tag{6.9}$$

Hence the parabolic version of Morrey's lemma, implies that $u \in C^\gamma(P_{\frac{r_0}{2}}(z_0), S^2)$ for any $0 < \gamma < 1$, and

$$\mathrm{osc}_{P_r(z_0)} u \le C \left(\frac{r}{r_0}\right)^\gamma \left(\epsilon_3^2 + C_6 r_0^3\right), \ 0 < r \le \frac{r_0}{2}. \tag{6.10}$$

Next we want to use the parabolic hole filling argument to show that $\nabla u \in C^\alpha(P_{\frac{r_0}{2}}(z_0))$ for some $\alpha \in (0, 1)$.

First we observe that the linear map $R(u)\xi = \alpha_1 \xi + \alpha_2 u \times \xi : \mathbb{R}^3 \to \mathbb{R}^3$ can be represented by

$$R(u) = \begin{pmatrix} \alpha_1 & -\alpha_2 u_3 & \alpha_2 u_2 \\ \alpha_2 u_3 & \alpha_1 & -\alpha_2 u_1 \\ -\alpha_2 u_2 & \alpha_2 u_1 & \alpha_1 \end{pmatrix}.$$

It is easy to check that $R(u)$ has an inverse $M(u)$, which is given by

$$M(u)^T = \frac{1}{\alpha_1} \begin{pmatrix} \alpha_1^2 + \alpha_2^2 u_1^2 & \alpha_2^2 u_1 u_2 - \alpha_1 \alpha_2 u_3 & \alpha_2^2 u_1 u_3 + \alpha_1 \alpha_2 u_2 \\ \alpha_2^2 u_1 u_2 + \alpha_1 \alpha_2 u_3 & \alpha_1^2 + \alpha_2^2 u_2^2 & \alpha_2 u_2 u_3 - \alpha_1 \alpha_2 u_1 \\ \alpha_2^2 u_1 u_3 - \alpha_1 \alpha_2 u_2 & \alpha_2^2 u_2 u_3 + \alpha_1 \alpha_2 u_1 & \alpha_1^2 + \alpha_2^2 u_3^2 \end{pmatrix}.$$

It is easy to see that $M(u)$ is an uniformly elliptic matrix. Now we can rewrite the equation of u as

$$\frac{\partial u}{\partial t} - \nabla \cdot (M(u)\nabla u) = M(u)\left(|\nabla u|^2 u + (H - \langle H, u\rangle u)\right) - \nabla(M(u)) \cdot \nabla u. \tag{6.11}$$

For any $z_1 \in P_{\frac{r_0}{2}}(z_0)$ and $0 < r < \frac{r_0}{2}$, consider an axillary equation for $v : P_r(z_1) \to \mathbb{R}^3$:

$$\frac{\partial v}{\partial t} - \nabla \cdot (M(u(z_1))\nabla v) = 0 \text{ in } P_r(z_1), \quad v = u \ \text{ on } \partial_p P_r(z_1), \tag{6.12}$$

where $\partial_p P_r(z_1)$ denotes the parabolic boundary of $P_r(z_1)$. It follows from the maximum principle, (6.10), and (6.9) that

$$\mathrm{osc}_{P_r(z_1)} v \le C_7 r^\gamma, \quad \int_{P_r(z_1)} |\nabla v|^2 \le \int_{P_r(z_1)} |\nabla u|^2 \le C_7 r^{3+2\gamma} \tag{6.13}$$

Multiplying (6.11) and (6.12) by $w \equiv u - v$ and integrating over $P_r(z_1)$, we obtain

$$\begin{aligned} &\int_{P_r(z_1)} \langle M(u(z_1))\nabla w, \nabla w\rangle \\ &\le C_8 \int_{P_r(z_1)} (|\nabla u|^2 + |H|)|w| + C_8 \int_{P_r(z_1)} |M(u) - M(u(z_1))||\nabla u||\nabla w| \\ &= I + II \end{aligned} \tag{6.14}$$

By the ellipticity of $M(u(z_1))$, we have

$$\int_{P_r(z_1)} \langle M(u(z_1))\nabla w, \nabla w\rangle \ge \alpha_1 \int_{P_r(z_1)} |\nabla w|^2.$$

By Hölder inequality, (6.10) and (6.13), we have

$$I \le C_9(\frac{r}{r_0})^{3+3\gamma}(\epsilon_0^2 + r_0^3),$$

and

$$\begin{aligned} II &\le \frac{\alpha_1}{2}\int_{P_r(z_1)} |\nabla u|^2 + C_{10}(\mathrm{osc}_{P_r(z_1)} u)^2 \int_{P_1(z_1)} |\nabla w|^2 \\ &\le \frac{\alpha_1}{2}\int_{P_r(z_1)} |\nabla u|^2 + C_{10}(\frac{r}{r_0})^{3+4\gamma}. \end{aligned} \tag{6.15}$$

Putting these estimates into (6.14), we obtain

$$\int_{P_r(z_1)} |\nabla w|^2 \le C_{11} r^{3+3\gamma}. \tag{6.16}$$

Since v solves (6.12), the standard parabolic theory implies that that for any $0 < \rho < r$ it holds

$$\int_{P_\rho(z_1)} \left|\nabla v - (\nabla v)_{P_\rho(z_1)}\right|^2 \le C_{12}\left(\frac{\rho}{r}\right)^7 \int_{P_r(z_1)} \left|\nabla v - (\nabla v)_{P_r(z_1)}\right|^2. \tag{6.17}$$

Combining (6.16) with (6.17), we obtain that

$$\begin{aligned} \int_{P_\rho(z_1)} \left|\nabla u - (\nabla u)_{P_\rho(z_1)}\right|^2 &\le \int_{P_\rho(z_1)} \left|\nabla v - (\nabla v)_{P_\rho(z_1)}\right|^2 + \int_{P_r(z_1)} |\nabla w|^2 \\ &\le C_{12}\left(\frac{\rho}{r}\right)^7 \int_{P_r(z_1)} |\nabla u|^2 + C_{12} r^{3+3\gamma}. \end{aligned} \tag{6.18}$$

We now choose some $\gamma \in (\frac{2}{3}, 1)$ whence $3 + 3\gamma > 5$. Applying the algebraic lemma 2.1 in Giaquinta [15] Chapter III, we conclude that

$$\rho^{-5}\int_{P_\rho(z_1)} \left|\nabla u - (\nabla u)_{P_\rho(z_1)}\right|^2 \le C_{13}\rho^{3\gamma-2}\left[1 + r^{-(3+3\gamma)}\int_{P_r(z_1)} |\nabla u|^2\right] \tag{6.19}$$

holds for any $z_1 \in P_{\frac{r_0}{2}}(z_0)$ and $0 < \rho \le r \le \frac{r_0}{2}$.

A well known characterization of Hölder continuous functions due to Campanato [2] yields that $\nabla u \in C^{\frac{3\gamma-2}{2}}(P_{\frac{r_0}{2}}(z_0))$. This completes the proof of lemma 6.1. □

Completion of proof of theorem 1.3:

It follows immediately from lemma 6.1 that $\nabla u \in C^\alpha(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $\alpha \in (0, 1)$. It remains to show that $\nabla^2 u, \frac{\partial u}{\partial t} \in L^6_{\mathrm{loc}}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$. To see this, observe that

$$\begin{aligned} \left|\frac{\partial u}{\partial t} - \nabla\cdot(M(u)\nabla u)\right| &= \left|M(u)|\nabla u|^2 u - \nabla(M(u))\cdot\nabla u + M(u)\left(H - \langle H, u\rangle u\right)\right| \\ &\le C_{14}(|\nabla u|^2 + |H|) \in L^6(P_R), \end{aligned}$$

for any $P_R \subset\subset \Omega \times \mathbb{R}_+ \setminus \Sigma$. Since $M(u)$ is Hölder continuous and uniformly elliptic, it follows from the $W_p^{2,1}$-estimate for the linear parabolic equation (see [24]) that we can conclude that $\nabla^2 u, \frac{\partial u}{\partial t} \in L^6(P_{\frac{R}{2}})$. This implies the second conclusion of theorem 1.3. $\Box$

Proof of theorem 1.4:

By applying lemma 2.4, we can conclude that H^ϵ is bounded in $L_t^\infty L_x^6(\mathbb{R}^3 \times [0,T])$ for any $0 < T < +\infty$, uniformly in ϵ. Hence we can apply the same argument of lemma 6.1 to conclude that $\nabla u \in C^\alpha(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $\alpha \in (0,1)$, and $\nabla^2 u, \frac{\partial u}{\partial t} \in L^6_{\text{loc}}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$. We leave the details to interested readers. $\Box$

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