

REGULARITY AND EXISTENCE OF GLOBAL SOLUTIONS TO THE ERICKSEN-LESLIE SYSTEM IN \mathbb{R}^2

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ABSTRACT. In this paper, we first establish the regularity theorem for suitable weak solutions to the Ericksen-Leslie system in \mathbb{R}^2 . Building on such a regularity, we then establish the existence of a global weak solution to the Ericksen-Leslie system in \mathbb{R}^2 for any initial data in the energy space, under the physical constraint conditions on the Leslie coefficients ensuring the dissipation of energy of the system, which is smooth away from at most finitely many times. This extends earlier works by Lin-Lin-Wang [23] on a simplified nematic liquid crystal flow to the general Ericksen-Leslie system.

1. INTRODUCTION

The hydrodynamic theory of nematic liquid crystals was developed by Ericksen and Leslie during the period of 1958 through 1968 [5, 6, 8, 17]. It is referred as the Ericksen-Leslie system in the literature. It reduces to the Osssen-Frank theory in the static case, which has been successfully studied (see, e.g., Hardt-Lin-Kinderlehrer [9]).

In \mathbb{R}^3 , let $u = (u_1, u_2, u_3)^T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denote the fluid vector field of the underlying incompressible fluid, and $d = (d_1, d_2, d_3)^T : \mathbb{R}^3 \rightarrow \mathbb{S}^2$ denote the orientation order parameter representing the macroscopic average of nematic liquid crystal molecular directors. Assume that the fluid is homogeneous (e.g., the fluid density $\rho \equiv 1$) and the inertial constant is zero (i.e., $\rho_1 = 0$), the general Ericksen-Leslie system in \mathbb{R}^3 consists of the following equations (cf. [7, 17, 18, 22]):

$$\begin{cases} \partial_t u + u \cdot \nabla u = \nabla \cdot \hat{\sigma}, \\ \nabla \cdot u = 0, \\ d \times \left(g + \nabla \cdot \left(\frac{\partial W}{\partial(\nabla d)} \right) - \frac{\partial W}{\partial d} \right) = 0, \end{cases} \quad (1.1)$$

where $\hat{\sigma} = (\hat{\sigma}_{ij})$ is the total stress given by

$$\hat{\sigma}_{ij} = -P\delta_{ij} - \frac{\partial W}{\partial d_{k,i}} d_{k,j} + \sigma_{ij}^L, \quad (1.2)$$

with P a scalar function representing the pressure, W being the Osssen-Frank energy density function given by

$$W := W(d, \nabla d) \equiv \frac{1}{2} \left(k_1 (\nabla \cdot d)^2 + k_2 |d \times (\nabla \times d)|^2 + k_3 |d \cdot (\nabla \times d)|^2 + (k_2 + k_4) [\text{tr}(\nabla d)^2 - (\nabla \cdot d)^2] \right) \quad (1.3)$$

for some elasticity constants k_1, k_2, k_3, k_4 , and $\sigma^L = \sigma^L(u, d) = (\sigma_{ij}^L(u, d))$ representing the Leslie stress tensor given by

$$\sigma_{ij}^L(u, d) = \mu_1 \sum_{k,p} d_k d_p A_{kp} d_i d_j + \mu_2 N_i d_j + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 \sum_k A_{ik} d_k d_j + \mu_6 \sum_k A_{jk} d_k d_i, \quad (1.4)$$

for six viscous coefficients μ_1, \dots, μ_6 , called Leslie's coefficients, and

$$g_i = \lambda_1 N_i + \lambda_2 \sum_j d_j A_{ji} = \lambda_1 \left(N_i + \frac{\lambda_2}{\lambda_1} \sum_j d_j A_{ji} \right). \quad (1.5)$$

Throughout this paper, we use

$$A_{ij} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), \quad \Omega_{ij} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right),$$

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$$\omega_i = \dot{d}_i = \partial_t d_i + (u \cdot \nabla) d_i, \quad N_i = \omega_i - \sum_j \Omega_{ij} d_j,$$

denote the rate of strain tensor, the skew-symmetric part of the strain rate, the material derivative of d and the rigid rotation part of the director changing rate by fluid vorticity, respectively.

Due to the temperature dependence of the Leslie coefficients, various coefficients have different behavior: μ_4 does not involve the alignment properties and hence is a smooth function of temperature, while all the other μ_i 's describe couplings between molecule orientation and the flow, and might be affected by the change of the nematic order parameter d . In this paper, we only consider the isothermal case in which all μ_i 's are assumed to be constants. The Leslie coefficients μ_i 's and λ_1, λ_2 satisfy the following relations:

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \quad (1.6)$$

$$\mu_2 + \mu_3 = \mu_6 - \mu_5. \quad (1.7)$$

The relation (1.6) is a necessary condition in order to satisfy the equation of motion identically, while (1.7) is called Parodi's relation. Under the assumption of Parodi's relation, the hydrodynamics of an incompressible nematic liquid crystal flow involves five independent Leslie's coefficients. To avoid the complexity arising from the general Ossen-Frank energy functional, we consider the elastically isotropic case $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$ so that the Ossen-Frank energy reduces to the Dirichlet energy: $W(d, \nabla d) = \frac{1}{2} |\nabla d|^2$. In this case, we have

$$\frac{\partial W}{\partial d_{k,i}} d_{k,j} = \left\langle \frac{\partial d}{\partial x_i}, \frac{\partial d}{\partial x_j} \right\rangle = (\nabla d \odot \nabla d)_{ij}, \quad \nabla \cdot \left(\frac{\partial W}{\partial (\nabla d)} \right) = \Delta d, \quad \frac{\partial W}{\partial d} = 0.$$

As a consequence, the Ericksen-Leslie system (1.1) can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot (\sigma^L(u, d)), \\ \nabla \cdot u = 0, \\ \partial_t d + u \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d = \frac{1}{|\lambda_1|} (\Delta d + |\nabla d|^2 d) + \frac{\lambda_2}{\lambda_1} (d^T A d) d. \end{cases} \quad (1.8)$$

Since the general Ericksen-Leslie system is very complicated, earlier attempts of rigorous mathematical analysis of (1.8) were made for a simplified system that preserve the crucial energy dissipation feature as in (1.8), pioneered by Lin [19] and Lin-Liu [20, 21]. More precisely, by adding the penalty term $\frac{1}{4\epsilon^2} (1 - |d|^2)^2$ ($\epsilon > 0$) in the energy functional W to remove the nonlinearities resulting from the nonlinear constraints $|d| = 1$, Lin and Liu have studied in [20, 21] the following Ginzburg-Landau approximate system:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \mu \Delta u - \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ \partial_t d + u \cdot \nabla d = \Delta d + \frac{1}{\epsilon^2} (1 - |d|^2) d. \end{cases} \quad (1.9)$$

They have established in [20] the existence of global weak solutions in dimensions 2 and 3, and global strong solutions of (1.9) in dimension 2, the local existence of strong solutions in dimension 3, and the existence of global strong solutions for large viscosity $\mu > 0$ in dimension 3. A partial regularity for suitable weak solutions of (1.9) in dimension 3, analogous to Caffarelli-Kohn-Nirenberg [2] on the Navier-Stokes equation, has been proved in [21]. As already pointed out by [20], it is still a challenging open problem as ϵ tends to zero, whether solutions (u^ϵ, d^ϵ) of (1.9) converge to that of the following simplified Ericksen-Leslie system:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \mu \Delta u - \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ \partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d. \end{cases} \quad (1.10)$$

Very recently, there have been some important advances on (1.10). For dimension 2, Lin-Lin-Wang [23] and Lin-Wang [24] have established the existence of a global unique weak solution of (1.10) in any smooth bounded domain, under the initial and boundary conditions, which is smooth away from possibly finitely many times (see also Hong [10], Xu-Zhang [35], Hong-Xin [11], and Lei-Li-Zhang[26] for related results in \mathbb{R}^2). It is an open problem whether there exists a global weak solution of (1.10) in dimension 3. There have been some partial results towards this problem. For example, the local existence and uniqueness of strong solutions of (1.10) has been proved by Ding-Wen [4], the blow-up criterion of local strong solutions of (1.10),

similar to Beale-Kato-Majda [1] for Navier-Stokes equations, has been established by Huang-Wang [12], the global well-posedness of (1.10) for rough initial data (u_0, d_0) with small $\text{BMO} \times \text{BMO}^{-1}$ -norm has been shown by Wang [32], and the local well-posedness of (1.10) for initial data (u_0, d_0) with small $L^3_{\text{uloc}}(\mathbb{R}^3)$ -norm of $(u_0, \nabla d_0)$ has been proved by Hineman-Wang [14] (here $L^3_{\text{uloc}}(\mathbb{R}^3)$ denotes the locally uniform L^3 -space on \mathbb{R}^3).

For the general Ericksen-Leslie system (1.8) in \mathbb{R}^3 , there have also been some recent works. For example, Lin-Liu [22] and Wu-Xu-Liu [34] have considered its Ginzburg-Landau approximation:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot (\sigma^L(u, d)), \\ \nabla \cdot u = 0, \\ \partial_t d + u \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d = \frac{1}{|\lambda_1|} (\Delta d + \frac{1}{\epsilon^2} (1 - |d|^2) d), \end{cases} \quad (1.11)$$

and have established the existence of global weak solutions, and the local existence and uniqueness of strong solutions of (1.11) under certain conditions on the Leslie coefficients μ_i 's. In particular, there have been results developed by [34] concerning the role of Parodi's condition (1.7) in the well-posedness and stability of (1.11). Most recently, Wang-Zhang-Zhang [33] have studied the general Ericksen-Leslie system (1.8) and established the local well-posedness, and the global well-posedness for small initial data under a seemingly optimal condition on the Leslie coefficients μ_i 's.

In this paper, we are mainly interested in both the regularity and existence of global weak solutions of the initial value problem of the general Ericksen-Leslie system (1.8) in \mathbb{R}^2 .

In \mathbb{R}^2 , however, we need to modify several terms inside the system (1.8) in order to make it into a closed system. Since u is a planar vector field in \mathbb{R}^2 , both Ω and A are horizontal 2×2 -matrices, henceforth we assume that

$$\Omega d := (\Omega \hat{d}, 0)^T, \quad A d := (A \hat{d}, 0)^T, \quad N := (\partial_t d + u \cdot \nabla d)^T - (\Omega \hat{d}, 0)^T, \quad (1.12)$$

as vectors in \mathbb{R}^3 , while $\hat{N} := (\partial_t \hat{d} + u \cdot \nabla \hat{d} - \Omega \hat{d})^T$ is a vector in \mathbb{R}^2 . Here

$$\hat{d} = (d_1, d_2, 0)^T \text{ for } d = (d_1, d_2, d_3)^T \in \mathbb{R}^3,$$

and $\sigma^L(u, d)$ is a 2×2 -matrix valued function given by

$$\sigma_{ij}^L(u, d) := \mu_1 \sum_{k,p=1}^2 d_k d_p A_{kp} d_i d_j + \mu_2 N_i d_j + \mu_3 N_j d_i + \mu_4 A_{ij} + \mu_5 \sum_{k=1}^2 A_{ik} d_k d_j + \mu_6 \sum_{k=1}^2 A_{jk} d_k d_i \quad (1.13)$$

for $1 \leq i, j \leq 2$, and

$$d^T A d := \tilde{d}^T A \hat{d} \equiv \sum_{i,j=1}^2 A_{ij} d_i d_j.$$

We will consider the initial value problem of (1.8) in \mathbb{R}^2 , i.e.,

$$(u, d) \Big|_{t=0} = (u_0, d_0) \text{ in } \mathbb{R}^2 \quad (1.14)$$

for any given $u_0 \in \mathbf{H}$, and $d_0 \in H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$. Here we denote the relevant function spaces

$$\mathbf{H} = \text{the closure of } C_0^\infty(\mathbb{R}^2, \mathbb{R}^2) \cap \{v \mid \nabla \cdot v = 0\} \text{ in } L^2(\mathbb{R}^2, \mathbb{R}^2),$$

$$H_{e_0}^k(\mathbb{R}^2, \mathbb{S}^2) = \left\{ d : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \mid d - e_0 \in H^k(\mathbb{R}^2, \mathbb{R}^3) \right\} \quad (k \in \mathbb{N}_+)$$

for some constant vector $e_0 \in \mathbb{S}^2$, and

$$\mathbf{J} = \text{the closure of } C_0^\infty(\mathbb{R}^2, \mathbb{R}^2) \cap \{v \mid \nabla \cdot v = 0\} \text{ in } H^1(\mathbb{R}^2, \mathbb{R}^2).$$

Definition 1.1. For $0 < T \leq \infty$, $u \in L^2([0, T], \mathbf{H})$ and $d \in L^2([0, T], H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2))$ is called a weak solution of the Ericksen-Leslie system (1.8) together with the initial condition (1.14) in \mathbb{R}^2 , if

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^2} \langle u, \psi' \phi \rangle - \int_0^T \int_{\mathbb{R}^2} u \otimes u : \psi \nabla \phi + \int_0^T \int_{\mathbb{R}^2} \sigma^L(u, d) : \psi \nabla \phi \\ & = -\psi(0) \int_{\mathbb{R}^2} \langle u_0, \phi \rangle + \int_0^T \int_{\mathbb{R}^2} \langle \nabla d \odot \nabla d, \psi \nabla \phi \rangle, \end{aligned}$$

and

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}^2} \langle d, \psi' \tilde{\phi} \rangle + \int_0^T \int_{\mathbb{R}^2} \left\langle u \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} Ad, \psi \tilde{\phi} \right\rangle \\ & = -\psi(0) \int_{\mathbb{R}^2} \langle d_0, \tilde{\phi} \rangle + \int_0^T \int_{\mathbb{R}^2} \left[-\frac{1}{|\lambda_1|} \langle \nabla d, \psi \nabla \tilde{\phi} \rangle + \frac{1}{|\lambda_1|} |\nabla d|^2 \langle d, \psi \tilde{\phi} \rangle + \frac{\lambda_2}{\lambda_1} (d^T Ad) \langle d, \psi \tilde{\phi} \rangle \right]. \end{aligned}$$

for any $\psi \in C^\infty([0, T])$ with $\psi(T) = 0$, $\phi \in \mathbf{J}$, and $\tilde{\phi} \in H^1(\mathbb{R}^2, \mathbb{R}^3)$.¹

In this paper, we will establish the regularity of *suitable* weak solutions of (1.8), and the existence and uniqueness of global weak solutions to (1.8) and (1.14) in \mathbb{R}^2 . As consequences, these extend the previous works by Lin-Lin-Wang [23] to the general case.

For $x_0 \in \mathbb{R}^2$, $t_0 \in (0, +\infty)$, $z_0 = (x_0, t_0)$ and $0 < r \leq \sqrt{t_0}$, denote

$$B_r(x_0) = \{x \in \mathbb{R}^2 \mid |x - x_0| \leq r\}, \quad P_r(z_0) = B_r(x_0) \times [t_0 - r^2, t_0].$$

When $x_0 = (0, 0)$ and $t_0 = 0$, we simply denote $B_r = B_r(0)$, $P_r = P_r(0)$.

Now we introduce the notion of suitable weak solutions of (1.8).

Definition 1.2. For $0 < T < +\infty$ and a domain $O \subset \mathbb{R}^2$, a weak solution $u \in L^2(O \times [0, T], \mathbb{R}^2)$, with $\nabla \cdot u = 0$, and $d \in L_t^2 H_x^1(O \times [0, T], \mathbb{S}^2)$ of (1.8) is called a *suitable weak solution* of (1.8) if, in addition, $u \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(O \times [0, T], \mathbb{R}^2)$, $d \in (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(O \times [0, T], \mathbb{S}^2)$, and $P \in L^2(O \times [0, T])$.

Theorem 1.3. For $0 < T < +\infty$ and a domain $O \subset \mathbb{R}^2$, assume that $u \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(O \times [0, T], \mathbb{R}^2)$, $d \in L^\infty([0, T], H^1(O, \mathbb{S}^2)) \cap L^2([0, T], H^2(O, \mathbb{S}^2))$, and $P \in L^2(O \times [0, T])$ is a suitable weak solution of (1.8). Assume both (1.6) and (1.7) hold. If, in additions, the Leslie coefficients μ_i 's satisfy

$$\lambda_1 < 0, \quad \mu_1 - \frac{\lambda_2^2}{\lambda_1} \geq 0, \quad \mu_4 > 0, \quad \mu_5 + \mu_6 \geq -\frac{\lambda_2^2}{\lambda_1}, \quad (1.15)$$

then $(u, d) \in C^\infty(O \times (0, T], \mathbb{R}^2 \times \mathbb{S}^2)$.

Employing the priori estimate given by the proof of Theorem 1.3, we will prove the existence of global weak solutions of (1.8) and (1.14) that enjoy partial smoothness properties.

Theorem 1.4. For any $u_0 \in \mathbf{H}$ and $d_0 \in H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$, assume the conditions (1.6), (1.7), and (1.15) hold. Then there is a global weak solution $u \in L^\infty([0, +\infty), \mathbf{H}) \cap L^2([0, +\infty), \mathbf{J})$ and $d \in L^\infty([0, +\infty), H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2))$ of the general Ericksen-Leslie system (1.8) and (1.14) such that the following properties hold:

(i) There exist a nonnegative integer L , depending only on (u_0, d_0) , and $0 < T_1 < \dots < T_L < +\infty$ such that

$$(u, d) \in C^\infty(\mathbb{R}^2 \times ((0, +\infty) \setminus \{T_i\}_{i=1}^L), \mathbb{R}^2 \times \mathbb{S}^2).$$

(ii) Each singular time T_i , $1 \leq i \leq L$, can be characterized by

$$\liminf_{t \uparrow T_i} \max_{x \in \mathbb{R}^2} \int_{B_r(x)} (|u|^2 + |\nabla d|^2)(y, t) \geq 8\pi, \quad \forall r > 0. \quad (1.16)$$

Moreover, there exist $x_m^i \rightarrow x_0^i \in \mathbb{R}^2$, $t_m^i \uparrow T_i$, $r_m^i \downarrow 0$ and a non-trivial smooth harmonic map $\omega_i : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ with finite energy such that as $m \rightarrow \infty$,

$$(u_m^i, d_m^i) \rightarrow (0, \omega_i) \text{ in } C_{\text{loc}}^2(\mathbb{R}^2 \times [-\infty, 0]),$$

where

$$u_m^i(x, t) = r_m^i u(x_m^i + r_m^i x, t_m^i + (r_m^i)^2 t), \quad d_m^i(x, t) = d(x_m^i + r_m^i x, t_m^i + (r_m^i)^2 t).$$

(iii) Set $T_0 = 0$. Then

$$(d_t, \nabla^2 d) \in \bigcap_{j=0}^{L-1} \bigcap_{\epsilon > 0} L^2(\mathbb{R}^2 \times [T_j, T_{j+1} - \epsilon]) \bigcap \left(\bigcap_{T_L < T < +\infty} L^2(\mathbb{R}^2 \times [T_L, T]) \right).$$

¹By using (1.8)₃, we see that $N \in L_t^2 H_x^{-1} + L_t^1 L_x^1$. Hence $\sigma^L(u, d) \in L_t^1(H_x^{-1} \cdot H_x^1) + L_t^1(L_x^1 \cdot L_x^\infty)$ and $\int_0^T \int_{\mathbb{R}^2} \sigma^L(u, d) : \psi \nabla \phi$ is well defined. Similarly, $\int_0^T \int_{\mathbb{R}^2} \langle -\Omega d + \frac{\lambda_2}{\lambda_1} Ad, \psi \tilde{\phi} \rangle$ and $\int_0^T \int_{\mathbb{R}^2} (d^T Ad) \langle d, \psi \tilde{\phi} \rangle$ are also defined as pairs between H^{-1} and H_0^1 .

(iv) There exist $t_k \uparrow +\infty$ and a smooth harmonic map $d_\infty \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ with finite energy such that $u(\cdot, t_k) \rightarrow 0$ in $H^1(\mathbb{R}^2)$ and $d(\cdot, t_k) \rightarrow d_\infty$ in $H^1(\mathbb{R}^2)$, and there exist a nonnegative integer l , $\{x_i\}_{i=1}^l \subset \mathbb{R}^2$, and nonnegative integers $\{m_i\}_{i=1}^l$ such that

$$|\nabla d(\cdot, t_k)|^2 dx \rightarrow |\nabla d_\infty|^2 dx + 8\pi \sum_{i=1}^l m_i \delta_{x_i}. \quad (1.17)$$

(v) If either the third component of d_0 , $(d_0)_3$, is nonnegative², or

$$\int_{\mathbb{R}^2} (|u_0|^2 + |\nabla d_0|^2) \leq 8\pi,$$

then $(u, d) \in C^\infty(\mathbb{R}^2 \times (0, +\infty), \mathbb{R}^2 \times \mathbb{S}^2)$. Moreover, there exist $t_k \uparrow +\infty$ and a smooth harmonic map $d_\infty \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ with finite energy such that

$$(u(\cdot, t_k), d(\cdot, t_k)) \rightarrow (0, d_\infty) \text{ in } C_{\text{loc}}^2(\mathbb{R}^2).$$

An important first step to prove Theorem 1.3 is to establish the decay lemma 3.1 under the small energy condition, which is proved by a blow-up argument. Here the local energy inequality (2.14) for suitable weak solutions to (1.8) plays a very important role, which depends on the conditions (1.6), (1.7), and (1.15) heavily. In contrast with earlier arguments developed by [23] on the simplified nematic liquid crystal equation (1.10), where the limiting equation resulting from the blow up process is the linear Stokes equation and the linear heat equation, the new linear system (3.9) arising from the blow-up process of the general Ericksen-Leslie system (1.8) is a coupling system. It is an interesting question to establish its smoothness. The proof of regularity of (3.9) is based on higher order local energy inequalities. The cancelation properties among the coupling terms play critical roles in the argument of various local or global energy inequalities for both the linear system (3.9) and the nonlinear system (1.8). The second step is to establish a higher integrability estimate of suitable weak solutions to (1.8) under the small energy condition, which is done by employing the techniques of Riesz potential estimates between parabolic Morrey spaces developed by [12] and [14]. The third step is to establish an arbitrary higher order energy estimate of (1.8) under the small energy condition. With the regularity theorem 1.3, we show the existence theorem 1.4 by adapting the scheme developed by [23].

Motivated by the uniqueness theorem proved by [23] and [35] on (1.10), we believe that the weak solution obtained in Theorem 1.4 is also unique in its own class and plan to address it in a forthcoming article.

The paper is organized as follows. In section two, we derive both local and global energy inequality for suitable weak solutions of (1.8). In section three, we prove an ϵ -regularity theorem for (1.8) first and then prove Theorem 1.3. In section four, we prove the existence theorem 1.4.

2. GLOBAL AND LOCAL ENERGY INEQUALITIES OF ERICKSEN-LESLIE' SYSTEM IN \mathbb{R}^2

In this section, we will establish both global and local energy inequality for suitable weak solutions of (1.8) in \mathbb{R}^2 under the conditions (1.6), (1.7), and (1.15). We begin with the global energy inequality.

Lemma 2.1. *For $0 < T \leq +\infty$, assume the conditions (1.6), (1.7), and (1.15) hold. If $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^2 \times [0, T], \mathbb{R}^2)$, $d \in L^\infty([0, T], H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)) \cap L^2([0, T], H_{e_0}^2(\mathbb{R}^2, \mathbb{S}^2))$, and $P \in L^2(\mathbb{R}^2 \times [0, T])$ is a suitable weak solution of the Ericksen-Leslie system (1.8). Then for any $0 \leq t_1 < t_2 \leq T$, it holds*

$$\int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2)(t_2) + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \left[\mu_4 |\nabla u|^2 + \frac{2}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right] \leq \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2)(t_1). \quad (2.1)$$

Proof. Since $|d| = 1$, we have $|\nabla d|^2 + \langle d, \Delta d \rangle = 0$. This, combined with the fact $d \in L^2([0, T], H_{e_0}^2(\mathbb{R}^2, \mathbb{S}^2))$, implies that $\nabla d \in L^4(\mathbb{R}^2 \times [0, T])$. Since $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^2 \times [0, T], \mathbb{R}^2)$, it follows from Ladyzhenskaya's inequality that $u \in L^4(\mathbb{R}^2 \times [0, T])$.

For $\eta \in C_0^\infty(\mathbb{R}^2)$ or $\eta \equiv 1$ in \mathbb{R}^2 , multiplying (1.8)₁ by $u\eta^2$, integrating the resulting equation over \mathbb{R}^2 , and using $\nabla \cdot u = 0$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} |u|^2 \eta^2 = 2 \int_{\mathbb{R}^2} \eta^2 [\nabla d \odot \nabla d : \nabla u - \sigma^L(u, d) : \nabla u]$$

²We remark that this condition was first brought out explicitly by Lei-Li-Zhang [26] for the simplified Ericksen-Leslie system (1.10).

$$+ \int_{\mathbb{R}^2} [(|u|^2 + 2P)u \cdot \nabla(\eta^2) + 2(\nabla d \odot \nabla d - \sigma^L(u, d)) : u \otimes \nabla(\eta^2)]. \quad (2.2)$$

Using (1.6), (1.7), the symmetry of A , and the skew-symmetry of Ω , we find

$$\begin{aligned} & \int_{\mathbb{R}^2} \eta^2 \sigma^L(u, d) : \nabla u \\ &= \int_{\mathbb{R}^2} \eta^2 \left[\mu_1 \hat{d}_k \hat{d}_p A_{kp} \hat{d}_i \hat{d}_j + \mu_2 \hat{N}_i \hat{d}_j + \mu_3 \hat{N}_j \hat{d}_i + \mu_4 A_{ij} + \mu_5 A_{ik} \hat{d}_k \hat{d}_j + \mu_6 A_{jk} \hat{d}_k \hat{d}_i \right] (A_{ij} + \Omega_{ij}) \\ &= \int_{\mathbb{R}^2} \eta^2 \left[\mu_1 |A : \hat{d} \otimes \hat{d}|^2 + \mu_4 |A|^2 + (\mu_2 + \mu_3) \hat{d} \cdot (A \cdot \hat{N}) + (\mu_2 - \mu_3) \hat{d} \cdot (\Omega \cdot \hat{N}) \right. \\ &\quad \left. + (\mu_5 + \mu_6) |A \cdot \hat{d}|^2 + (\mu_5 - \mu_6) (A \cdot \hat{d}) (\Omega \cdot \hat{d}) \right] \\ &= \int_{\mathbb{R}^2} \eta^2 \left[\mu_1 |A : \hat{d} \otimes \hat{d}|^2 + \mu_4 |A|^2 + (\mu_5 + \mu_6) |A \cdot \hat{d}|^2 \right. \\ &\quad \left. + \lambda_1 \hat{N} \cdot (\Omega \cdot \hat{d}) - \lambda_2 \hat{N} \cdot (A \cdot \hat{d}) + \lambda_2 (A \cdot \hat{d}) (\Omega \cdot \hat{d}) \right]. \end{aligned} \quad (2.3)$$

Putting (2.3) into (2.2), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} |u|^2 \eta^2 &= 2 \int_{\mathbb{R}^2} \eta^2 \left[\nabla d \odot \nabla d : \nabla u - \mu_1 |A : \hat{d} \otimes \hat{d}|^2 - \mu_4 |A|^2 - (\mu_5 + \mu_6) |A \cdot \hat{d}|^2 \right. \\ &\quad \left. - \lambda_1 \hat{N} \cdot (\Omega \cdot \hat{d}) + \lambda_2 \hat{N} \cdot (A \cdot \hat{d}) - \lambda_2 (A \cdot \hat{d}) (\Omega \cdot \hat{d}) \right] \\ &\quad + \int_{\mathbb{R}^2} [(|u|^2 + 2P)u \cdot \nabla(\eta^2) + 2(\nabla d \odot \nabla d - \sigma^L(u, d)) : u \otimes \nabla(\eta^2)]. \end{aligned} \quad (2.4)$$

Since $(\Delta d + |\nabla d|^2 d) \in L^2(\mathbb{R}^2 \times [0, T])$, it follows from the equation (1.8)₃ that $(\partial_t d + u \cdot \nabla d) \in L^2(\mathbb{R}^2 \times [0, T])$. Multiplying (1.8)₃ by $\eta^2 \Delta d$ and integrating the resulting equation over \mathbb{R}^2 yields that

$$\begin{aligned} & \frac{d}{dt} \int_{B_1} \frac{1}{2} |\nabla d|^2 \eta^2 + \int_{B_1} \frac{1}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \eta^2 \\ &= - \int_{\mathbb{R}^2} \langle \partial_t d, \nabla d \rangle \cdot \nabla(\eta^2) + \int_{\mathbb{R}^2} \eta^2 \left[\langle u \cdot \nabla d, \Delta d \rangle + \langle \frac{\lambda_2}{\lambda_1} A \hat{d} - \Omega \hat{d}, \Delta \hat{d} \rangle + \frac{\lambda_2}{\lambda_1} (\hat{d}^T A \hat{d}) |\nabla d|^2 \right]. \end{aligned} \quad (2.5)$$

Using integration by parts and $\nabla \cdot u = 0$, we see

$$\int_{\mathbb{R}^2} \eta^2 \langle u \cdot \nabla d, \Delta d \rangle = - \int_{\mathbb{R}^2} \eta^2 \nabla d \odot \nabla d : \nabla u + \int_{\mathbb{R}^2} \left[\frac{1}{2} |\nabla d|^2 u \cdot \nabla(\eta^2) - \nabla d \odot \nabla d : u \otimes \nabla(\eta^2) \right].$$

Substituting this into (2.5), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{B_1} \frac{1}{2} |\nabla d|^2 \eta^2 + \int_{B_1} \frac{1}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \eta^2 \\ &= \int_{\mathbb{R}^2} \eta^2 \left[\langle \frac{\lambda_2}{\lambda_1} A \hat{d} - \Omega \hat{d}, \Delta \hat{d} \rangle + \frac{\lambda_2}{\lambda_1} (\hat{d}^T A \hat{d}) |\nabla d|^2 - \eta^2 \nabla d \odot \nabla d : \nabla u \right] \\ &\quad + \int_{\mathbb{R}^2} \left[\frac{1}{2} |\nabla d|^2 u \cdot \nabla(\eta^2) - \nabla d \odot \nabla d : u \otimes \nabla(\eta^2) - \langle \partial_t d, \nabla d \rangle \cdot \nabla(\eta^2) \right]. \end{aligned} \quad (2.6)$$

Adding (2.4) together with (2.6), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2) \eta^2 + 2 \int_{\mathbb{R}^2} \left[\mu_4 |A|^2 + \frac{1}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right] \eta^2 \\ &= -2 \int_{\mathbb{R}^2} \eta^2 \left[\mu_1 |A : \hat{d} \otimes \hat{d}|^2 + (\mu_5 + \mu_6) |A \cdot \hat{d}|^2 + \lambda_1 \hat{N} \cdot (\Omega \cdot \hat{d}) - \lambda_2 \hat{N} \cdot (A \cdot \hat{d}) \right. \\ &\quad \left. + \lambda_2 (A \cdot \hat{d}) (\Omega \cdot \hat{d}) - \langle \frac{\lambda_2}{\lambda_1} A \hat{d} - \Omega \hat{d}, \Delta \hat{d} \rangle - \frac{\lambda_2}{\lambda_1} (\hat{d}^T A \hat{d}) |\nabla d|^2 \right] \\ &\quad + \int_{\mathbb{R}^2} [(|u|^2 + 2P)u \cdot \nabla(\eta^2) + 2(\nabla d \odot \nabla d - \sigma^L(u, d)) : u \otimes \nabla(\eta^2)] \end{aligned}$$

$$+ \int_{\mathbb{R}^2} [|\nabla d|^2 u \cdot \nabla(\eta^2) - 2\nabla d \odot \nabla d : u \otimes \nabla(\eta^2) - 2\langle \partial_t d, \nabla d \rangle \cdot \nabla(\eta^2)]. \quad (2.7)$$

Denote the first term in the right hand side of (2.7) as \mathbf{I} . In order to estimate \mathbf{I} , we need to use (1.8)₃ to make crucial cancelations among terms of \mathbf{I} as follows.

$$\begin{aligned} \lambda_1 \hat{N} \cdot (\Omega \cdot \hat{d}) &= \lambda_1 \hat{N}_i \Omega_{ij} \hat{d}_j \\ &= \left[-\lambda_2 A_{ik} \hat{d}_k - \Delta \hat{d}_i + \lambda_2 (\hat{d}^T A \hat{d}) \hat{d}_i \right] \Omega_{ij} \hat{d}_j \\ &= \left[-\lambda_2 A_{ik} \hat{d}_k - \Delta \hat{d}_i \right] \Omega_{ij} \hat{d}_j = -\lambda_2 (A \cdot \hat{d}) (\Omega \cdot \hat{d}) - \langle \Omega \hat{d}, \Delta \hat{d} \rangle, \end{aligned} \quad (2.8)$$

while

$$\begin{aligned} -\lambda_2 \hat{N} \cdot (A \cdot \hat{d}) &= -\lambda_2 \hat{N}_i A_{ij} \hat{d}_j \\ &= -\lambda_2 \left[-\frac{\lambda_2}{\lambda_1} A_{ij} \hat{d}_j - \frac{1}{\lambda_1} (\Delta \hat{d}_i + |\nabla d|^2 \hat{d}_i) + \frac{\lambda_2}{\lambda_1} (\hat{d}^T A \hat{d}) \hat{d}_i \right] A_{ij} \hat{d}_j \\ &= \frac{\lambda_2^2}{\lambda_1} |A \cdot \hat{d}|^2 - \frac{\lambda_2^2}{\lambda_1} |\hat{d}^T A \hat{d}|^2 + \frac{\lambda_2}{\lambda_1} \langle A \hat{d}, \Delta \hat{d} \rangle + \frac{\lambda_2}{\lambda_1} (\hat{d}^T A \hat{d}) |\nabla d|^2. \end{aligned} \quad (2.9)$$

Since $A : \hat{d} \otimes \hat{d} = \hat{d}^T A \hat{d}$, we have, by substituting (2.8) and (2.9) into \mathbf{I} , that

$$\mathbf{I} = -2 \int_{\mathbb{R}^2} \left[\left(\mu_1 - \frac{\lambda_2^2}{\lambda_1} \right) |A : \hat{d} \otimes \hat{d}|^2 + \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) |A \cdot \hat{d}|^2 \right]. \quad (2.10)$$

Substituting (2.10) into (2.7), we finally obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2) \eta^2 + \int_{\mathbb{R}^2} \left[\mu_4 |\nabla u|^2 + \frac{2}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right] \eta^2 \\ &= -2 \int_{\mathbb{R}^2} \eta^2 \left[\left(\mu_1 - \frac{\lambda_2^2}{\lambda_1} \right) |A : \hat{d} \otimes \hat{d}|^2 + \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) |A \cdot \hat{d}|^2 \right] \\ &+ \int_{\mathbb{R}^2} [(|u|^2 + 2P) u \cdot \nabla(\eta^2) + 2(\nabla d \odot \nabla d - \sigma^L(u, d)) : u \otimes \nabla(\eta^2)] \\ &+ \int_{\mathbb{R}^2} [|\nabla d|^2 u \cdot \nabla(\eta^2) - 2\nabla d \odot \nabla d : u \otimes \nabla(\eta^2) - 2\langle \partial_t d, \nabla d \rangle \cdot \nabla(\eta^2) + \mu_4 \langle u \cdot \nabla u, \nabla(\eta^2) \rangle]. \end{aligned} \quad (2.11)$$

Here we have used the fact $\nabla \cdot u = 0$ and the following identity:

$$\int_{\mathbb{R}^2} |A|^2 \eta^2 = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \eta^2 - \frac{1}{2} \int_{\mathbb{R}^2} \langle (u \cdot \nabla) u, \nabla(\eta^2) \rangle. \quad (2.12)$$

If $\eta \equiv 1$, then (2.11) and the condition (1.15) imply

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2) + \int_{\mathbb{R}^2} \left[\mu_4 |\nabla u|^2 + \frac{2}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right] \\ &= -2 \int_{\mathbb{R}^2} \left[\left(\mu_1 - \frac{\lambda_2^2}{\lambda_1} \right) |A : \hat{d} \otimes \hat{d}|^2 + \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \right) |A \cdot \hat{d}|^2 \right] \leq 0. \end{aligned} \quad (2.13)$$

Integrating (2.13) over $0 \leq t_1 \leq t_2 \leq T$ yields (2.1). This completes the proof of lemma 2.1. \square

We also need the following local energy inequality in the proofs of our main theorems.

Lemma 2.2. *For $0 < T \leq +\infty$, assume the conditions (1.6), (1.7), and (1.15) hold. If $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^2 \times [0, T], \mathbb{R}^2)$, $d \in L^\infty([0, T], H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)) \cap L^2([0, T], H_{e_0}^2(\mathbb{R}^2, \mathbb{S}^2))$, and $P \in L^2(\mathbb{R}^2 \times [0, T])$ is a suitable weak solution of the Ericksen-Leslie system (1.8). Then for any $0 \leq t_1 < t_2 \leq T$ and $\eta \in C_0^\infty(\mathbb{R}^2)$, it holds*

$$\begin{aligned} & \int_{\mathbb{R}^2} \eta^2 (|u|^2 + |\nabla d|^2)(t_2) + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \eta^2 \left[\mu_4 |\nabla u|^2 + \frac{2}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right] \\ & \leq \int_{\mathbb{R}^2} \eta^2 (|u|^2 + |\nabla d|^2)(t_1) + C \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \left[(|u|^2 + |\nabla u| + |u| |\nabla d| + |\nabla d|^2 + |\nabla^2 d| + |P|) |u| |\nabla(\eta^2)| \right. \\ & \quad \left. + (|\nabla u| + |u| |\nabla d| + |\nabla d|^2 + |\nabla^2 d|) |\nabla d| |\nabla(\eta^2)| \right]. \end{aligned} \quad (2.14)$$

Proof. It suffices to estimate the last two terms in the right hand of (2.11). Denote these two terms by **II** and **III**. To do so, first observe that by (1.8)₃ it holds that

$$|\partial_t d| \leq C(|u||\nabla d| + |\nabla u| + |\nabla^2 d| + |\nabla d|^2),$$

and hence

$$|\sigma^L(u, d)| \leq C(|A| + |N|) \leq C(|\nabla u| + |\partial_t d| + |u||\nabla d|) \leq C(|\nabla u| + |u||\nabla d| + |\nabla^2 d| + |\nabla d|^2).$$

With these estimates, we can show that

$$\begin{aligned} \text{II} &\leq C \int_{\mathbb{R}^2} (|u|^2 + |P| + |\nabla d|^2 + |\sigma^L(u, d)|) |u||\nabla(\eta^2)| \\ &\leq C \int_{\mathbb{R}^2} (|u|^2 + |P| + |\nabla d|^2 + |u||\nabla d| + |\nabla u| + |\nabla^2 d|) |u||\nabla(\eta^2)|, \end{aligned}$$

and

$$\begin{aligned} \text{III} &\leq C \int_{\mathbb{R}^2} (|\nabla d|^2 |u| + |\partial_t d||\nabla d| + |u||\nabla u|) |\nabla(\eta^2)| \\ &\leq C \int_{\mathbb{R}^2} (|\nabla d|^2 |u| + |u||\nabla u| + (|\nabla u| + |u||\nabla d| + |\nabla d|^2 + |\nabla^2 d|) |\nabla d|) |\nabla(\eta^2)|. \end{aligned}$$

Putting these estimates of **II** and **III** into (2.11) yields (2.14). This completes the proof. \square

3. ϵ -REGULARITY OF THE ERICKSEN-LESLIE SYSTEM (1.8) IN \mathbb{R}^2

In this section, we will establish the regularity of suitable weak solutions of the Ericksen-Leslie system (1.8) in \mathbb{R}^2 , under a smallness condition. The crucial step is the following decay lemma under the smallness condition.

Lemma 3.1. *Assume that the conditions (1.6), (1.7), and (1.15) hold. There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that for $0 < T < +\infty$ and a bounded domain $O \subset \mathbb{R}^2$, if $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(O \times [0, T], \mathbb{R}^2)$, $d \in L^\infty([0, T], H^1(O, \mathbb{S}^2)) \cap L^2([0, T], H^2(O, \mathbb{S}^2))$, and $P \in L^2(O \times [0, T])$ is a suitable weak solution of the Ericksen-Leslie system (1.8) in $O \times [0, T]$, which satisfies, for some $z_0 = (x_0, t_0) \in O \times (0, T]$ and $0 < r_0 \leq \min\{d(x_0, \partial O), \sqrt{t_0}\}$,*

$$\Phi(u, d, P, z_0, r) \leq \epsilon_0,$$

then

$$\Phi(u, d, P, z_0, \theta_0 r) \leq \frac{1}{2} \Phi(u, d, P, z_0, r). \quad (3.1)$$

Here we denote

$$\begin{aligned} &\Phi(u, d, P, z_0, r) \\ &:= \left(\int_{P_r(z_0)} |u|^4 \right)^{\frac{1}{4}} + \left(\int_{P_r(z_0)} |\nabla u|^2 \right)^{\frac{1}{2}} + \left(\int_{P_r(z_0)} |\nabla d|^4 \right)^{\frac{1}{4}} + \left(\int_{P_r(z_0)} |\Delta d|^2 \right)^{\frac{1}{2}} + \left(\int_{P_r(z_0)} |P|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Proof. First observe that since (1.8) is invariant under translations and dilations, we have that

$$u_r(x, t) = ru(x_0 + rx, t_0 + r^2 t), \quad d_r(x, t) = d(x_0 + rx, t_0 + r^2 t), \quad P_r(x, t) = r^2 P(x_0 + rx, t_0 + r^2 t),$$

is a suitable weak solution of (1.8) in $P_1(0)$. Thus it suffices to prove the lemma for $z_0 = (0, 0)$ and $r = 1$.

We argue it by contradiction. Suppose that the lemma were false. Then there would exist $\epsilon_i \downarrow 0$ and a sequence of suitable weak solutions $(u_i, d_i, P_i) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(P_1, \mathbb{R}^2) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(P_1, \mathbb{S}^2) \times L^2(P_1)$ of (1.8) such that

$$\Phi(u_i, d_i, P_i, (0, 0), 1) = \epsilon_i \downarrow 0, \quad (3.2)$$

but, for any $\theta \in (0, \frac{1}{2})$, it holds

$$\Phi(u_i, d_i, P_i, (0, 0), \theta) > \frac{1}{2} \epsilon_i. \quad (3.3)$$

Now we define a blow-up sequence:

$$\tilde{u}_i = \frac{u_i}{\epsilon_i}, \quad \tilde{d}_i = \frac{d_i - (d_i)_1}{\epsilon_i}, \quad \tilde{P}_i = \frac{P_i}{\epsilon_i}, \quad (3.4)$$

where $(d_i)_1 = \frac{1}{|P_1|} \int_{P_1} d_i$ is the average of d_i over P_1 .

It is easy to see that $(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i)$ satisfies

$$\begin{cases} \partial_t \tilde{u}_i + \epsilon_i \tilde{u}_i \cdot \nabla \tilde{u}_i + \nabla \tilde{P}_i = -\epsilon_i \nabla \cdot (\nabla \tilde{d}_i \odot \nabla \tilde{d}_i) + \epsilon_i^{-1} \nabla \cdot (\sigma^L(u_i, d_i)), \\ \nabla \cdot \tilde{u}_i = 0, \\ \partial_t \tilde{d}_i + \epsilon_i \tilde{u}_i \cdot \nabla \tilde{d}_i - \epsilon_i^{-1} \Omega^i \hat{d}_i + \frac{\lambda_2}{\lambda_1} \epsilon_i^{-1} A^i \hat{d}_i = \frac{1}{|\lambda_1|} \left(\Delta \tilde{d}_i + \epsilon_i |\nabla \tilde{d}_i|^2 d_i \right) + \frac{\lambda_2}{\lambda_1} \epsilon_i^{-1} (\hat{d}_i^T A^i \hat{d}_i) d_i. \end{cases} \quad (3.5)$$

Moreover, we have

$$\begin{cases} \Phi(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i, (0, 0), 1) = 1, \\ \Phi(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i, (0, 0), \theta) > \frac{1}{2}. \end{cases} \quad (3.6)$$

It follows from the equation (3.5)₃ that

$$\left\| \partial_t \tilde{d}_i \right\|_{L^2(P_{\frac{3}{4}})} \leq C \left(\|\tilde{u}_i \cdot \nabla \tilde{d}_i\|_{L^2(P_1)} + \|\nabla \tilde{u}_i\|_{L^2(P_1)} + \|\nabla \tilde{d}_i\|_{L^4(P_1)}^2 \right) \leq C. \quad (3.7)$$

After taking possible subsequences, we may assume that there exists

$$(\tilde{u}, \tilde{d}, \tilde{P}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1(P_1, \mathbb{R}^2)) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2(P_1, \mathbb{R}^3)) \times L^2(P_1)$$

and a point $d_0 \in \mathbb{S}^2$ such that

$$\begin{cases} \tilde{u}_i \rightharpoonup \tilde{u} & \text{in } L_t^4 L_x^4(P_1) \cap L_t^2 H_x^1(P_1), \\ \tilde{P}_i \rightharpoonup \tilde{P} & \text{in } L_t^2 L_x^2(P_1), \\ \tilde{d}_i \rightharpoonup \tilde{d} & \text{in } L_t^4 W_x^{1,4}(P_1) \cap L_t^2 H_x^2(P_1), \\ \tilde{d}_i \rightharpoonup \tilde{d} & \text{in } H^1(P_{\frac{3}{4}}), \\ d_i \rightarrow d_0 & \text{a.e. } P_{\frac{3}{4}}. \end{cases}$$

It is easy to check that

$$\begin{aligned} \epsilon_i^{-1} \Omega^i &= \frac{1}{2} (\nabla \tilde{u}_i - (\nabla \tilde{u}_i)^T) \rightharpoonup \tilde{\Omega} := \frac{1}{2} (\nabla \tilde{u} - (\nabla \tilde{u})^T) \quad \text{in } L^2(P_{\frac{3}{4}}), \\ \epsilon_i^{-1} A^i &= \frac{1}{2} (\nabla \tilde{u}_i + (\nabla \tilde{u}_i)^T) \rightharpoonup \tilde{A} := \frac{1}{2} (\nabla \tilde{u} + (\nabla \tilde{u})^T) \quad \text{in } L^2(P_{\frac{3}{4}}), \\ \epsilon_i^{-1} (\hat{d}_i^T A^i \hat{d}_i) &\rightharpoonup \hat{d}_0^T \tilde{A} \hat{d}_0, \quad \epsilon_i^{-1} A^i \hat{d}_i \rightharpoonup \tilde{A} \hat{d}_0 \quad \text{in } L^2(P_{\frac{3}{4}}), \\ \epsilon_i^{-1} N^i &= \epsilon_i^{-1} \left(\partial_t d_i + u_i \cdot \nabla d_i - \Omega^i \hat{d}_i \right) \rightharpoonup \tilde{\mathbf{N}} := \partial_t \tilde{d} - \tilde{\Omega} \hat{d}_0 \quad \text{in } L^2(P_{\frac{3}{4}}), \end{aligned}$$

while

$$\begin{aligned} \epsilon_i^{-1} \sigma^L(u_i, d_i) &:= \mu_1 (\hat{d}_i \otimes \hat{d}_i : (\epsilon_i^{-1} A^i)) \hat{d}_i \otimes \hat{d}_i + \mu_2 (\epsilon_i^{-1} N^i) \otimes \hat{d}_i + \mu_3 \hat{d}_i \otimes (\epsilon_i^{-1} N^i) + \mu_4 (\epsilon_i^{-1} A^i) \\ &\quad + \mu_5 ((\epsilon_i^{-1} A^i) \cdot \hat{d}_i) \otimes \hat{d}_i + \mu_6 \hat{d}_i \otimes ((\epsilon_i^{-1} A^i) \cdot \hat{d}_i) \\ \rightharpoonup \tilde{\sigma}^L(\tilde{u}, d_0) &:= \mu_1 (\hat{d}_0 \otimes \hat{d}_0 : \tilde{A}) \hat{d}_0 \otimes \hat{d}_0 + \mu_2 \tilde{\mathbf{N}} \otimes \hat{d}_0 + \mu_3 \hat{d}_0 \otimes \tilde{\mathbf{N}} + \mu_4 \tilde{A} \\ &\quad + \mu_5 (\tilde{A} \cdot \hat{d}_0) \otimes \hat{d}_0 + \mu_6 \hat{d}_0 \otimes (\tilde{A} \cdot \hat{d}_0) \quad \text{in } L^2(P_{\frac{3}{4}}). \end{aligned}$$

Since $|d_i| = 1$, an elementary argument from the differential geometry implies that

$$\tilde{d}(x, t) \in T_{d_0} \mathbb{S}^2 \quad \text{a.e. } (x, t) \in P_{\frac{3}{4}}. \quad (3.8)$$

Therefore $(\tilde{u}, \tilde{d}, \tilde{P})$ satisfies (3.8) and the following linear system in $P_{\frac{3}{4}}$:

$$\begin{cases} \partial_t \tilde{u} + \nabla \tilde{P} = \nabla \cdot (\tilde{\sigma}^L(\tilde{u}, d_0)), \\ \nabla \cdot \tilde{u} = 0, \\ \partial_t \tilde{d} - \tilde{\Omega} \hat{d}_0 + \frac{\lambda_2}{\lambda_1} \tilde{A} \hat{d}_0 = \frac{1}{|\lambda_1|} \Delta \tilde{d} + \frac{\lambda_2}{\lambda_1} (\hat{d}_0^T \tilde{A} \hat{d}_0) d_0. \end{cases} \quad (3.9)$$

By the lower semicontinuity, we have

$$\Phi(\tilde{u}, \tilde{d}, \tilde{P}, (0, 0), 1) \leq 1. \quad (3.10)$$

By the regularity lemma 3.2 below, we know that $(\tilde{u}, \tilde{d}, \tilde{P})$ is smooth in $P_{\frac{1}{2}}$ and there exists $0 < \theta_0 < \frac{1}{2}$ such that

$$\Phi(\tilde{u}, \tilde{d}, \tilde{P}, (0, 0), \theta_0) \leq C\theta_0 < \frac{1}{4}. \quad (3.11)$$

In order to reach the desired contradiction, we need to apply the local energy inequality (2.14) for $(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i)$. First, observe that the equation (3.5)₁ can be written as

$$\partial_t \tilde{u}_i + \nabla \tilde{P}_i - \frac{\mu_4}{2} \Delta \tilde{u}_i = g_i := \left[-\epsilon_i \nabla \cdot (\tilde{u}_i \otimes \tilde{u}_i + \nabla \tilde{d}_i \odot \nabla \tilde{d}_i) + \epsilon_i^{-1} \nabla \cdot (\sigma^L(u_i, d_i) - \mu_4 A^i) \right]. \quad (3.12)$$

It follows from (3.6) that $g_i \in L^2([-1, 0], H^{-1}(B_1))$ and

$$\begin{aligned} \|g_i\|_{L^2([-1, 0], H^{-1}(B_1))} &\lesssim \left[\|\tilde{u}_i\|_{L^4(P_1)}^2 + \|\nabla \tilde{d}_i\|_{L^4(P_1)}^2 + \|\nabla \tilde{u}_i\|_{L^2(P_1)} + \|\nabla^2 \tilde{d}_i\|_{L^2(P_1)} \right] \\ &\lesssim \Phi(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i, (0, 0), 1) \leq C. \end{aligned}$$

Hence by the standard estimate on Stokes' system (cf. [31]) we have that $\partial_t \tilde{u}_i \in L^2([-\frac{3}{4}, 0], H^{-1}(B_{\frac{3}{4}}))$ and

$$\|\partial_t \tilde{u}_i\|_{L^2([-\frac{3}{4}, 0], H^{-1}(B_{\frac{3}{4}}))} \lesssim \left[\Phi(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i, (0, 0), 1) + \|g_i\|_{L^2([-1, 0], H^{-1}(B_1))} \right] \leq C. \quad (3.13)$$

It follows from (3.6), (3.7), and (3.13) that we can apply the Aubin-Lions compactness lemma (cf. [31]) to conclude that, after taking possible subsequences,

$$\tilde{u}_i \rightarrow \tilde{u}, \quad \nabla \tilde{d}_i \rightarrow \nabla \tilde{d} \quad \text{in } L^2(P_{\frac{3}{4}}). \quad (3.14)$$

By Fubini's theorem, for any $\theta \in (0, \frac{1}{2})$ there exists $\tau_0 \in (\theta^2, 4\theta^2)$ such that

$$\int_{B_{2\theta}} (|\tilde{u}_i - \tilde{u}|^2 + |\nabla \tilde{d}_i - \nabla \tilde{d}|^2)(x, -\tau_0) dx \leq C\theta^{-2} \int_{P_{2\theta}} (|\tilde{u}_i - \tilde{u}|^2 + |\nabla \tilde{d}_i - \nabla \tilde{d}|^2) \leq C\theta^{-2} o(1).$$

Here $o(1)$ denotes the constant such that $\lim_{i \rightarrow +\infty} o(1) = 0$. Since

$$\int_{B_{2\theta}} (|\tilde{u}|^2 + |\nabla \tilde{d}|^2)(x, -\tau_0) dx \leq C\theta^2,$$

we have

$$\int_{B_{2\theta}} (|\tilde{u}_i|^2 + |\nabla \tilde{d}_i|^2)(x, -\tau_0) dx \leq C[\theta^2 + \theta^{-2} o(1)]. \quad (3.15)$$

Since (u_i, d_i, P_i) satisfies the local energy inequality (2.14), we see that by rescalings $(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i)$ satisfies the following local energy inequality: for any $-\tau_0 \leq t \leq 0$ and any $\eta \in C_0^\infty(B_1)$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \eta^2 (|\tilde{u}_i|^2 + |\nabla \tilde{d}_i|^2)(t) + \int_{-\tau_0}^t \int_{\mathbb{R}^2} \eta^2 \left[\mu_4 |\nabla \tilde{u}_i|^2 + \frac{2}{|\lambda_1|} |\Delta \tilde{d}_i + \epsilon_i |\nabla \tilde{d}|^2 d_i|^2 \right] \\ &\leq \int_{\mathbb{R}^2} \eta^2 (|\tilde{u}_i|^2 + |\nabla \tilde{d}_i|^2)(-\tau_0) \\ &\quad + C \int_{-\tau_0}^t \int_{\mathbb{R}^2} \left[(\epsilon_i |\tilde{u}_i|^2 + |\nabla \tilde{u}_i| + \epsilon_i |\tilde{u}_i| |\nabla \tilde{d}_i| + \epsilon_i |\nabla \tilde{d}_i|^2 + |\nabla^2 \tilde{d}_i| + |\tilde{P}_i|) |\tilde{u}_i| |\nabla(\eta^2)| \right] \end{aligned}$$

$$+(\|\nabla\tilde{u}_i\| + \epsilon_i|\tilde{u}_i|\|\nabla\tilde{d}_i\| + \epsilon_i|\nabla\tilde{d}_i|^2 + |\nabla^2\tilde{d}_i|)|\nabla\tilde{d}_i|\|\nabla(\eta^2)\|]. \quad (3.16)$$

By the weak and strong convergence properties for $(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i)$ listed as above, we have that, as $i \rightarrow +\infty$,

$$\begin{aligned} \mathbf{E}_i &:= \int_{-\tau_0}^t \int_{\mathbb{R}^2} \left[(\epsilon_i|\tilde{u}_i|^2 + |\nabla\tilde{u}_i\| + \epsilon_i|\tilde{u}_i|\|\nabla\tilde{d}_i\| + \epsilon_i|\nabla\tilde{d}_i|^2 + |\nabla^2\tilde{d}_i\| + |\tilde{P}_i|)|\tilde{u}_i|\|\nabla(\eta^2)\| \right. \\ &\quad \left. + (|\nabla\tilde{u}_i\| + \epsilon_i|\tilde{u}_i|\|\nabla\tilde{d}_i\| + \epsilon_i|\nabla\tilde{d}_i|^2 + |\nabla^2\tilde{d}_i\|)|\nabla\tilde{d}_i|\|\nabla(\eta^2)\| \right] \\ &\rightarrow \mathbf{E} := \int_{-\tau_0}^t \int_{\mathbb{R}^2} \left[(|\nabla\tilde{u}\| + |\nabla^2\tilde{d}\| + |\tilde{P}\|)|\tilde{u}\|\nabla(\eta^2)\| + (|\nabla\tilde{u}\| + |\nabla^2\tilde{d}\|)|\nabla\tilde{d}\|\nabla(\eta^2)\| \right]. \end{aligned} \quad (3.17)$$

Now we choose $\eta \in C_0^\infty(B_1)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{\sqrt{\tau_0}}, \quad \eta \equiv 0 \text{ outside } B_{2\sqrt{\tau_0}}, \quad \text{and } |\nabla\eta| \leq C\tau_0^{-\frac{1}{2}} \leq C\theta^{-1}.$$

Then we have that for $\theta \in (0, \frac{1}{4})$,

$$\begin{aligned} |\mathbf{E}| &\lesssim \theta^{-1} \int_{P_{2\theta}} \left[(|\nabla\tilde{u}\| + |\nabla^2\tilde{d}\| + |\tilde{P}\|)|\tilde{u}\| + (|\nabla\tilde{u}\| + |\nabla^2\tilde{d}\|)|\nabla\tilde{d}\| \right] \\ &\lesssim \left[(\|\nabla^2\tilde{d}\|_{L^2(P_{2\theta})} + \|\nabla\tilde{u}\|_{L^2(P_{2\theta})} + \|\tilde{P}\|_{L^2(P_{2\theta})}) (\|\nabla\tilde{d}\|_{L^4(P_{2\theta})} + \|\tilde{u}\|_{L^4(P_{2\theta})}) \right] \\ &\lesssim \left[\Phi(\tilde{u}, \tilde{d}, \tilde{P}, (0, 0), 2\theta) \right]^2 \leq C\theta^2 \end{aligned}$$

so that

$$|\mathbf{E}_i| \leq C(\theta^2 + o(1)). \quad (3.18)$$

Substituting (3.15) and (3.18) into (3.16) yields

$$\sup_{-\theta^2 \leq t \leq 0} \int_{B_\theta} (|\tilde{u}_i|^2 + |\nabla\tilde{d}_i|^2)(t) + \int_{P_\theta} \left[\mu_4 |\nabla\tilde{u}_i|^2 + \frac{2}{|\lambda_1|} |\Delta\tilde{d}_i + \epsilon_i |\nabla\tilde{d}|^2 d_i|^2 \right] \leq C(\theta^2 + \theta^{-2}o(1)). \quad (3.19)$$

Since

$$\int_{P_\theta} |\Delta\tilde{d}_i|^2 \leq \int_{P_\theta} |\Delta\tilde{d}_i + \epsilon_i |\nabla\tilde{d}|^2 d_i|^2 + \epsilon_i^2 \int_{P_\theta} |\nabla\tilde{d}_i|^4 \leq \int_{P_\theta} |\Delta\tilde{d}_i + \epsilon_i |\nabla\tilde{d}|^2 d_i|^2 + C\epsilon_i^2,$$

and by the H^2 -estimate

$$\int_{P_{\frac{\theta}{2}}} |\nabla^2\tilde{d}_i|^2 \lesssim \int_{P_\theta} |\Delta\tilde{d}_i|^2 + \theta^{-2} \int_{P_\theta} |\nabla\tilde{d}_i|^2 \lesssim \int_{P_\theta} |\Delta\tilde{d}_i|^2 + \theta^2 + \theta^{-2}o(1),$$

we obtain

$$\sup_{-\frac{1}{4}\theta^2 \leq t \leq 0} \int_{B_{\frac{\theta}{2}}} (|\tilde{u}_i|^2 + |\nabla\tilde{d}_i|^2)(t) + \int_{P_{\frac{\theta}{2}}} (|\nabla\tilde{u}_i|^2 + |\nabla^2\tilde{d}_i|^2) \leq C(\theta^2 + \epsilon_i^2 + \theta^{-2}o(1)). \quad (3.20)$$

Recall Ladyzhenskaya's inequality in \mathbb{R}^2 (cf. [15]):

$$\int_{B_r} |f|^4 \lesssim \int_{B_{2r}} |f|^2 \int_{B_{2r}} (r^{-2}|f|^2 + |\nabla f|^2), \quad \forall f \in H^1(B_{2r}). \quad (3.21)$$

Applying (3.21) to \tilde{u}_i and $\nabla\tilde{d}_i$ and integrating over t -variable and using (3.20), we obtain

$$\begin{aligned} \int_{P_{\frac{\theta}{4}}} (|\tilde{u}_i|^4 + |\nabla\tilde{d}_i|^4) &\lesssim \left(\sup_{t \in [-\frac{1}{4}\theta^2, 0]} \int_{B_{\frac{\theta}{2}}} |\tilde{u}_i|^2 + |\nabla\tilde{d}_i|^2 \right) \int_{P_{\frac{\theta}{2}}} \left[\theta^{-2}(|\tilde{u}_i|^2 + |\nabla\tilde{d}_i|^2) + |\nabla\tilde{u}_i|^2 + |\nabla^2\tilde{d}_i|^2 \right] \\ &\leq C(\theta^4 + \epsilon_i^4 + \theta^{-4}o(1)). \end{aligned} \quad (3.22)$$

To estimate \tilde{P}_i , let's take divergence of the equation (3.5)₁ to get

$$\Delta\tilde{P}_i = -\epsilon_i(\nabla \cdot)^2 (\tilde{u}_i \otimes \tilde{u}_i + \nabla\tilde{d}_i \odot \nabla\tilde{d}_i) + \epsilon_i^{-1}(\nabla \cdot)^2 (\sigma^L(u_i, d_i)) \quad \text{in } B_\theta. \quad (3.23)$$

Let $\phi \in C_0^\infty(\mathbb{R}^2)$ such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B_{\frac{\theta}{8}}, \quad \phi \equiv 0 \text{ outside } B_{\frac{\theta}{4}}, \quad \text{and } |\nabla\phi| \lesssim \theta^{-1}.$$

Define \tilde{Q}_i by

$$\tilde{Q}_i(x, t) = - \int_{\mathbb{R}^2} \nabla_y^2 G(x - y) : \phi^2(y) \left[\epsilon_i(\tilde{u}_i \otimes \tilde{u}_i + \nabla \tilde{d}_i \odot \nabla \tilde{d}_i) - \epsilon_i^{-1} \sigma^L(u_i, d_i) \right] (y, t) dy,$$

where G is the fundamental solution of the Laplace equation on \mathbb{R}^2 . Then we have

$$\Delta \tilde{Q}_i = -(\nabla \cdot)^2 \left[\phi^2 \left(\epsilon_i(\tilde{u}_i \otimes \tilde{u}_i + \nabla \tilde{d}_i \odot \nabla \tilde{d}_i) - \epsilon_i^{-1} \sigma^L(u_i, d_i) \right) \right] \text{ in } \mathbb{R}^2.$$

By Calderon-Zygmund's L^2 -theory we have

$$\begin{aligned} \|\tilde{Q}_i\|_{L^2(\mathbb{R}^2)}^2 &\lesssim \left[\epsilon_i(\|\tilde{u}_i\|_{L^4(B_{\frac{\theta}{4}})}^4 + \|\nabla \tilde{d}_i\|_{L^4(B_{\frac{\theta}{4}})}^4) + \|\epsilon_i^{-1} \sigma^L(u_i, d_i)\|_{L^2(B_{\frac{\theta}{4}})}^2 \right] \\ &\lesssim \left[\epsilon_i(\|\tilde{u}_i\|_{L^4(B_{\frac{\theta}{4}})}^4 + \|\nabla \tilde{d}_i\|_{L^4(B_{\frac{\theta}{4}})}^4) + \|\nabla \tilde{u}_i\|_{L^2(B_{\frac{\theta}{4}})}^2 + \|\epsilon_i^{-1} N^i\|_{L^2(B_{\frac{\theta}{4}})}^2 \right] \\ &\lesssim \left[\|\tilde{u}_i\|_{L^4(B_{\frac{\theta}{4}})}^4 + \|\nabla \tilde{d}_i\|_{L^4(B_{\frac{\theta}{4}})}^4 + \|\nabla \tilde{u}_i\|_{L^2(B_{\frac{\theta}{4}})}^2 + \|\nabla^2 \tilde{d}_i\|_{L^2(B_{\frac{\theta}{4}})}^2 \right]. \end{aligned} \quad (3.24)$$

Integrating (3.24) over $t \in [-\theta^2, 0]$, and using (3.20) and (3.22), we obtain

$$\int_{-\theta^2}^0 \int_{\mathbb{R}^2} |\tilde{Q}_i|^2 \lesssim \int_{P_{\frac{\theta}{4}}} \left(|\tilde{u}_i|^4 + |\nabla \tilde{d}_i|^4 + |\nabla \tilde{u}_i|^2 + |\nabla^2 \tilde{d}_i|^2 \right) \leq C(\theta^4 + \epsilon_i^4 + \theta^{-4} o(1)). \quad (3.25)$$

Set $\tilde{R}_i = \tilde{P}_i - \tilde{Q}_i$ in P_θ . Then we have

$$\Delta \tilde{R}_i(t) = 0 \text{ in } B_{\frac{\theta}{4}}, \quad \forall t \in [-\theta^2, 0],$$

so that by the standard estimate of harmonic functions and (3.25) we have

$$\int_{P_{\theta^2}} |\tilde{R}_i|^2 \lesssim \theta^2 \int_{P_{\frac{\theta}{4}}} |\tilde{R}_i|^2 \leq C\theta^2 \int_{P_{\frac{\theta}{4}}} (|\tilde{P}_i|^2 + |\tilde{Q}_i|^2) \leq C[\theta^2 + (\theta^4 + \epsilon_i^4 + \theta^{-4} o(1))]. \quad (3.26)$$

Putting (3.25) together with (3.26) yields

$$\int_{P_{\theta^2}} |\tilde{P}_i|^2 \leq C[\theta^2 + (\theta^4 + \epsilon_i^4 + \theta^{-4} o(1))]. \quad (3.27)$$

Combining all these estimates (3.20), (3.22), and (3.27), we obtain

$$\Phi(\tilde{u}_i, \tilde{d}_i, \tilde{P}_i, (0, 0), \theta^2) \leq C[\theta + \epsilon_i + \theta^{-1} o(1)] \leq \frac{1}{4}, \quad (3.28)$$

provided that we first choose sufficiently small θ and then choose sufficiently large i . This gives the desired contradiction. The proof is complete. \square

The following lemma plays an important role in the blow-up process, which may have its own interest.

Lemma 3.2. *Assume (1.6), (1.7), and (1.15) hold. For any point $d_0 \in \mathbb{S}^2$, if $\tilde{u} \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(P_{\frac{3}{4}}, \mathbb{R}^2)$, $\tilde{d} \in (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(P_{\frac{3}{4}}, T_{d_0} \mathbb{S}^2)$, and $\tilde{P} \in L^2(P_{\frac{3}{4}})$ solves the linear system (3.9) and satisfies the condition (3.10), then $(\tilde{u}, \tilde{d}, \tilde{P}) \in C^\infty(P_{\frac{1}{2}})$ and satisfies the following estimate:*

$$\Phi(\tilde{u}, \tilde{d}, \tilde{P}, (0, 0), \theta) \leq C\theta, \quad \forall \theta \in (0, \frac{1}{2}). \quad (3.29)$$

Proof. To simplify the notations, we write (u, d, P) for $(\tilde{u}, \tilde{d}, \tilde{P})$ in the proof below. The argument is based on the higher order local energy inequality argument.

Taking $\frac{\partial}{\partial x_i}$ of the linear system (3.9) yields

$$\begin{cases} \partial_t u_{x_i} + \nabla P_{x_i} = \nabla \cdot (\tilde{\sigma}^L(u, d_0))_{x_i}, \\ \nabla \cdot u_{x_i} = 0, \\ \partial_t d_{x_i} - \Omega_{x_i} \hat{d}_0 + \frac{\lambda_2}{\lambda_1} A_{x_i} \hat{d}_0 = \frac{1}{|\lambda_1|} \Delta d_{x_i} + \frac{\lambda_2}{\lambda_1} \left(\hat{d}_0^T A_{x_i} \hat{d}_0 \right) d_0. \end{cases} \quad (3.30)$$

For any $\eta \in C_0^\infty(B_1)$, multiplying the equation (3.30)₁ by $u_{x_i}\eta^2$ and the equation (3.30)₃ by $\Delta d_{x_i}\eta^2$ and integrating the resulting equations over B_1 , we obtain³

$$\begin{aligned}
& \frac{d}{dt} \int_{B_1} |\nabla u|^2 \eta^2 = \\
& -2 \left[\int_{B_1} P [\Delta u \cdot \nabla(\eta^2) + \nabla u : \nabla^2(\eta^2)] + \int_{B_1} (\tilde{\sigma}^L(u, d_0))_{x_i} : u_{x_i} \otimes \nabla(\eta^2) + \int_{B_1} \eta^2 (\tilde{\sigma}^L(u, d_0))_{x_i} : \nabla u_{x_i} \right], \\
& \frac{d}{dt} \int_{B_1} |\nabla^2 d|^2 \eta^2 + \frac{2}{|\lambda_1|} \int_{B_1} |\Delta \nabla d|^2 \eta^2 \\
& = -2 \int_{B_1} \partial_t d_{x_i} \cdot \nabla d_{x_i} \cdot \nabla(\eta^2) - 2 \int_{B_1} \left(\langle \Omega_{x_i} \hat{d}_0 - \frac{\lambda_2}{\lambda_1} A_{x_i} \hat{d}_0, \Delta \hat{d}_{x_i} \rangle + \langle \frac{\lambda_2}{\lambda_1} (\hat{d}_0^T A_{x_i} \hat{d}_0) d_0, \Delta d_{x_i} \rangle \right) \eta^2. \\
& = -2 \int_{B_1} \partial_t d_{x_i} \cdot \nabla d_{x_i} \cdot \nabla(\eta^2) - 2 \int_{B_1} \left(\langle \Omega_{x_i} \hat{d}_0 - \frac{\lambda_2}{\lambda_1} A_{x_i} \hat{d}_0, \Delta \hat{d}_{x_i} \rangle \right) \eta^2,
\end{aligned} \tag{3.31}$$

where we have used in the last step the fact $d \in T_{d_0} \mathbb{S}^2$ in order to deduce that $\langle d_0, \Delta d_{x_i} \rangle = 0$ a.e. in B_1 .

Similar to the calculations in the proof of lemma 2.1, we have

$$\begin{aligned}
& \int_{B_1} \eta^2 (\tilde{\sigma}^L(u, d_0))_{x_i} : \nabla u_{x_i} \\
& = \int_{B_1} \eta^2 \left[\mu_1 (\hat{d}_0 \otimes \hat{d}_0 : A_{x_i}) \hat{d}_0 \otimes \hat{d}_0 + \mu_2 \tilde{\mathbf{N}}_{x_i} \otimes \hat{d}_0 + \mu_3 \hat{d}_0 \otimes \tilde{\mathbf{N}}_{x_i} + \mu_4 A_{x_i} \right. \\
& \quad \left. + \mu_5 (A_{x_i} \cdot \hat{d}_0) \otimes \hat{d}_0 + \mu_6 \hat{d}_0 \otimes (A_{x_i} \cdot \hat{d}_0) \right] : (A_{x_i} + \Omega_{x_i}) \\
& = \int_{B_1} \eta^2 \left[\mu_1 (\hat{d}_0^T A_{x_i} \hat{d}_0)^2 + \mu_4 |A_{x_i}|^2 - \lambda_2 \tilde{\mathbf{N}}_{x_i} \cdot (A_{x_i} \cdot \hat{d}_0) + \lambda_1 \tilde{\mathbf{N}}_{x_i} \cdot (\Omega_{x_i} \cdot \hat{d}_0) \right. \\
& \quad \left. + (\mu_5 + \mu_6) |A_{x_i} \cdot \hat{d}_0|^2 + \lambda_2 (A_{x_i} \cdot \hat{d}_0) (\Omega_{x_i} \cdot \hat{d}_0) \right].
\end{aligned} \tag{3.32}$$

By the equation (3.30)₃, we have

$$\lambda_1 \tilde{\mathbf{N}}_{x_i} \cdot (\Omega_{x_i} \cdot \hat{d}_0) = -\lambda_2 (A_{x_i} \cdot \hat{d}_0) (\Omega_{x_i} \cdot \hat{d}_0) - \langle \Omega_{x_i} \hat{d}_0, \Delta \hat{d}_{x_i} \rangle,$$

and

$$-\lambda_2 \tilde{\mathbf{N}}_{x_i} \cdot (A_{x_i} \cdot \hat{d}_0) = \frac{\lambda_2^2}{\lambda_1} |A_{x_i} \cdot \hat{d}_0|^2 - \frac{\lambda_2^2}{\lambda_1} (\hat{d}_0^T A_{x_i} \hat{d}_0)^2 + \frac{\lambda_2}{\lambda_1} \langle A_{x_i} \hat{d}_0, \Delta \hat{d}_{x_i} \rangle.$$

Substituting these identities into (3.32), we obtain

$$\begin{aligned}
& \int_{B_1} \eta^2 (\tilde{\sigma}^L(u, d_0))_{x_i} : \nabla u_{x_i} \\
& = \int_{B_1} \eta^2 \left[\left(\mu_1 - \frac{\lambda_2^2}{\lambda_1} \right) (\hat{d}_0^T A_{x_i} \hat{d}_0)^2 + \mu_4 |A_{x_i}|^2 + \left\langle \frac{\lambda_2}{\lambda_1} A_{x_i} \hat{d}_0 - \Omega_{x_i} \hat{d}_0, \Delta \hat{d}_{x_i} \right\rangle \right. \\
& \quad \left. + (\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}) |A_{x_i} \cdot \hat{d}_0|^2 \right].
\end{aligned} \tag{3.33}$$

Putting (3.33) into (3.31) and adding the resulting (3.31) with (3.32), we have, by (1.15),

$$\begin{aligned}
& \frac{d}{dt} \int_{B_1} (|\nabla u|^2 + |\nabla^2 d|^2) \eta^2 + \int_{B_1} \left(\mu_4 |\nabla^2 u|^2 + \frac{2}{|\lambda_1|} |\nabla^3 d|^2 \right) \eta^2 \\
& = -2 \int_{B_1} \eta^2 \left[\left(\mu_1 - \frac{\lambda_2^2}{\lambda_1} \right) (\hat{d}_0^T A_{x_i} \hat{d}_0)^2 + (\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}) |A_{x_i} \cdot \hat{d}_0|^2 \right] \\
& \quad - 2 \int_{B_1} P [\Delta u \cdot \nabla(\eta^2) + \nabla u : \nabla^2(\eta^2)] - 2 \int_{B_1} (\tilde{\sigma}^L(u, d_0))_{x_i} : u_{x_i} \otimes \nabla(\eta^2)
\end{aligned}$$

³Strictly speaking, we first need to take finite quotient D_h^i of the system (3.9) and then multiply the first equation and the third equation of the resulting equations by $D_h^i u(x, t) \eta^2 = \frac{u(x+he_i, t) - u(x, t)}{h}$ and $D_h^i (\Delta d) \eta^2$ respectively for $h > 0$ and $i = 1, 2$, with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then the desired estimate follows from the estimates on the finite quotients by sending h to zero.

$$\begin{aligned}
& + \int_{B_1} \left(\mu_4 \nabla^2 u \cdot \nabla(\eta^2) : \nabla u - 2\partial_t d_{x_i} \cdot \nabla d_{x_i} \cdot \nabla(\eta^2) - \frac{2}{|\lambda_1|} \nabla(\nabla d) \otimes \nabla(\nabla d) : \nabla^2(\eta^2) \right) \\
\leq & -2 \int_{B_1} P [\Delta u \cdot \nabla(\eta^2) + \nabla u : \nabla^2(\eta^2)] - 2 \int_{B_1} (\tilde{\sigma}^L(u, d_0))_{x_i} : u_{x_i} \otimes \nabla(\eta^2) + \\
& \int_{B_1} \left[\mu_4 \nabla^2 u \cdot \nabla(\eta^2) : \nabla u - 2\partial_t d_{x_i} \cdot \nabla d_{x_i} \cdot \nabla(\eta^2) + \frac{2}{|\lambda_1|} (\nabla^3 d \cdot (\nabla d)_{x_i} + \Delta \nabla d \cdot (\nabla d)_{x_i})(\eta^2)_{x_i} \right] \\
:= & R_1 + R_2 + R_3. \tag{3.35}
\end{aligned}$$

Now we estimate each term of the right hand side of (3.35) as follows.

$$\begin{aligned}
|R_1| & \lesssim \int_{B_1} |P| (|\nabla^2 u|^2 \eta |\nabla \eta| + |\nabla u| (|\nabla^2 \eta| + |\nabla \eta|^2)) \\
& \leq \frac{\mu_4}{6} \int_{B_1} |\nabla^2 u|^2 \eta^2 + C \int_{B_1} (|P|^2 + |\nabla u|^2) (|\nabla \eta|^2 + |\nabla^2 \eta|), \\
|R_2| & \lesssim \int_{B_1} (|\nabla^2 u| + |\nabla^3 d|) |\nabla u| \eta |\nabla \eta| \\
& \leq \frac{\mu_4}{6} \int_{B_1} |\nabla^2 u|^2 \eta^2 + \frac{1}{2|\lambda_1|} \int_{B_1} |\nabla^3 d|^2 \eta^2 + C \int_{B_1} |\nabla u|^2 |\nabla \eta|^2,
\end{aligned}$$

and

$$\begin{aligned}
|R_3| & \lesssim \int_{B_1} |\nabla^2 u| |\nabla u| \eta |\nabla \eta| + (|\nabla^2 u| + |\nabla^3 d|) |\nabla^2 d| \eta |\nabla \eta| \\
& \leq \frac{\mu_4}{6} \int_{B_1} |\nabla^2 u|^2 \eta^2 + \frac{1}{2|\lambda_1|} \int_{B_1} |\nabla^3 d|^2 \eta^2 + C \int_{B_1} (|\nabla u|^2 + |\nabla^2 d|^2) |\nabla \eta|^2.
\end{aligned}$$

Putting these estimates into (3.35), we obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{B_1} (|\nabla u|^2 + |\nabla^2 d|^2) \eta^2 + \int_{B_1} \left(\frac{\mu_4}{2} |\nabla^2 u|^2 + \frac{1}{|\lambda_1|} |\nabla^3 d|^2 \right) \eta^2 \\
& \leq C \int_{B_1} \left[(|P|^2 + |\nabla u|^2) (|\nabla \eta|^2 + |\nabla^2 \eta|) + |\nabla^2 d|^2 |\nabla \eta|^2 \right], \tag{3.36}
\end{aligned}$$

By Fubini's theorem, there exists $t_* \in [-\frac{1}{4}, 0]$ such that

$$\int_{B_1} (|\nabla u|^2 + |\nabla^2 d|^2) \eta^2(t_*) \leq 8 \int_{P_1} (|\nabla u|^2 + |\nabla^2 d|^2) \eta^2.$$

Integrating (3.36) over $t \in [t_*, 0]$ yields that

$$\begin{aligned}
& \sup_{-\frac{1}{4} \leq t \leq 0} \int_{B_1} (|\nabla u|^2 + |\nabla^2 d|^2) \eta^2(t) + \int_{-\frac{1}{4}}^0 \int_{B_1} (|\nabla^2 u|^2 + |\nabla^3 d|^2) \eta^2 \\
& \leq C \int_{P_1} \left[(|P|^2 + |\nabla u|^2) (|\nabla \eta|^2 + |\nabla^2 \eta|) + |\nabla^2 d|^2 |\nabla \eta|^2 \right] + C \int_{P_1} (|\nabla u|^2 + |\nabla^2 d|^2) \eta^2. \tag{3.37}
\end{aligned}$$

For the pressure P , taking divergence of the equation (3.30)₁ yields that for any $-1 \leq t \leq 0$,

$$\Delta P_{x_i} = (\nabla \cdot)^2 (\tilde{\sigma}^L(u, d_0))_{x_i} \quad \text{in } B_{\frac{3}{4}}. \tag{3.38}$$

Similar to the pressure estimates obtained in the proof of lemma 3.1, we have

$$\int_{P_{\frac{1}{4}}} |\nabla P|^2 \lesssim \int_{P_{\frac{5}{16}}} |\tilde{\sigma}^L(u, d_0)_{x_i}|^2 + |P|^2 \lesssim \int_{P_{\frac{3}{8}}} (|\nabla^2 u|^2 + |\nabla^3 d|^2 + |P|^2). \tag{3.39}$$

Let $\eta \in C_0^1(B_1)$ be a cut-off function of $B_{\frac{3}{8}}$, i.e. $\eta \equiv 1$ in $B_{\frac{3}{8}}$, $\eta \equiv 0$ outside $B_{\frac{3}{4}}$, $0 \leq \eta \leq 1$, and $|\nabla \eta| + |\nabla^2 \eta| \leq 16$. Then, by combining (3.37) with (3.39), we obtain

$$\sup_{-(\frac{1}{4})^2 \leq t \leq 0} \int_{B_{\frac{1}{4}}} (|\nabla u|^2 + |\nabla^2 d|^2)(t) + \int_{P_{\frac{1}{4}}} (|\nabla^2 u|^2 + |\nabla^3 d|^2 + |\nabla P|^2)$$

$$\leq C \int_{P_{\frac{3}{4}}} \left[|P|^2 + |\nabla u|^2 + |\nabla^2 d|^2 \right]. \quad (3.40)$$

It turns out that the above energy method can be extended to any high order. Here we only give a sketch of the proof. In fact, if we denote $\nabla^\alpha = \frac{\partial^k}{\partial x^\alpha}$ as the k -th order derivative for any multiple index $\alpha = (\alpha_1, \alpha_2)$ ($k = |\alpha| = \alpha_1 + \alpha_2 \geq 2$), and take ∇^α of the system (3.9), then we obtain

$$\begin{cases} \partial_t(\nabla^\alpha u) + \nabla(\nabla^\alpha P) = \nabla \cdot (\nabla^\alpha(\tilde{\sigma}^L(u, d_0))), \\ \nabla \cdot (\nabla^\alpha u) = 0, \\ \partial_t(\nabla^\alpha d) - (\nabla^\alpha \Omega) \hat{d}_0 + \frac{\lambda_2}{\lambda_1} (\nabla^\alpha A) \hat{d}_0 = \frac{1}{|\lambda_1|} \Delta(\nabla^\alpha d) + \frac{\lambda_2}{\lambda_1} \left(\hat{d}_0^T (\nabla^\alpha A) \hat{d}_0 \right) d_0. \end{cases} \quad (3.41)$$

Multiplying (3.41)₁ by $(\nabla^\alpha u) \eta^2$ and (3.41)₃ by $\Delta(\nabla^\alpha d) \eta^2$ and integrating the resulting equations over B_1 , and repeating the above calculations and cancelations, we would obtain

$$\begin{aligned} & \frac{d}{dt} \int_{B_1} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2 + \int_{B_1} \left(\frac{\mu_4}{2} |\nabla^{k+1} u|^2 + \frac{1}{|\lambda_1|} |\nabla^{k+2} d|^2 \right) \eta^2 \\ & \leq C \int_{B_1} \left[(|\nabla^{k-1} P|^2 + |\nabla^k u|^2) (|\nabla \eta|^2 + |\nabla^2 \eta|) + |\nabla^{k+1} d|^2 |\nabla \eta|^2 \right]. \end{aligned} \quad (3.42)$$

For P , since

$$\Delta(\nabla^\alpha P) = (\nabla \cdot)^2 (\nabla^\alpha(\tilde{\sigma}^L(u, d_0))) \quad \text{in } B_{\frac{3}{4}}, \quad (3.43)$$

we have

$$\int_{P_{\frac{1}{4}}} |\nabla^k P|^2 \lesssim \int_{P_{\frac{3}{8}}} (|\nabla^{k-1} P|^2 + |\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2). \quad (3.44)$$

Following the same lines of proof as above, we can choose suitable time slice $t_* \in (-\frac{1}{4}, 0)$ such that

$$\int_{B_1} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(t_*) \leq 8 \int_{P_1} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2.$$

By choosing suitable test functions similar to the above ones, we can reach that for any $k \geq 2$, it holds

$$\begin{aligned} & \sup_{-(\frac{1}{4})^2 \leq t \leq 0} \int_{B_{\frac{1}{4}}} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) (t) + \int_{P_{\frac{1}{4}}} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2 + |\nabla^k P|^2) \\ & \leq C \int_{P_{\frac{3}{4}}} \left[|\nabla^{k-1} P|^2 + |\nabla^k u|^2 + |\nabla^{k+1} d|^2 \right]. \end{aligned} \quad (3.45)$$

It is clear that with suitable adjusting of the radius, we see that (3.45) and (3.40) implies that

$$\begin{aligned} & \sup_{-(\frac{1}{4})^2 \leq t \leq 0} \int_{B_{\frac{1}{4}}} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) (t) + \int_{P_{\frac{1}{4}}} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2 + |\nabla^k P|^2) \\ & \leq C \int_{P_{\frac{3}{4}}} \left[|P|^2 + |\nabla u|^2 + |\nabla^2 d|^2 \right] \end{aligned} \quad (3.46)$$

holds for all $k \geq 1$.

Now we can apply the regularity theory for both the linear Stokes equations (c.f. [31]) and the linear heat equation (cf. [15]) to conclude that $(u, d) \in C^\infty(P_{\frac{1}{2}})$. Furthermore, apply the elliptic estimate for the pressure equation (3.38), we see that $P \in C^\infty(P_{\frac{1}{2}})$ (first we have $\nabla^k P \in C^0(P_{\frac{1}{2}})$, then note that $\partial_t^l P$ also satisfies a similar elliptic equation, so that $\nabla^k \partial_t^l P \in C^0(P_{\frac{1}{2}})$). Therefore $(u, d, P) \in C^\infty(P_{\frac{1}{2}})$ and the desired estimate (3.29) holds. The proof of lemma 3.2 is complete. \square

In order to show the smoothness of solutions to (1.8) under the condition (3.1), we need to iterate the decay inequality (3.1) and establish higher integrability of $(u, \nabla d)$ by applying the techniques of Morrey space estimates for Riesz potentials, similar to that by Hineman-Wang [14].

Lemma 3.3. *Assume that the conditions (1.6), (1.7), and (1.15) hold. For any $0 < T \leq +\infty$ and a bounded domain $O \subset \mathbb{R}^2$, there exists $\epsilon_0 > 0$ such that $(u, P, d) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(O \times [0, T]) \times L^2(O \times [0, T]) \times L_t^2 H_x^2(O \times [0, T], \mathbb{S}^2)$ is a suitable weak solution of (1.8), and satisfies, for $z_0 = (x_0, t_0) \in O \times (0, T)$ and $P_{r_0}(z_0) \subset O \times (0, T)$,*

$$\Phi(u, d, P, z_0, r_0) \leq \epsilon_0, \quad (3.47)$$

then $(u, \nabla d) \in L_{\text{loc}}^q(P_{r_0}(z_0))$ for any $1 < q < +\infty$. Moreover, it holds

$$\|(u, \nabla d)\|_{L^q(P_{\frac{r_0}{4}}(z_0))} \leq C(q, r_0)\epsilon_0. \quad (3.48)$$

Proof. Set $r_1 = \frac{r_0}{2}$. Then it is easy to see that (3.47) also holds for (u, P, d) with z_0, r_0 replaced by z_1, r_1 for any $z_1 \in P_{\frac{r_0}{2}}(z_0)$. Applying lemma 3.1 for (u, P, d) on $P_{r_1}(z_1)$, we conclude that there exists $\theta_0 \in (0, \frac{1}{2})$ such that for any $0 < r \leq r_1$, it holds that

$$\Phi(u, d, P, z_1, \theta_0 r) \leq \frac{1}{2} \Phi(u, d, P, z_1, r)$$

Iterating this inequality k -times, $k \geq 1$, yields

$$\Phi(u, d, P, z_1, \theta_0^k r) \leq 2^{-k} \Phi(u, d, P, z_1, r).$$

It is well known that this implies that there exists $\alpha \in (0, 1)$ such that for any $0 < \tau < r \leq r_1$, it holds

$$\Phi(u, d, P, z_1, \tau) \leq \left(\frac{\tau}{r}\right)^\alpha \Phi(u, d, P, z_1, r) \quad (3.49)$$

holds for any $z_1 \in P_{\frac{r_0}{2}}(z_0)$ and $0 < r \leq \frac{r_0}{2}$.

Now we proceed with the Riesz potential estimates of $(u, \nabla d)$ between Morrey spaces as follows. First, let's recall the notion of Morrey spaces on $\mathbb{R}^2 \times \mathbb{R}$, equipped with the parabolic metric δ :

$$\delta((x, t), (y, s)) = \max\{|x - y|, \sqrt{|t - s|}\}, \quad \forall (x, t), (y, s) \in \mathbb{R}^2 \times \mathbb{R}.$$

For any open set $U \subset \mathbb{R}^{2+1}$, $1 \leq p < +\infty$, and $0 \leq \lambda \leq 4$, define the Morrey Space $M^{p, \lambda}(U)$ by

$$M^{p, \lambda}(U) := \left\{ v \in L_{\text{loc}}^p(U) : \|v\|_{M^{p, \lambda}(U)}^p \equiv \sup_{z \in U, r > 0} r^{\lambda-4} \int_{P_r(z) \cap U} |v|^p < \infty \right\}. \quad (3.50)$$

It follows from (3.49) that for some $\alpha \in (0, 1)$,

$$u, \nabla d \in M^{4, 4(1-\alpha)}\left(P_{\frac{r_0}{2}}(z_0)\right), \quad (\nabla u, \nabla^2 d, P) \in M^{2, 4-2\alpha}\left(P_{\frac{r_0}{2}}(z_0)\right). \quad (3.51)$$

Write the equation (1.8)₃ as

$$\partial_t d - \frac{1}{|\lambda_1|} \Delta d = f, \quad \text{with } f := \left(-u \cdot \nabla d + \Omega \hat{d} - \frac{\lambda_2}{\lambda_1} A \hat{d} + \frac{1}{|\lambda_1|} |\nabla d|^2 d + \frac{\lambda_2}{\lambda_1} (\hat{d}^T A \hat{d}) d \right). \quad (3.52)$$

By (3.51), we see that

$$f \in M^{2, 2(2-\alpha)}\left(P_{\frac{r_0}{2}}(z_0)\right).$$

As in [24] and [13], let $\eta \in C_0^\infty(\mathbb{R}^{2+1})$ be a cut-off function of $P_{\frac{r_0}{2}}(z_0)$: $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $P_{\frac{r_0}{2}}(z_0)$, and $|\partial_t \eta| + |\nabla^2 \eta| \leq C r_0^{-2}$. Set $w = \eta^2 d$. Then we have

$$\partial_t w - \frac{1}{|\lambda_1|} \Delta w = F, \quad F := \eta^2 f + (\partial_t \eta^2 - \frac{1}{|\lambda_1|} \Delta \eta^2)(d - d_{z_0, \frac{r_0}{2}}) - \frac{2}{|\lambda_1|} \nabla \eta^2 \cdot \nabla d, \quad (3.53)$$

where $d_{z_0, \frac{r_0}{2}}$ is the average of d over $P_{\frac{r_0}{2}}(z_0)$. It is easy to check that $F \in M^{2, 2(2-\alpha)}(\mathbb{R}^{2+1})$ and satisfies the estimate

$$\|F\|_{M^{2, 2(2-\alpha)}(\mathbb{R}^{2+1})} \leq C \left[\Phi(u, d, P, z_0, r_0) + \|f\|_{M^{2, 2(2-\alpha)}(P_{\frac{r_0}{2}}(z_0))} \right] \leq C \epsilon_0. \quad (3.54)$$

Let $\Gamma(x, t)$ denote the fundamental solution of the heat operator $(\partial_t - \frac{1}{|\lambda_1|} \Delta)$ on \mathbb{R}^2 . Then by the Duhamel formula for (3.53) and the estimate (see also [13] lemma 3.1):

$$|\nabla \Gamma|(x, t) \lesssim \frac{1}{\delta^3((x, t), (0, 0))}, \quad \forall (x, t) \neq (0, 0),$$

we have

$$|\nabla w(x, t)| \leq \int_0^t \int_{\mathbb{R}^2} |\nabla \Gamma(x - y, t - s)| |F(y, s)| \leq C \int_{\mathbb{R}^3} \frac{|F(y, s)|}{\delta^3((x, t), (y, s))} := C\mathcal{I}_1(|F|)(x, t), \quad (3.55)$$

where \mathcal{I}_β is the Riesz potential of order β on \mathbb{R}^3 ($\beta \in [0, 4]$), defined by

$$\mathcal{I}_\beta(g) = \int_{\mathbb{R}^3} \frac{|g(y, s)|}{\delta((x, t), (y, s))^{4-\beta}}, \quad \forall g \in L^p(\mathbb{R}^3). \quad (3.56)$$

Applying the Riesz potential estimates (see [13] Theorem 3.1), we conclude that $\nabla w \in M^{\frac{2(2-\alpha)}{1-\alpha}, 2(2-\alpha)}(\mathbb{R}^3)$ and

$$\|\nabla w\|_{M^{\frac{2(2-\alpha)}{1-\alpha}, 2(2-\alpha)}(\mathbb{R}^3)} \lesssim \|F\|_{M^{2, 2(2-\alpha)}(\mathbb{R}^3)} \leq C\epsilon_0. \quad (3.57)$$

Choosing $\alpha \uparrow 1$ and using $\lim_{\alpha \uparrow 1} \frac{2(2-\alpha)}{1-\alpha} = +\infty$, we can conclude that for any $1 < q < \infty$, $\nabla w \in L^q(P_{r_0}(z_0))$ and

$$\|\nabla w\|_{L^q(P_{r_0}(z_0))} \leq C(q, r_0)\epsilon_0. \quad (3.58)$$

Since $(d - w)$ solves

$$\partial_t(d - w) - \frac{1}{|\lambda_1|} \Delta(d - w) = 0 \text{ in } P_{\frac{r_0}{2}}(z_0),$$

it follows from the standard estimate on the heat equation that for any $1 < q < +\infty$, $\nabla d \in L^q(P_{\frac{r_0}{4}}(z_0))$ and

$$\|\nabla d\|_{L^q(P_{\frac{r_0}{4}}(z_0))} \leq C(q, r_0)\epsilon_0. \quad (3.59)$$

Now we proceed with the estimation of u . Let $v : \mathbb{R}^2 \times [0, +\infty) \rightarrow \mathbb{R}^2$ solve the Stokes equation:

$$\begin{cases} \partial_t v - \frac{\mu_4}{2} \Delta v + \nabla Q = -\nabla \cdot [\eta^2(\nabla d \odot \nabla d + u \otimes u)] + \nabla \cdot [\eta^2(\sigma^L(u, d) - \mu_4 A)] & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \nabla \cdot v = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v(\cdot, 0) = 0 & \text{in } \mathbb{R}^2. \end{cases} \quad (3.60)$$

By using the Oseen kernel (see Leray [16]), an estimate for v , similar to (3.55), can be given by

$$|v(x, t)| \leq C \int_0^t \int_{\mathbb{R}^2} \frac{|X(y, s)|}{\delta((x, t), (y, s))} \leq C\mathcal{I}_1(|X|)(x, t), \quad (x, t) \in \mathbb{R}^2 \times (0, +\infty), \quad (3.61)$$

where $X = -\eta^2(\nabla d \odot \nabla d + u \otimes u) + \eta^2(\sigma^L(u, d) - \mu_4 A)$. As above, we can check that $X \in M^{2, 2(2-\alpha)}(\mathbb{R}^3)$ and

$$\|X\|_{M^{2, 2(2-\alpha)}(\mathbb{R}^3)} \leq C \left[\| |u| + |\nabla d| \|_{M^{4, 4-2\alpha}(P_{\frac{r_0}{2}}(z_0))}^2 + \| |\nabla u| + |\nabla^2 d| \|_{M^{2, 4-2\alpha}(P_{\frac{r_0}{2}}(z_0))} \right] \leq C\epsilon_0.$$

Hence, by [13] Theorem 3.1, we have that $v \in M^{\frac{2(2-\alpha)}{1-\alpha}, 2(2-\alpha)}(\mathbb{R}^3)$, and

$$\|v\|_{M^{\frac{2(2-\alpha)}{1-\alpha}, 2(2-\alpha)}(\mathbb{R}^3)} \leq C \|X\|_{M^{2, 2(2-\alpha)}(\mathbb{R}^3)} \leq C\epsilon_0. \quad (3.62)$$

By sending $\alpha \uparrow 1$, (3.62) implies that for any $1 < q < +\infty$, $v \in L^q(P_{r_0}(z_0))$ and

$$\|v\|_{L^q(P_{r_0}(z_0))} \leq C(q, r_0)\epsilon_0. \quad (3.63)$$

Since $(u - v)$ satisfies the linear homogeneous Stokes equation in $P_{\frac{r_0}{2}}(z_0)$:

$$\partial_t(u - v) - \frac{\mu_4}{2} \Delta(u - v) + \nabla(P - Q) = 0, \quad \nabla \cdot (u - v) = 0 \quad \text{in } P_{\frac{r_0}{2}}(z_0).$$

It is well-known that $(u - v) \in L^\infty(P_{\frac{r_0}{4}}(z_0))$. Therefore we conclude that for any $1 < q < +\infty$, $u \in L^q(P_{\frac{r_0}{4}}(z_0))$, and

$$\|u\|_{L^q(P_{\frac{r_0}{4}}(z_0))} \leq C(q, r_0)\epsilon_0. \quad (3.64)$$

The estimate (3.48) follows from (3.59) and (3.64). This completes the proof. \square

Now we utilize the integrability estimate (3.48) of $(u, \nabla d)$ to prove the smoothness of (u, d) .⁴ The argument is based on local inequalities of higher order energy of $(u, \nabla d)$.

Lemma 3.4. *Assume that the conditions (1.6), (1.7), and (1.15) hold. For any $0 < T \leq +\infty$ and a bounded domain $O \subset \mathbb{R}^2$, there exists $\epsilon_0 > 0$ such that $(u, P, d) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(O \times [0, T]) \times L^2(O \times [0, T]) \times (L_t^\infty H_x^1 \cap L_t^2 H^2(O \times [0, T], \mathbb{S}^2))$ is a suitable weak solution of (1.8), and satisfies, for $z_0 = (x_0, t_0) \in O \times (0, T)$ and $P_{r_0}(z_0) \subset O \times (0, T)$,*

$$\Phi(u, d, P, z_0, r_0) \leq \epsilon_0, \quad (3.65)$$

then $(\nabla^l u, \nabla^{l+1} d) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(P_{\frac{1+2^{-(l+1)}}{2}r_0}(z_0))$ for any $l \geq 0$, and the following estimate holds

$$\begin{aligned} & \sup_{t_0 - \frac{((1+2^{-(l+1)}r_0)^2)}{4} \leq t \leq t_0} \int_{B_{\frac{1+2^{-(l+1)}}{2}r_0}(x_0)} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2) \\ & \quad + \int_{P_{\frac{1+2^{-(l+1)}}{2}r_0}(z_0)} (|\nabla^{l+1} u|^2 + |\nabla^{l+2} d|^2 + |\nabla^l P|^2) \\ & \leq C(l)\epsilon_0. \end{aligned} \quad (3.66)$$

Proof. For simplicity, assume $z_0 = (0, 0)$ and $r_0 = 2$. We will prove (3.66) by an induction on $l \geq 0$.

(i) $l = 0$: (3.66) follows from the local energy inequality (2.14), similar to that given by lemma 3.1.

(ii) $l \geq 1$: Suppose that (3.66) holds for $l \leq k-1$. We want to show (3.66) also holds for $l = k$. From the hypothesis of induction, we have that for all $0 \leq l \leq k-1$,

$$\begin{aligned} & \sup_{-(1+2^{-(l+1)})^2 \leq t \leq 0} \int_{B_{1+2^{-(l+1)}}} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2) \\ & \quad + \int_{P_{1+2^{-(l+1)}}} (|\nabla^{l+1} u|^2 + |\nabla^{l+2} d|^2 + |\nabla^l P|^2) \leq C(l)\epsilon_0. \end{aligned} \quad (3.67)$$

Hence by the Ladyzhenskaya inequality (3.21) we have

$$\int_{P_{1+2^{-(l+1)}}} (|\nabla^l u|^4 + |\nabla^{l+1} d|^4) \leq C(l)\epsilon_0, \quad \forall 0 \leq l \leq k-1. \quad (3.68)$$

By lemma 3.3, we also have

$$\|u\|_{L^q(P_{\frac{3}{2}})} + \|\nabla d\|_{L^q(P_{\frac{3}{2}})} \leq C(q)\epsilon_0, \quad \forall 1 < q < +\infty. \quad (3.69)$$

Take k -th order spatial derivative ∇^k of the equation (1.8)₁, we have⁵

$$\partial_t(\nabla^k u) + \nabla^k \nabla \cdot (u \otimes u) + \nabla^k \nabla P = -\nabla^k \nabla \cdot (\nabla d \odot \nabla d) + \nabla^k \nabla \cdot (\sigma^L(u, d)). \quad (3.70)$$

Let $\eta \in C_0^\infty(B_2)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{1+2^{-(k+1)}}, \quad \eta = 0 \text{ outside } B_{1+2^{-k}}, \quad |\nabla \eta| + |\nabla^2 \eta| \leq 2^{k+4}.$$

Multiplying (3.70) by $\nabla^k u \eta^2$ and integrating over B_2 , we obtain⁶

$$\begin{aligned} \frac{d}{dt} \int_{B_2} \frac{1}{2} |\nabla^k u|^2 \eta^2 &= \int_{B_2} \nabla^k (u \otimes u) : \nabla(\nabla^k u \eta^2) + \int_{B_2} \nabla^k P \cdot \nabla^k u \cdot \nabla(\eta^2) \\ &\quad + \int_{B_2} \nabla^k (\nabla d \odot \nabla d) : \nabla(\nabla^k u \eta^2) - \int_{B_2} \nabla^k (\sigma^L(u, d)) : \nabla(\nabla^k u \eta^2) \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (3.71)$$

We estimate I_1, I_2, I_3 as follows. Applying Hölder's inequality and the following interpolation inequality:

$$\int_{B_2} |f|^4 \eta^4 \lesssim \left(\int_{B_2} |f|^2 \eta^2 \right) \left(\int_{B_2} |\nabla f|^2 \eta^2 + \int_{B_2} |f|^2 |\nabla \eta|^2 \right), \quad \forall f \in H^1(\mathbb{R}^2), \quad (3.72)$$

⁴In fact, we only need to use $(u, \nabla d) \in L^8$ in the proof for the cases $k = 1, 2$.

⁵Strictly speaking we need to take the finite quotient $D_h^i \nabla^{k-1}$ of (1.8)₁ and then taking limit as h tends to zero.

⁶Strictly speaking, we need to multiply the equation by $D_h^i \nabla^{k-1} u \eta^2$.

we have

$$\begin{aligned}
|I_1| &\lesssim \int_{B_2} [|u| |\nabla^k u| + \sum_{j=1}^{k-1} |\nabla^j u| |\nabla^{k-j} u|] (|\nabla^{k+1} u| \eta^2 + |\nabla^k u| |\nabla \eta|) \\
&\leq \left(\delta + C \int_{B_2} |\nabla^k u|^2 \eta^2 \right) \int_{B_2} |\nabla^{k+1} u|^2 \eta^2 + C \sum_{j=0}^{k-1} \int_{\text{spt} \eta} |\nabla^j u|^4 \\
&\quad + C \left(\int_{\text{spt} \eta} |\nabla^k u|^2 + \int_{\text{spt} \eta} |\nabla^k u|^2 \int_{B_2} |\nabla^k u|^2 \eta^2 \right)
\end{aligned} \tag{3.73}$$

where $\delta > 0$ is a small constant to be chosen later. For I_2 and I_3 , we have

$$\begin{aligned}
|I_2| &\lesssim \int_{B_2} |\nabla^{k-1} P| (|\nabla^{k+1} u| |\eta| |\nabla \eta| + |\nabla^k u| |\nabla^2(\eta^2)|) \\
&\leq \delta \int_{B_2} |\nabla^{k+1} u|^2 \eta^2 + C \int_{\text{spt} \eta} (|\nabla^{k-1} P|^2 + |\nabla^k u|^2).
\end{aligned} \tag{3.74}$$

$$\begin{aligned}
|I_3| &\lesssim \int_{B_2} (|\nabla d| |\nabla^{k+1} d| + \sum_{j=1}^{k-1} |\nabla^{j+1} d| |\nabla^{k+1-j} d|) (|\nabla^{k+1} u| \eta^2 + |\nabla^k u| |\nabla(\eta^2)|) \\
&\leq \left(\delta + C \int_{B_2} |\nabla^{k+1} d|^2 \eta^2 \right) \int_{B_2} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2) \eta^2 + C \int_{\text{spt} \eta} (|\nabla u|^2 + \sum_{j=1}^k |\nabla^j d|^4) \\
&\quad + C \left(\int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) + \int_{\text{spt} \eta} |\nabla^{k+1} d|^2 \int_{B_2} |\nabla^{k+1} d|^2 \eta^2 \right).
\end{aligned} \tag{3.75}$$

For I_4 , we need to proceed as follows. Set

$$\sigma_k^L(u, d) = \mu_1(\hat{d} \otimes \hat{d} : \nabla^k A) \hat{d} \otimes \hat{d} + \mu_2 \nabla^k \hat{N} \otimes \hat{d} + \mu_3 \hat{d} \otimes \nabla^k \hat{N} + \mu_4 \nabla^k A + \mu_5(\nabla^k A \cdot \hat{d}) \otimes \hat{d} + \mu_6 \hat{d} \otimes (\nabla^k A \cdot \hat{d}),$$

and

$$\omega_k^L(u, d) := \nabla^k(\sigma^L(u, d)) - \sigma_k^L(u, d).$$

Then we have

$$\begin{aligned}
- \int_{B_2} \nabla^k(\sigma^L(u, d)) : \nabla(\nabla^k u \eta^2) &= - \int_{B_2} \sigma_k^L(u, d) : (\nabla^{k+1} u) \eta^2 - \int_{B_2} \sigma_k^L(u, d) : \nabla^k u \otimes \nabla(\eta^2) \\
&\quad - \int_{B_2} \omega_k^L(u, d) : (\nabla^{k+1} u \eta^2 + \nabla^k u \otimes \nabla(\eta^2)) \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{3.76}$$

To estimate J_1, J_2, J_3 , we take ∇^k of the equation (1.8)₃ to get⁷

$$\nabla^k N + \frac{\lambda_2}{\lambda_1} \nabla^k(A \hat{d}) = \frac{1}{|\lambda_1|} (\Delta \nabla^k d + \nabla^k(|\nabla d|^2 d)) + \frac{\lambda_2}{\lambda_1} \nabla^k((\hat{d}^\Gamma A \hat{d}) d). \tag{3.77}$$

Since $|d| = 1$, we have that $|\nabla d|^2 = -\langle \Delta d, d \rangle$. Denote by $\#$ the multi-linear map with constant coefficients. It is well known (see [1]) that for any $l \geq 0$,

$$\|\nabla^l(d\#d)\|_{L^2} \lesssim \|d\|_{L^\infty} \|\nabla^l d\|_{L^2} \lesssim \|\nabla^l d\|_{L^2},$$

and

$$\|\nabla^l(d\#d\#d)\|_{L^2} \lesssim \|d\|_{L^\infty} \|\nabla^l(d\#d)\|_{L^2} + \|d\#d\|_{L^\infty} \|\nabla^l d\|_{L^2} \lesssim \|\nabla^l d\|_{L^2}.$$

Therefore by (3.67) and (3.68) we have that for any $0 \leq l \leq k-1$,

$$\nabla(d\#d), \nabla(d\#d\#d) \in \left(L_t^\infty H_x^l \cap L_t^2 H_x^{l+1} \cap L^4 \right) \left(P_{1+2^{-(l+1)}} \right).$$

and

$$\left\| |\nabla(d\#d)| + |\nabla(d\#d\#d)| \right\|_{L^4 \cap L_t^\infty H_x^l \cap L_t^2 H_x^{l+1} (P_{1+2^{-(l+1)}})} \leq C(l) \epsilon_0. \tag{3.78}$$

⁷Strictly speaking, we need to take $D_h^i \nabla^{k-1}$ of the equation.

The estimate (3.78) also holds for $d\#d\#d\#d$. Applying the equation (3.77) we have

$$\begin{aligned}
|J_2| &\lesssim \int_{B_2} (|\nabla^{k+1}u| + |\nabla^k N|) |\nabla^k u| \eta |\nabla \eta| \\
&\lesssim \int_{B_2} \left[|\nabla^{k+1}u| + |\nabla^{k+2}d| + \sum_{j=1}^k |\nabla^j u| |\nabla^{k-j+1}d| \right] |\nabla^k u| \eta |\nabla \eta| \\
&\quad + \left[\sum_{l=0}^{k-1} |\nabla^{l+2}d| |\nabla^{k-l}(d\#d)| \right] |\nabla^k u| \eta |\nabla \eta| \\
&\quad + \left[\sum_{l=0}^{k-1} |\nabla^{k-l}(d\#d\#d)| |\nabla^{l+1}u| \right] |\nabla^k u| \eta |\nabla \eta| := J_{21} + J_{22} + J_{23}.
\end{aligned}$$

Direct calculations imply

$$\begin{aligned}
J_{21} &\leq \left[\delta + C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \eta^2 \right] \int_{B_2} (|\nabla^{k+1}u|^2 + |\nabla^{k+2}d|^2) \eta^2 \\
&\quad + C \int_{\text{spt}\eta} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \eta^2 + C \sum_{i=1}^k \int_{\text{spt}\eta} (|\nabla^i d|^4 + |\nabla^i u|^2), \\
J_{22} &\leq \int_{B_2} \left[|\nabla^k d| |\nabla d|^2 + |\nabla^{k+1}d| |\nabla d| + \sum_{l=0}^{k-2} |\nabla^{l+2}d| |\nabla^{k-l}(d\#d)| \right] |\nabla^k u| \eta |\nabla \eta| \\
&\leq C \int_{B_2} |\nabla^k u|^2 \eta^2 \int_{B_2} |\nabla^{k+1}u|^2 \eta^2 + C \int_{\text{spt}\eta} |\nabla^k u|^2 \int_{B_2} |\nabla^k u|^2 \eta^2 \\
&\quad + C \int_{\text{spt}\eta} \left[|\nabla d|^8 + |\nabla^k d|^2 + |\nabla^{k+1}d|^2 + \sum_{i=1}^k |\nabla^i d|^4 \right],
\end{aligned}$$

and

$$\begin{aligned}
J_{23} &\leq \int_{B_2} \left[|\nabla^k u| |\nabla d| + \sum_{l=1}^{k-1} |\nabla^{k+1-l}(d\#d\#d)| |\nabla^l u| \right] |\nabla^k u| \eta |\nabla \eta| \\
&\lesssim C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \eta^2 \int_{B_2} (|\nabla^{k+1}u|^2 + |\nabla^{k+2}d|^2) \eta^2 \\
&\quad + C \int_{\text{spt}\eta} (|\nabla d|^8 + |\nabla^k u|^2 + |\nabla^{k+1}d|^2) + C \sum_{j=0}^{k-1} \int_{\text{spt}\eta} (|\nabla^j u|^4 + |\nabla^{j+1}d|^4) \\
&\quad + C \int_{\text{spt}\eta} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \eta^2.
\end{aligned}$$

By the definition of $\omega_k^L(u, d)$ and the equation (1.8)₃, we have

$$\begin{aligned}
|\omega_k^L(u, d)| &\lesssim \sum_{l=1}^k |\nabla^l u| (|\nabla^{k+1-l}(d\#d)| + |\nabla^{k+1-l}(d\#d\#d)|) + \sum_{l=0}^{k-1} |\nabla^l \hat{N}| |\nabla^{k-l}d| \\
&\lesssim \sum_{l=1}^k |\nabla^l u| (|\nabla^{k+1-l}(d\#d)| + |\nabla^{k+1-l}(d\#d\#d)|) + |\nabla d| |\nabla^{k+1}d| + \sum_{l=2}^k |\nabla^l d|^2 \\
&\quad + \sum_{l=0}^{k-1} \left[|\nabla^l(Ad)| + |\nabla^l(\nabla^2 d\#d\#d)| + |\nabla^l(A\#d\#d\#d)| \right] |\nabla^{k-l}d| \\
&\lesssim (|\nabla^k u| + |\nabla^{k+1}d|) (|\nabla d| + |\nabla(d\#d)| + |\nabla(d\#d\#d)|) \\
&\quad + \sum_{l=1}^k (|\nabla^{l-1}u|^2 + |\nabla^l d|^2 + |\nabla^l(d\#d)|^2 + |\nabla^l(d\#d\#d)|^2). \tag{3.79}
\end{aligned}$$

Hence we can estimate

$$\begin{aligned}
|J_3| &\lesssim \int_{B_2} |\omega_k^L(u, d)| (|\nabla^{k+1}u|\eta^2 + |\nabla^k u|\eta|\nabla\eta|) \\
&\leq \left[\delta + C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2)\eta^2 \right] \int_{B_2} (|\nabla^{k+1}u|^2 + |\nabla^{k+2}d|^2)\eta^2 + C \int_{\text{spt}\eta} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \\
&\quad + C \sum_{l=1}^k \int_{\text{spt}\eta} (|\nabla^{l-1}u|^4 + |\nabla^l d|^4 + |\nabla^l(d\#d)|^4 + |\nabla^l(d\#d\#d)|^4) \\
&\quad + C \int_{\text{spt}\eta} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1}d|^2)\eta^2. \tag{3.80}
\end{aligned}$$

The most difficult term to handle is J_1 , since the integrands involve terms consisting of the highest order factors $\nabla^{k+1}u$ and $\nabla^{k+2}d$. Here we need to apply (1.8) to cancel some of those terms and employ the condition (1.15) to argue that other terms are non-positive. For this, we proceed as follows (similar to lemma 3.2).

$$\begin{aligned}
J_1 &= - \int_{B_2} \left[\mu_1(\hat{d} \otimes \hat{d} : \nabla^k A)\hat{d} \otimes \hat{d} + \mu_2 \nabla^k \hat{N} \otimes \hat{d} + \mu_3 \hat{d} \otimes \nabla^k \hat{N} \right. \\
&\quad \left. + \mu_4 \nabla^k A + \mu_5 (\nabla^k A \hat{d}) \otimes \hat{d} + \mu_6 \hat{d} \otimes (\nabla^k A \hat{d}) \right] : \left[\nabla^k A \eta^2 + \nabla^k \Omega \eta^2 \right] \\
&= - \int_{B_2} \mu_1 |\hat{d}^T \nabla^k A \hat{d}|^2 \eta^2 + \mu_4 |\nabla^k A|^2 \eta^2 + (\mu_5 + \mu_6) |\nabla^k A \hat{d}|^2 \eta^2 \\
&\quad + \lambda_1 \nabla^k \hat{N} \cdot (\nabla^k \Omega \hat{d}) \eta^2 - \lambda_2 \nabla^k \hat{N} \cdot (\nabla^k A \hat{d}) \eta^2 + \lambda_2 (\nabla^k A \hat{d}) (\nabla^k \Omega \hat{d}) \eta^2. \tag{3.81}
\end{aligned}$$

Applying the equation (3.77), we obtain

$$\begin{aligned}
\lambda_1 \nabla^k \hat{N} \cdot (\nabla^k \Omega \hat{d}) + \lambda_2 (\nabla^k A \hat{d}) (\nabla^k \Omega \hat{d}) &= - \langle \nabla^k \Omega \hat{d}, \Delta \nabla^k \hat{d} \rangle - \langle \nabla^k (|\nabla d|^2 \hat{d}), \nabla^k \Omega \hat{d} \rangle \\
&\quad + \lambda_2 \langle \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}), \nabla^k \Omega \hat{d} \rangle - \lambda_2 \langle \nabla^k (A \hat{d}) - \nabla^k A \hat{d}, \nabla^k \Omega \hat{d} \rangle, \tag{3.82}
\end{aligned}$$

and

$$\begin{aligned}
-\lambda_2 \nabla^k \hat{N} (\nabla^k A \hat{d}) &= \frac{\lambda_2}{\lambda_1} \langle \Delta \nabla^k \hat{d}, \nabla^k A \hat{d} \rangle - \frac{\lambda_2^2}{\lambda_1} |\hat{d}^T \nabla^k A \hat{d}|^2 + \frac{\lambda_2^2}{\lambda_1} |\nabla^k A \hat{d}|^2 \\
&\quad + \frac{\lambda_2^2}{\lambda_1} \langle \nabla^k (A \hat{d}) - \nabla^k A \hat{d}, \nabla^k A \hat{d} \rangle + \frac{\lambda_2}{\lambda_1} \langle \nabla^k (|\nabla d|^2 \hat{d}), \nabla^k A \hat{d} \rangle \\
&\quad + \frac{\lambda_2^2}{\lambda_1} \langle (\hat{d}^T \nabla^k A) \hat{d} - \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}), \nabla^k A \hat{d} \rangle. \tag{3.83}
\end{aligned}$$

Putting (3.82) and (3.83) into (3.81) yields

$$\begin{aligned}
J_1 &= - \int_{B_2} \left[(\mu_1 - \frac{\lambda_2^2}{\lambda_1}) |\hat{d}^T \nabla^k A \hat{d}|^2 + \mu_4 |\nabla^k A|^2 + (\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}) |\nabla^k A \hat{d}|^2 + \langle \frac{\lambda_2}{\lambda_1} \nabla^k A \hat{d} - \nabla^k \Omega \hat{d}, \Delta \nabla^k \hat{d} \rangle \right] \eta^2 \\
&\quad - \int_{B_2} \left[\langle \lambda_2 \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}) - \nabla^k (|\nabla d|^2 \hat{d}) - \lambda_2 (\nabla^k (A \hat{d}) - \nabla^k A \hat{d}), \nabla^k \Omega \hat{d} \rangle + \frac{\lambda_2}{\lambda_1} \langle \nabla^k (|\nabla d|^2 \hat{d}), \nabla^k A \hat{d} \rangle \right] \\
&\quad + \frac{\lambda_2^2}{\lambda_1} \langle \nabla^k (A \hat{d}) - \nabla^k A \hat{d}, \nabla^k A \hat{d} \rangle + \frac{\lambda_2^2}{\lambda_1} \langle (\hat{d}^T \nabla^k A) \hat{d} - \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}), \nabla^k A \hat{d} \rangle \eta^2. \tag{3.84}
\end{aligned}$$

Multiplying the equation (3.77) by $\Delta \nabla^k d \eta^2$ and integrating over B_2 , we have⁸

$$\begin{aligned}
\frac{d}{dt} \int_{B_2} \frac{1}{2} |\nabla^{k+1}d|^2 \eta^2 + \frac{1}{|\lambda_1|} \int_{B_2} |\Delta \nabla^k d|^2 \eta^2 &= \int_{B_2} \langle \frac{\lambda_2}{\lambda_1} \nabla^k A \hat{d} - \nabla^k \Omega \hat{d}, \Delta \nabla^k \hat{d} \rangle \eta^2 \\
&\quad + \int_{B_2} \langle \nabla^k (u \cdot \nabla d) + (\nabla^k \Omega \hat{d} - \nabla^k (\Omega \hat{d})) + \frac{\lambda_2}{\lambda_1} (\nabla^k (A \hat{d}) - \nabla^k A \hat{d}) + \frac{1}{\lambda_1} \nabla^k (|\nabla d|^2 d), \Delta \nabla^k d \rangle \eta^2
\end{aligned}$$

⁸Strictly speaking, we need to multiply the equation by $\Delta D_h^i \nabla^{k-1} d \eta^2$.

$$- \int_{B_2} \partial_t \nabla^k d \cdot \nabla \nabla^k d \cdot \nabla \eta^2 - \frac{\lambda_2}{\lambda_1} \int_{B_2} \left\langle \nabla^k ((\hat{d}^T A \hat{d}) d), \Delta \nabla^k d \right\rangle \eta^2. \quad (3.85)$$

Adding (3.71) and (3.85), using (1.15), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2 + \int_{B_2} \left(2\mu_4 |\nabla^k A|^2 + \frac{2}{|\lambda_1|} |\nabla^{k+2} d|^2 \right) \eta^2 \leq I_1 + I_2 + I_3 + J_2 + J_3 \\ & + \int_{B_2} \left\langle \nabla^k (u \cdot \nabla d) + (\nabla^k \Omega \hat{d} - \nabla^k (\Omega \hat{d})) + \frac{\lambda_2}{\lambda_1} (\nabla^k (A \hat{d}) - \nabla^k A \hat{d}), \Delta \nabla^k d \right\rangle \eta^2 \\ & - \int_{B_2} \partial_t \nabla^k d \cdot \nabla \nabla^k d \cdot \nabla \eta^2 + \int_{B_2} \left\langle \frac{1}{\lambda_1} \nabla^k (|\nabla d|^2 d) - \frac{\lambda_2}{\lambda_1} \nabla^k ((\hat{d}^T A \hat{d}) d), \Delta \nabla^k d \right\rangle \eta^2 \\ & - \int_{B_2} \left[\left\langle \lambda_2 \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}) - \nabla^k (|\nabla d|^2 \hat{d}) - \lambda_2 (\nabla^k (A \hat{d}) - \nabla^k A \hat{d}), \nabla^k \Omega \hat{d} \right\rangle + \frac{\lambda_2}{\lambda_1} \left\langle \nabla^k (|\nabla d|^2 \hat{d}), \nabla^k A \hat{d} \right\rangle \right. \\ & \left. + \frac{\lambda_2^2}{\lambda_1} \left\langle \nabla^k A \hat{d} - \nabla^k (A \hat{d}), \nabla^k A \hat{d} \right\rangle - \frac{\lambda_2^2}{\lambda_1} \left\langle \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}) - (\hat{d}^T \nabla^k A \hat{d}) \hat{d}, \nabla^k A \hat{d} \right\rangle \right] \eta^2 \\ := & I_1 + I_2 + I_3 + J_2 + J_3 + K_1 + K_2 + K_3 + K_4. \end{aligned} \quad (3.86)$$

The terms K_1, K_2, K_3, K_4 can be estimated as follows.

$$\begin{aligned} |K_1| & \lesssim \int_{B_2} [|\nabla d| |\nabla^k u| + |u| |\nabla^{k+1} d| + \sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2)] |\nabla^{k+2} d| \eta^2 \\ & \leq [\delta + C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2] \int_{B_2} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2) \eta^2 + C \sum_{l=0}^{k-1} \int_{\text{spt} \eta} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2) \\ & + C \int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2, \end{aligned} \quad (3.87)$$

For K_2 , by the equation (3.77) and the fact $|\nabla d|^2 = -\langle d, \Delta d \rangle$ we have

$$|\partial_t \nabla^k d| \lesssim |\nabla^k (u \cdot \nabla d) + |\nabla^k (\Omega \hat{d})| + |\nabla^k (A \hat{d})| + |\nabla^{k+2} d| + |\nabla^k ((\hat{d}^T A \hat{d}) d)| + |\nabla^k (\langle d, \Delta d \rangle d)|,$$

so that

$$\begin{aligned} |K_2| & \lesssim \int_{B_2} [|\nabla^{k+2} d| + |\nabla^{k+1} u| + |\nabla^{k+1} d| (|u| + |\nabla d|) + |\nabla^k u| (|\nabla d| + |\nabla (d \# d \# d)|)] \\ & + \sum_{l=2}^k (|\nabla^{l-1} u|^2 + |\nabla^l d|^2 + |\nabla^l (d \# d)|^2 + |\nabla^l (d \# d \# d)|^2) |\nabla^{k+1} d| \eta |\nabla \eta| \\ & \leq [\delta + C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2] \int_{B_2} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2) \eta^2 \\ & + \sum_{l=1}^k \int_{\text{spt} \eta} (|\nabla^{l-1} u|^4 + |\nabla^l d|^4 + |\nabla^l (d \# d)|^4 + |\nabla^l (d \# d \# d)|^4) \\ & + C \int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) + C \int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2. \end{aligned} \quad (3.88)$$

For K_4 , we first estimate the terms inside the integrand. Since

$$\langle \nabla^k (\hat{d}^T A \hat{d}) \hat{d}, \nabla^k \Omega \hat{d} \rangle = 0,$$

it follows

$$\begin{aligned} & |\langle \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}), \nabla^k \Omega \hat{d} \rangle| = |\langle \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}) - \nabla^k (\hat{d}^T A \hat{d}) \hat{d}, \nabla^k \Omega \hat{d} \rangle| \\ & \lesssim (|\nabla^k u| |\nabla d| + \sum_{l=0}^{k-2} |\nabla^{l+1} u| |\nabla^{k-l} (d \# d \# d)|) |\nabla^{k+1} u| \\ & \lesssim [|\nabla^k u| |\nabla d| + \sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^{l+1} (d \# d \# d)|^2)] |\nabla^{k+1} u|. \end{aligned}$$

We also have

$$\begin{aligned}
& |\langle \nabla^k (|\nabla d|^2 d), \nabla \Omega \hat{d} \rangle| = |\langle \nabla^k (|\nabla d|^2 d) - \nabla^k (|\nabla d|^2) \hat{d}, \nabla^k \Omega \hat{d} \rangle| \\
& \lesssim [|\nabla^k d| |\nabla d|^2 + \sum_{l=0}^{k-2} \nabla^l (|\nabla d|^2) |\nabla^{k-l} d|] |\nabla^{k+1} u| \\
& \lesssim [|\nabla^k d| |\nabla d|^2 + \sum_{l=2}^k (|\nabla^l d|^2 + |\nabla^l (d \# d)|^2)] |\nabla^{k+1} u|, \\
& |\langle \nabla^k (A \hat{d}) - \nabla^k A \hat{d}, \nabla^k \Omega \hat{d} \rangle| \lesssim [|\nabla^k u| |\nabla d| + \sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2)] |\nabla^{k+1} u|, \\
& |\langle \nabla^k (|\nabla d|^2 d), \nabla^k A \hat{d} \rangle| \lesssim [|\nabla^{k+1} d| |\nabla d| + |\nabla^k d| |\nabla d|^2 + \sum_{l=2}^k (|\nabla^l d|^2 + |\nabla^l (d \# d)|^2)] |\nabla^{k+1} u|,
\end{aligned}$$

and

$$\begin{aligned}
& |\langle \nabla^k A \hat{d} - \nabla^k (A \hat{d}), \nabla^k A \hat{d} \rangle| \lesssim [|\nabla^k u| |\nabla d| + \sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2)] |\nabla^{k+1} u|, \\
& |\langle \nabla^k ((\hat{d}^T A \hat{d}) \hat{d}) - (\hat{d}^T \nabla^k A \hat{d}) \hat{d}, \nabla^k A \hat{d} \rangle| \\
& \lesssim [|\nabla^k u| |\nabla (d \# d \# d)| + \sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^{l+1} (d \# d \# d)|^2)] |\nabla^{k+1} u|.
\end{aligned}$$

Putting all these estimates together, we would have

$$\begin{aligned}
|K_4| & \lesssim \int_{B_2} [(|\nabla^k u| + |\nabla^{k+1} d|) |\nabla d| + |\nabla^k d| |\nabla d|^2 + |\nabla^k u| |\nabla (d \# d \# d)|] |\nabla^{k+1} u| \eta^2 \\
& + \int_{B_2} [\sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2 + |\nabla^{l+1} (d \# d)|^2 + |\nabla^{l+1} (d \# d \# d)|^2)] |\nabla^{k+1} u| \eta^2 \\
& \leq [\delta + C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2] \int_{B_2} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2) \eta^2 \\
& + C \sum_{l=0}^{k-1} \int_{\text{spt} \eta} (|\nabla^l u|^4 + |\nabla^{l+1} d|^4 + |\nabla^{l+1} (d \# d)|^4 + |\nabla^{l+1} (d \# d \# d)|^4) \\
& + C \int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2 + C \int_{\text{spt} \eta} |\nabla d|^8. \tag{3.89}
\end{aligned}$$

To estimate K_3 , we first estimate both terms inside the integrand.

$$|\langle \nabla^k (|\nabla d|^2 d), \Delta \nabla^k d \rangle| \lesssim [|\nabla^k d| |\nabla d|^2 + |\nabla^{k+1} d| |\nabla d| + \sum_{l=2}^k (|\nabla^l d|^2 + |\nabla^l (d \# d)|^2)] |\nabla^{k+2} d|.$$

Since $|d| = 1$, it follows $\langle d, \Delta d \rangle = -|\nabla d|^2$ and

$$\langle d, \Delta \nabla^k d \rangle = -\nabla^k (|\nabla d|^2) - \sum_{j=0}^{k-1} \langle \Delta \nabla^j d, \nabla^{k-j} d \rangle.$$

Therefore we have

$$\begin{aligned}
& \left| \langle \nabla^k ((\hat{d}^T A \hat{d}) d), \Delta \nabla^k d \rangle \right| \\
& = \left| (\hat{d}^T \nabla^k A \hat{d}) \langle d, \Delta \nabla^k d \rangle + \left\langle \left(\sum_{j=0}^{k-1} \nabla^j A \nabla^{k-j} (d \# d) \right) d, \Delta \nabla^k d \right\rangle + \left\langle \left(\sum_{j=0}^{k-1} \nabla^j (\hat{d}^T A \hat{d}) \nabla^{k-j} \right) d, \Delta \nabla^k d \right\rangle \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| -(\hat{d}^T \nabla^k A \hat{d}) [\nabla^k (|\nabla d|^2) + \sum_{j=0}^{k-1} \langle \Delta \nabla^j d, \nabla^{k-j} d \rangle] + \left\langle \sum_{j=0}^{k-1} \nabla^j A \nabla^{k-j} (d \# d), \Delta \nabla^k d \right\rangle \right. \\
&\quad \left. + \left\langle \left(\sum_{j=0}^{k-1} \nabla^j (\hat{d}^T A \hat{d}) \nabla^{k-j} d, \Delta \nabla^k d \right) \right| \right. \\
&\lesssim (|\nabla^{k+1} d| |\nabla d| + |\nabla^k d| |\nabla d|^2) |\nabla^{k+1} u| + |\nabla^k u| (|\nabla (d \# d)| + |\nabla (d \# d \# d)|) |\nabla^{k+2} d| \\
&\quad + \left(\sum_{l=2}^k |\nabla^l d|^2 \right) |\nabla^{k+1} u| + \left[\sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^{l+1} (d \# d)|^2 + |\nabla^{l+1} (d \# d \# d)|^2) \right] |\nabla^{k+2} d|.
\end{aligned}$$

Substituting these two estimates into K_3 , we would obtain

$$\begin{aligned}
&|K_3| \\
&\lesssim \int_{B_2} [|\nabla^{k+1} d| |\nabla d| + |\nabla^k d| |\nabla d|^2 + |\nabla^k u| (|\nabla (d \# d)| + |\nabla (d \# d \# d)|)] (|\nabla^{k+1} u| + |\nabla^{k+2} d|) \eta^2 \\
&\quad + \int_{B_2} \left[\sum_{l=1}^{k-1} (|\nabla^l u|^2 + |\nabla^l d|^2 + |\nabla^{l+1} (d \# d)|^2 + |\nabla^{l+1} (d \# d \# d)|^2) \right] (|\nabla^{k+1} u| + |\nabla^{k+2} d|) \eta^2 \\
&\leq [\delta + \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2] \int_{B_2} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2) \eta^2 \\
&\quad + C \int_{\text{spt} \eta} [|\nabla d|^4 + |\nabla d|^8 + \sum_{l=1}^{k-1} (|\nabla^l u|^4 + |\nabla^{l+1} d|^4 + |\nabla^{l+1} (d \# d)|^4 + |\nabla^{l+1} (d \# d \# d)|^4)] \\
&\quad + C \int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2. \tag{3.90}
\end{aligned}$$

Finally, by substituting all these estimates on I_i 's, J_i 's, K_i 's, and L_i 's into (3.86) we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2 + \int_{B_2} \left(\mu_4 |\nabla^{k+1} u|^2 + \frac{2}{|\lambda_1|} |\nabla^{k+2} d|^2 \right) \eta^2 \\
&\leq \left[\delta + C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2 \right] \int_{B_2} (|\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2) \eta^2 \\
&\quad + C \int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2 \\
&\quad + C \int_{\text{spt} \eta} [|\nabla d|^8 + |\nabla d|^4 + \sum_{l=0}^{k-1} (|\nabla^l u|^4 + |\nabla^{l+1} d|^4 + |\nabla^{l+1} (d \# d)|^4 + |\nabla^{l+1} (d \# d \# d)|^4)] \\
&\quad + C \int_{\text{spt} \eta} \left[\sum_{l=1}^k (|\nabla^l u|^2 + |\nabla^{l+1} d|^2) + |\nabla^{l-1} P|^2 \right]. \tag{3.91}
\end{aligned}$$

By Fubini's theorem, we can choose $t_* \in [-4, -\frac{9}{4}]$ such that

$$\int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(t_*) \leq 8 \int_{P_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2 \leq 8\epsilon_0.$$

For a large constant $C_1 > 8$ to be chosen later, define $T_* \in [t_*, 0]$ by

$$T_* = \sup_{t_* \leq t \leq 0} \left\{ \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(s) < C_1 \epsilon_0, \forall t_* \leq s \leq t \right\}.$$

Claim 1. If $\epsilon_0 > 0$ is sufficiently small, then $T_* = 0$.

First, by continuity we know that $T_* > t_*$. Suppose that $T_* < 0$. Then we have

$$\int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(s) < C_1 \epsilon_0 \quad \forall t_* \leq s < T_*; \quad \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(T_*) = C_1 \epsilon_0. \tag{3.92}$$

By choosing sufficiently small $\epsilon_0 > 0$ and $\delta > 0$, we may assume

$$\left[\delta + C \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(t) \right] \leq \delta + CC_1 \epsilon_0 \leq \frac{1}{2} \min \left\{ \mu_4, \frac{2}{|\lambda_1|} \right\}, \quad \forall t_* \leq t \leq T_*.$$

Set

$$\phi(t) := \int_{t_*}^t \int_{\text{spt} \eta} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2)(s) ds, \quad \forall t_* \leq t \leq T_*.$$

Then by integrating (3.91) over $t \in [t_*, T_*]$ and applying Gronwall's inequality, we have

$$\begin{aligned} & \int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(T_*) + \frac{1}{2} \int_{t_*}^{T_*} \int_{B_2} \left(\mu_4 |\nabla^{k+1} u|^2 + \frac{2}{|\lambda_1|} |\nabla^{k+2} d|^2 \right) \eta^2 \\ & \leq e^{C\phi(T_*)} \left[\int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(t_*) + C \sum_{l=1}^k \int_{P_{1+2^{-(l+1)}}} (|\nabla^l u|^2 + |\nabla^{l+1} d|^2 + |\nabla^{l-1} P|^2) \right] \\ & + C \int_{P_{\frac{3}{2}}} [|u|^8 + |\nabla d|^8] + C \sum_{l=0}^{k-1} \int_{P_{1+2^{-(l+1)}}} [|\nabla^l u|^4 + |\nabla^{l+1} d|^4 + |\nabla^{l+1}(d \# d)|^4 + |\nabla^{l+1}(d \# d \# d)|^4] \\ & \leq e^{C\epsilon_0} (8\epsilon_0 + C\epsilon_0), \end{aligned} \tag{3.93}$$

where we have used both (3.67), (3.68), and (3.69) in the last step.

It is easy to see that we can choose $C_1 > (C+8)e^{C\epsilon_0}$ so that $e^{C\epsilon_0}(8\epsilon_0 + C\epsilon_0) < C_1\epsilon_0$. Hence (3.93) implies

$$\int_{B_2} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \eta^2(T_*) < C_1 \epsilon_0,$$

which contradicts the definition of T_* . Thus the claim holds true.

Since $\nabla^k P$ satisfies

$$\Delta \nabla^k P = -(\nabla \cdot)^2 [\nabla^k (u \otimes u + \nabla d \odot \nabla d - \sigma^L(u, d))], \tag{3.94}$$

the elliptic theory and (3.93) (with $T_* = 0$) then yield

$$\begin{aligned} \int_{P_{1+2^{-(k+2)}}} |\nabla^k P|^2 & \lesssim \sup_{-(1+2^{-(k+1)})^2 \leq t \leq 0} \int_{B_{1+2^{-(k+1)}}} (|\nabla^k u|^2 + |\nabla^{k+1} d|^2) \\ & + \int_{P_{1+2^{-(k+1)}}} (|P|^2 + |\nabla^{k+1} u|^2 + |\nabla^{k+2} d|^2) \leq C\epsilon_0. \end{aligned} \tag{3.95}$$

This yields that the conclusion holds for $l = k$. Thus the proof is complete. \square

Completion of Proof of Theorem 1.3: It is readily seen that by the Sobolev embedding theorem, lemma 3.4 implies that $(\nabla^k u, \nabla^{k+1} d) \in L^\infty(P_{\frac{3r_0}{4}}(z_0))$ for any $k \geq 1$. This, combined with the theory of linear Stokes' equation and heat equation, would imply the smoothness of (u, d) in $P_{\frac{r_0}{2}}(z_0)$. \square

4. EXISTENCE OF GLOBAL WEAK SOLUTIONS OF ERICKSEN-LESLIE'S SYSTEM (1.8)

In this section, by utilizing both the local energy inequality (2.14) for suitable weak solutions of (1.8) and the regularity Theorem 1.3 for suitable weak solutions to (1.8), we will establish the existence of global weak solutions to (1.8) and (1.14) that enjoy the regularity described as in Theorem 1.4. The argument is similar to [23] Section 5.

First, we recall the following version of Ladyzhenskaya's inequality (see [30] lemma 3.1 for the proof).

Lemma 4.1. *There exists $C_0 > 0$ such that for any $T > 0$, if $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^2 \times [0, T])$, then for $0 < R \leq +\infty$, it holds*

$$\int_{\mathbb{R}^2 \times [0, T]} |u|^4 \leq C_0 \left(\sup_{(x,t) \in \mathbb{R}^2 \times [0, T]} \int_{B_R(x)} |u|^2(\cdot, t) \right) \left[\int_{\mathbb{R}^2 \times [0, T]} |\nabla u|^2 + \frac{1}{R^2} \int_{\mathbb{R}^2 \times [0, T]} |u|^2 \right]. \tag{4.1}$$

Similar to [23] lemma 5.2, we can estimate the life span of smooth solutions to (1.8) in term of Sobolev profiles of smooth initial data.

Lemma 4.2. *Assume (1.6), (1.7), and (1.15) hold. There exist $\epsilon_1 > 0$ and $\theta_1 > 0$ depending on (u_0, d_0) such that if $(u_0, d_0) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times \mathbb{S}^2) \cap_{k \geq 0} (H^k(\mathbb{R}^2, \mathbb{R}^2) \times H_{\epsilon_0}^{k+1}(\mathbb{R}^2, \mathbb{S}^2))$ satisfies*

$$\sup_{x \in \mathbb{R}^2} \int_{B_{2R_0}(x)} (|u_0|^2 + |\nabla d_0|^2) \leq \epsilon_1^2 \quad (4.2)$$

for some $R_0 > 0$. Then there exist $T_0 \geq \theta_1 R_0^2$ and a unique solution $(u, d) \in C^\infty(\mathbb{R}^2 \times [0, T_0], \mathbb{R}^2 \times \mathbb{S}^2)$ of (1.8) and (1.14) in \mathbb{R}^2 , satisfying

$$\sup_{(x,t) \in \mathbb{R}^2 \times [0, T_0]} \int_{B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(t) \leq 2\epsilon_1^2. \quad (4.3)$$

Proof. By the theorem of Wang-Zhang-Zhang [33] on the local existence of smooth solutions, there exist $T_0 > 0$ and a unique smooth solution $(u, d) \in C^\infty(\mathbb{R}^2 \times [0, T_0], \mathbb{R}^2 \times \mathbb{S}^2)$ to (1.8) and (1.14). Let $0 < t_0 \leq T_0$ be the maximal time such that

$$\sup_{x \in \mathbb{R}^2} \int_{B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(t) \leq 2\epsilon_1^2, \quad 0 \leq t \leq t_0. \quad (4.4)$$

Hence we have

$$\sup_{x \in \mathbb{R}^2} \int_{B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(t_0) = 2\epsilon_1^2. \quad (4.5)$$

Without loss of generality, we assume $t_0 \leq R_0^2$. Set

$$E(t) = \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2)(t), \quad E_0 = \int_{\mathbb{R}^2} (|u_0|^2 + |\nabla d_0|^2).$$

Then by lemma 2.1 we have that for any $0 < t \leq t_0$,

$$E(t) + \int_{\mathbb{R}^2 \times [0, t]} \left(\mu_4 |\nabla u|^2 + \frac{2}{|\lambda_1|} |\Delta d + |\nabla d|^2 d|^2 \right) \leq E_0. \quad (4.6)$$

By lemma 4.1, we have that for all $0 \leq t \leq t_0$,

$$\begin{aligned} \int_{\mathbb{R}^2 \times [0, t]} |\nabla d|^4 &\leq C_0 \left(\sup_{(x,s) \in \mathbb{R}^2 \times [0, t]} \int_{B_{R_0}(x)} |\nabla d|^2(s) \right) \left[\int_{\mathbb{R}^2 \times [0, t]} |\Delta d|^2 + \frac{1}{R_0^2} \int_{\mathbb{R}^2 \times [0, t]} |\nabla d|^2 \right] \\ &\leq C_0 \mathcal{E}_{R_0}^2(t) \left[\int_{\mathbb{R}^2 \times [0, t]} |\Delta d|^2 + \frac{tE_0}{R_0^2} \right] \end{aligned} \quad (4.7)$$

where

$$\mathcal{E}_{R_0}^1(t) = \sup_{(x,s) \in \mathbb{R}^2 \times [0, t]} \int_{B_{R_0}(x)} |u|^2(s), \quad \mathcal{E}_{R_0}^2(t) = \sup_{(x,s) \in \mathbb{R}^2 \times [0, t]} \int_{B_{R_0}(x)} |\nabla d|^2(s), \quad \mathcal{E}_{R_0}(t) = \sum_{i=1}^2 \mathcal{E}_{R_0}^i(t).$$

By (4.4), we have $\mathcal{E}_{R_0}(t) \leq 2\epsilon_1^2$, $\forall 0 \leq t \leq t_0$ so that

$$\int_{\mathbb{R}^2 \times [0, t_0]} |\nabla d|^4 \leq C_0 \epsilon_1^2 \left[\int_{\mathbb{R}^2 \times [0, t_0]} |\Delta d|^2 + \frac{t_0 E_0}{R_0^2} \right]. \quad (4.8)$$

Hence we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times [0, t_0]} |\Delta d|^2 &= \int_{\mathbb{R}^2 \times [0, t_0]} (|\Delta d + |\nabla d|^2 d|^2 + |\nabla d|^4) \leq \frac{|\lambda_1|}{2} E_0 + \int_{\mathbb{R}^2 \times [0, t_0]} |\nabla d|^4 \\ &\leq C_0 \epsilon_1^2 \int_{\mathbb{R}^2 \times [0, t_0]} |\Delta d|^2 + \left(\frac{C_0 t_0 \epsilon_1^2}{R_0^2} + \frac{|\lambda_1|}{2} \right) E_0. \end{aligned}$$

Choosing $0 < \epsilon_1^2 \leq \frac{1}{2C_0}$, we would have

$$\int_{\mathbb{R}^2 \times [0, t_0]} |\nabla^2 d|^2 = \int_{\mathbb{R}^2 \times [0, t_0]} |\Delta d|^2 \leq \left(|\lambda_1| + \frac{C_0 \epsilon_1^2 t_0}{R_0^2} \right) E_0 \leq C_1 E_0. \quad (4.9)$$

This, combined with (4.8), also yields

$$\int_{\mathbb{R}^2 \times [0, t_0]} |\nabla d|^4 \leq C_1 \epsilon_1^2 E_0. \quad (4.10)$$

We can also estimate

$$\begin{aligned} \int_{\mathbb{R}^2 \times [0, t_0]} |u|^4 &\leq C_0 \mathcal{E}_{R_0}^1(t_0) \left[\int_{\mathbb{R}^2 \times [0, t_0]} |\nabla u|^2 + \frac{1}{R_0^2} \int_{\mathbb{R}^2 \times [0, t_0]} |u|^2 \right] \\ &\leq C_0 \mathcal{E}_{R_0}^1(t_0) \left(\frac{E_0}{\mu_4} + \frac{t_0 E_0}{R_0^2} \right) \leq C_1 \epsilon_1^2 E_0. \end{aligned} \quad (4.11)$$

Now we need to estimate $\mathcal{E}_{R_0}(t)$. Before we do it, we need to recall the following global L^2 -estimate of P :

$$\int_0^{t_0} \int_{\mathbb{R}^2} |P|^2 \lesssim \int_0^{t_0} \int_{\mathbb{R}^2} (|u|^4 + |\nabla d|^4 + |\sigma^L(u, d)|^2) \lesssim \int_0^{t_0} \int_{\mathbb{R}^2} (|u|^4 + |\nabla d|^4 + |\nabla u|^2 + |\nabla^2 d|^2). \quad (4.12)$$

For any $x \in \mathbb{R}^2$, let $\eta \in C_0^\infty(B_{2R_0}(x))$ be a cut-off function of $B_{R_0}(x)$:

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{R_0}(x), \quad \eta \equiv 0 \text{ outside } B_{2R_0}(x), \quad |\nabla \eta| \leq 4R_0^{-1}.$$

Then, by applying lemma 2.2 with this η and the estimates (4.6), (4.10), (4.9), (4.11), and (4.12), we have

$$\begin{aligned} &\sup_{0 \leq t \leq t_0} \int_{B_{R_0}(x)} (|u|^2 + |\nabla d|^2) + c_0 \int_0^{t_0} \int_{B_{R_0}(x)} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) - \mathcal{E}_{2R_0}(0) \\ &\lesssim \int_{\mathbb{R}^2 \times [0, t_0]} \left[|u|^3 + |u| |\nabla u| + |u|^2 |\nabla d| + |u| |\nabla^2 d| + |u| |P| + |\nabla d| |\nabla^2 d| + |u| |\nabla d|^2 + |\nabla d| |\nabla u| \right] |\nabla \eta|^2 \\ &\lesssim \frac{1}{R_0} \left\| |u| + |\nabla d| \right\|_{L^2(\mathbb{R}^2 \times [0, t_0])} \left[\left\| |\nabla u| + |\nabla^2 d| + |P| \right\|_{L^2(\mathbb{R}^2 \times [0, t_0])} + \left\| |u| + |\nabla d| \right\|_{L^4(\mathbb{R}^2 \times [0, t_0])}^2 \right] \\ &\leq C \left(\frac{t_0}{R_0^2} \right)^{\frac{1}{2}} E_0^{\frac{1}{2}}. \end{aligned} \quad (4.13)$$

Thus, by taking supremum over $x \in \mathbb{R}^2$ we obtain

$$\begin{aligned} 2\epsilon_1^2 &= \sup_{x \in \mathbb{R}^2, 0 \leq t \leq t_0} \int_{B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(t) \\ &\leq \mathcal{E}_{2R_0}(0) + C \left(\frac{t_0}{R_0^2} \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \leq \epsilon_1^2 + C_0 \left(\frac{t_0}{R_0^2} \right)^{\frac{1}{2}} E_0^{\frac{1}{2}}. \end{aligned} \quad (4.14)$$

This implies

$$t_0 \geq \frac{\epsilon_1^2}{C_0^2 E_0} R_0^2 = \theta_1 R_0^2, \quad \text{with } \theta_1 \equiv \frac{\epsilon_1^2}{C_0^2 E_0}.$$

Set $T_0 = t_0$. Then we have that $T_0 \geq \theta_1 R_0^2$ and (4.3) holds. This completes the proof. \square

Before proving Theorem 1.4, we need the following density property of Sobolev maps (see [28] for the proof).

Lemma 4.3. *For $n = 2$ and any given map $f \in H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$, there exist $\{f_k\} \subset C^\infty(\mathbb{R}^2, \mathbb{S}^2) \cap_{l \geq 1} H_{e_0}^l(\mathbb{R}^2, \mathbb{S}^2)$ such that*

$$\lim_{k \rightarrow \infty} \|f_k - f\|_{H^1(\mathbb{R}^2)} = 0.$$

Proof of Theorem 1.4:

Since $u_0 \in \mathbf{H}$, there exists $u_0^k \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, with $\nabla \cdot u_0^k = 0$, such that

$$\lim_{k \rightarrow \infty} \|u_0^k - u_0\|_{L^2(\mathbb{R}^2)} = 0.$$

Since $d_0 \in H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$, lemma 4.3 implies that there exist $\{d_0^k\} \subset C^\infty(\mathbb{R}^2, \mathbb{S}^2) \cap_{l \geq 1} H_{e_0}^l(\mathbb{R}^2, \mathbb{S}^2)$ such that

$$\lim_{k \rightarrow \infty} \|d_0^k - d_0\|_{H^1(\mathbb{R}^2)} = 0.$$

By the absolute continuity of $\int (|u_0|^2 + |\nabla d_0|^2)$, there exists $R_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^2} \int_{B_{2R_0}(x)} (|u_0|^2 + |\nabla d_0|^2) \leq \frac{\epsilon_1^2}{2}, \quad (4.15)$$

where $\epsilon_1 > 0$ is given by lemma 4.2. By the strong convergence of $(u_0^k, \nabla d_0^k)$ to $(u_0, \nabla d_0)$ in $L^2(\mathbb{R}^2)$, we have that

$$\sup_{x \in \mathbb{R}^2} \int_{B_{2R_0}(x)} (|u_0^k|^2 + |\nabla d_0^k|^2) \leq \epsilon_1^2, \quad \forall k \gg 1. \quad (4.16)$$

For simplicity, we assume (4.16) holds for all $k \geq 1$. By lemma 4.2, there exist $\theta_0 = \theta_0(\epsilon_1, E_0) \in (0, 1)$ and $T_0^k \geq \theta_0 R_0^2$ such that there exist solutions $(u^k, d^k) \in C^\infty(\mathbb{R}^2 \times [0, T_0^k], \mathbb{R}^2 \times \mathbb{S}^2)$ to (1.8) and (1.14) with the initial condition:

$$(u^k, d^k)|_{t=0} = (u_0^k, d_0^k), \quad (4.17)$$

that satisfies

$$\sup_{(x,t) \in \mathbb{R}^2 \times [0, T_0^k]} \int_{B_{R_0}(x)} (|u^k|^2 + |\nabla d^k|^2)(t) \leq 2\epsilon_1^2, \quad \forall k \geq 1. \quad (4.18)$$

By lemma 2.1, we have that for all $k \geq 1$,

$$\begin{aligned} & \sup_{0 \leq t \leq T_0^k} \int_{\mathbb{R}^2} (|u^k|^2 + |\nabla d^k|^2)(t) + \int_{\mathbb{R}^2 \times [0, T_0^k]} (\mu_4 |\nabla u^k|^2 + \frac{2}{|\lambda_1|} |\nabla^2 d^k|^2) \\ & \leq \int_{\mathbb{R}^2} (|u_0^k|^2 + |\nabla d_0^k|^2) \leq CE_0. \end{aligned} \quad (4.19)$$

Combining (4.18) and (4.19) with lemma 4.2, we conclude that

$$\int_{\mathbb{R}^2 \times [0, T_0^k]} (|u^k|^4 + |\nabla d^k|^4) \leq C\epsilon_1^2 E_0, \quad (4.20)$$

$$\int_{\mathbb{R}^2 \times [0, T_0^k]} (|\partial_t d^k|^2 + |\nabla^2 d^k|^2 + |P^k|^2) \leq CE_0, \quad (4.21)$$

and

$$\sup_{x \in \mathbb{R}^2} \int_0^{T_0^k} \int_{B_{R_0}(x)} (|\nabla u^k|^2 + |\nabla^2 d^k|^2 + |P^k|^2) \leq C\epsilon_1^2. \quad (4.22)$$

Furthermore, (1.8) implies that for any $\phi \in \mathbf{J}$,

$$\langle \partial_t u^k, \phi \rangle = - \int_{\mathbb{R}^2} \nabla u^k \cdot \nabla \phi + \int_{\mathbb{R}^2} (u^k \otimes u^k + \nabla d^k \odot \nabla d^k - \sigma^L(u^k, d^k)) : \nabla \phi,$$

where $\langle \cdot, \cdot \rangle$ denotes the pair between $H^{-1}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$, we conclude that $\partial_t u^k \in L^2([0, T_0^k], H^{-1}(\mathbb{R}^2))$ and

$$\|\partial_t u^k\|_{L^2([0, T_0^k], H^{-1}(\mathbb{R}^2))} \leq CE_0. \quad (4.23)$$

It follows from (4.20) and (4.22) that

$$\Phi(u^k, d^k, P^k, (x, t), R_0) \leq C\epsilon_1^2, \quad \forall x \in \mathbb{R}^2, R_0^2 \leq t \leq T_0^k.$$

Hence, by Theorem 1.3, we conclude that for any $\delta > 0$,

$$\|(u^k, d^k)\|_{C^l(\mathbb{R}^2 \times [\delta, T_0^k])} \leq C(l, \epsilon_1, E_0, \delta), \quad \forall l \geq 1. \quad (4.24)$$

After passing to a subsequence, we may assume that there exist $T_0 \geq \theta_0 R_0^2$, $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^2 \times [0, T_0], \mathbb{R}^2)$, $d \in L_t^\infty H_{e_0}^1 \cap L_t^2 H_{e_0}^2(\mathbb{R}^2 \times [0, T_0], \mathbb{S}^2)$, and $P \in L^2(\mathbb{R}^2 \times [0, T_0])$ such that

$$u^k \rightharpoonup u \text{ in } L_t^2 H_x^1(\mathbb{R}^2 \times [0, T_0], \mathbb{R}^2), \quad d^k \rightharpoonup d \text{ in } L_t^2 H_{e_0}^2(\mathbb{R}^2 \times [0, T_0], \mathbb{S}^2), \quad P^k \rightharpoonup P \text{ in } L^2(\mathbb{R}^2 \times [0, T_0]).$$

It follows from (4.21) and (4.23) that we can apply Aubin-Lions' lemma to conclude that

$$u^k \rightarrow u, \quad \nabla d^k \rightarrow \nabla d \text{ in } L_{\text{loc}}^4(\mathbb{R}^2 \times [0, T_0]).$$

By (4.24), we may assume that for any $0 < \delta < T_0, 0 < R < +\infty$ and $l \geq 1$,

$$\lim_{k \rightarrow \infty} \|(u^k, d^k) - (u, d)\|_{C^l(B_R \times [\delta, T_0])} = 0.$$

It is clear that $u \in C^\infty(\mathbb{R}^2 \times (0, T_0], \mathbb{R}^2) \cap (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{R}^2 \times [0, T_0], \mathbb{R}^2)$, $d \in C^\infty(\mathbb{R}^2 \times (0, T_0], \mathbb{S}^2) \cap (L_t^2 H_{e_0}^1 \cap L_t^2 H_{e_0}^2)(\mathbb{R}^2 \times [0, T_0], \mathbb{S}^2)$ solves (1.8) in $\mathbb{R}^2 \times (0, T_0]$. It follows from (4.23) and (4.21) that we can assume

$$(u, \nabla d)(t) \rightharpoonup (u_0, \nabla d_0) \text{ in } L^2(\mathbb{R}^2), \text{ as } t \downarrow 0.$$

In particular, by the lower semicontinuity we have that

$$E(0) \leq \liminf_{t \downarrow 0} E(t).$$

On the other hand, (4.19) implies

$$E(0) \geq \limsup_{t \downarrow 0} E(t).$$

This implies that $(u, \nabla d)(t) \rightarrow (u_0, \nabla d_0)$ in $L^2(\mathbb{R}^2)$ and hence (u, d) satisfies the initial condition (1.14).

Let $T_1 \in [T_0, +\infty)$ be the first finite singular time of (u, d) , i.e.,

$$(u, d) \in C^\infty(\mathbb{R}^2 \times (0, T_1), \mathbb{R}^2 \times \mathbb{S}^2) \text{ but } (u, d) \notin C^\infty(\mathbb{R}^2 \times (0, T_1], \mathbb{R}^2 \times \mathbb{S}^2).$$

Then we must have

$$\limsup_{t \uparrow T_1} \max_{x \in \mathbb{R}^2} \int_{B_R(x)} (|u|^2 + |\nabla d|^2)(t) \geq \epsilon_1^2, \quad \forall R > 0. \quad (4.25)$$

Now we look for an extension of this weak solution beyond T_1 . To do it, we define the new initial data at $t = T_1$.

Claim 2. $(u, d) \in C^0([0, T_1], L^2(\mathbb{R}^2))$.

This follows easily from (4.23) and (4.21). By Claim 2, we can define

$$(u(T_1), d(T_1)) = \lim_{t \uparrow T_1} (u(t), d(t)) \text{ in } L^2(\mathbb{R}^2).$$

By lemma 2.1 we have that $\nabla d \in L^\infty([0, T_1], L^2(\mathbb{R}^2))$ so that $\nabla d(t) \rightarrow \nabla d(T_1)$ in $L^2(\mathbb{R}^2)$. In particular, $u(T_1) \in \mathbf{H}$ and $d(T_1) \in H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$.

Use $(u(T_1), d(T_1))$ as an initial data to obtain a continuation of (u, d) beyond T_1 as a weak solution of (1.8) and (1.14), we will show that this procedure will cease in finite steps and afterwards we will have constructed a global weak solution. In fact, at any such singular time there is at least a loss of energy amount of ϵ_1^2 . By (4.25), there exist $t_i \uparrow T_1$ and $x_0 \in \mathbb{R}^2$ such that

$$\limsup_{t_i \uparrow T_1} \int_{B_R(x_0)} (|u|^2 + |\nabla d|^2)(t_i) \geq \epsilon_1^2 \text{ for all } R > 0.$$

This implies

$$\begin{aligned} \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2)(T_1) &= \lim_{R \downarrow 0} \int_{\mathbb{R}^2 \setminus B_R(x_0)} (|u|^2 + |\nabla d|^2)(T_1) \leq \lim_{R \downarrow 0} \liminf_{t_i \uparrow T_1} \int_{\mathbb{R}^2 \setminus B_R(x_0)} (|u|^2 + |\nabla d|^2)(t_i) \\ &\leq \lim_{R \downarrow 0} \left[\liminf_{t_i \uparrow T_1} \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2)(t_i) - \limsup_{t_i \uparrow T_1} \int_{B_R(x_0)} (|u|^2 + |\nabla d|^2)(t_i) \right] \leq E_0 - \epsilon_1^2. \end{aligned}$$

Hence the number of finite singular times must be bounded by $L = \left\lceil \frac{E_0}{\epsilon_1^2} \right\rceil$. If $0 < T_L < +\infty$ is the largest finite singular time, then we can use $(u(T_L), d(T_L))$ as the initial data to construct a weak solution (u, d) to (1.8) and (1.14) in $\mathbb{R}^2 \times [T_L, +\infty)$. Thus (i) of Theorem 1.4 is established. It is also clear that (iii) of Theorem 1.4 holds for the solution constructed.

Now, we perform a blow-up analysis at each finite singular time. It follows from (4.25) that there exist $0 < t_0 < T_1, t_m \uparrow T_1, r_m \downarrow 0$ such that

$$\epsilon_1^2 = \sup_{x \in \mathbb{R}^2, t_0 \leq t \leq t_m} \int_{B_{r_m}(x)} (|u|^2 + |\nabla d|^2)(t). \quad (4.26)$$

By lemma 4.2, there exist θ_0 , depending only on ϵ_1 and E_0 , and $x_m \in \mathbb{R}^2$ such that

$$\int_{B_{2r_m}(x_m)} (|u|^2 + |\nabla d|^2)(t_m - \theta_0 r_m^2) \geq \frac{1}{2} \sup_{x \in \mathbb{R}^2} \int_{B_{2r_m}(x)} (|u|^2 + |\nabla d|^2)(t_m - \theta_0 r_m^2) \geq \frac{\epsilon_1^2}{4}. \quad (4.27)$$

By lemma 4.2 and (4.26), we have

$$\int_{\mathbb{R}^2 \times [t_0, t_m]} (|u|^4 + |\nabla d|^4) \leq C\epsilon_1^2. \quad (4.28)$$

Define the blow-up sequence $(u_m, d_m, P_m) : \mathbb{R}^2 \times [\frac{t_0 - t_m}{r_m^2}, 0] \rightarrow \mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R}$ by

$$u_m(x, t) = r_m u(x_m + r_m x, t_m + r_m^2 t), \quad d_m(x, t) = d(x_m + r_m x, t_m + r_m^2 t), \quad P_m = r_m^2 P(x_m + r_m x, t_m + r_m^2 t).$$

Then (u_m, d_m, P_m) solves (1.8) in $\mathbb{R}^2 \times [\frac{t_0 - t_m}{r_m^2}, 0]$. Moreover,

$$\begin{aligned} \int_{B_2(0)} (|u_m|^2 + |\nabla d_m|^2)(-\theta_0) &\geq \frac{\epsilon_1^2}{2}, \\ \int_{B_1(x)} (|u_m|^2 + |\nabla d_m|^2)(t) &\leq \epsilon_1^2, \quad \forall x \in \mathbb{R}^2, \quad \frac{t_0 - t_m}{r_m^2} \leq t \leq 0, \\ \int_{\mathbb{R}^2 \times [\frac{t_0 - t_m}{r_m^2}, 0]} (|u_m|^4 + |\nabla d_m|^4) &\leq C\epsilon_1^2, \end{aligned}$$

and

$$\int_{P_2(z)} (|\nabla u_m|^2 + |\nabla^2 d_m|^2 + |P_m|^2) \leq C\epsilon_1^2, \quad \forall z = (x, t) \in \mathbb{R}^2 \times [\frac{t_0 - t_m}{r_m^2}, 0].$$

It is easy to see $\frac{t_0 - t_m}{r_m^2} \rightarrow -\infty$. Hence, by Theorem 1.3, we can assume that there exists a smooth solution $(u_\infty, d_\infty) : \mathbb{R}^2 \times (-\infty, 0] \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ to (1.8) such that for any $l \geq 1$,

$$(u_m, d_m) \rightarrow (u_\infty, d_\infty) \text{ in } C_{\text{loc}}^l(\mathbb{R}^2 \times [-\infty, 0]).$$

Claim 3. $u_\infty \equiv 0$.

In fact, since $u \in L^4(\mathbb{R}^2 \times [0, T_1])$, we have

$$\int_{P_R} |u_\infty|^4 = \lim_{m \rightarrow \infty} \int_{P_R} |u_m|^4 = \lim_{m \rightarrow \infty} \int_{B_{Rr_m}(x_m)} \int_{t_m - R^2 r_m^2}^{t_m} |u|^4 = 0.$$

Claim 4. $d_\infty \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ is a nontrivial harmonic map with finite energy.

Since $(\Delta d + |\nabla d|^2 d) \in L^2(\mathbb{R}^2 \times [0, T_1])$, we have, for any compact $K \subset \mathbb{R}^2$,

$$\begin{aligned} \int_{-2\theta_0}^0 \int_K |\Delta d_\infty + |\nabla d_\infty|^2 d_\infty|^2 &\leq \liminf_m \int_{-2\theta_0}^0 \int_{\Omega_m} |\Delta d_m + |\nabla d_m|^2 d_m|^2 \\ &= \lim_m \int_{t_m - 2\theta_0 r_m^2}^{t_m} \int_{\mathbb{R}^2} |\Delta d + |\nabla d|^2 d|^2 = 0. \end{aligned}$$

By the equation (1.8)₃ and $u_\infty \equiv 0$, this implies $\partial_t d_\infty \equiv 0$ on $\mathbb{R}^2 \times [-2\theta_0, 0]$. Hence $d_\infty(t) \equiv d_\infty \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ is a harmonic map. Since

$$\int_{B_2} |\nabla d_\infty|^2 = \lim_m \int_{B_2} (|u_m|^2 + |\nabla d_m|^2)(-\theta_0) \geq \frac{\epsilon_1^2}{4},$$

d_∞ is a non-constant map. By the lower semicontinuity, we have that for any ball $B_R \subset \mathbb{R}^2$,

$$\int_{B_R} |\nabla d_\infty|^2 \leq \liminf_m \int_{B_R} |\nabla d_m|^2(-\theta_0) = \liminf_m \int_{B_{r_m R}(x_m)} |\nabla d|^2(t_m - \theta_0 r_m^2) \leq E_0,$$

and hence d_∞ has finite energy. It is well-known ([29] [30]) that d_∞ can be lifted to be a non-constant harmonic map from \mathbb{S}^2 to \mathbb{S}^2 . In particular, d_∞ has a non zero degree and

$$\int_{\mathbb{R}^2} |\nabla d_\infty|^2 \geq 8\pi |\deg(d_\infty)| \geq 8\pi.$$

It follows from the above argument that for any $r > 0$,

$$\limsup_{t \uparrow T_1} \max_{x \in \mathbb{R}^2} \int_{B_r(x)} (|u|^2 + |\nabla d|^2)(t) \geq \int_{\mathbb{R}^2} (|u_\infty|^2 + |\nabla d_\infty|^2) \geq 8\pi.$$

To show (iv). By lemma 2.1, there exists $t_k \uparrow +\infty$ such that for $(u_k, d_k) = (u(t_k), d(t_k))$,

$$\int_{\mathbb{R}^2} (|u_k|^2 + |\nabla d_k|^2) \leq E_0, \quad \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} (|\nabla u_k|^2 + |\Delta d_k + |\nabla d_k|^2 d_k|^2) = 0.$$

It is easy to see that $u_k \rightarrow 0$ in $H^1(\mathbb{R}^2)$, and $\{d_k\} \subset H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$ is a bounded sequence of approximate harmonic maps, with tension fields $\tau(d_k) = \Delta d_k + |\nabla d_k|^2 d_k$ converging to zero in $L^2(\mathbb{R}^2)$. By the energy identity result by Qing [27] and Lin-Wang [25], we can conclude that there exist a harmonic map $d_\infty \in C^\infty \cap H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$, and finitely many points $\{x_i\}_{i=1}^l, \{m_i\}_{i=1}^l \subset \mathbb{N}$, such that

$$|\nabla d_k|^2 dx \rightharpoonup |\nabla d_\infty|^2 dx + \sum_{i=1}^l 8\pi m_i \delta_{x_i}$$

as convergence of Radon measures. This yields (iv).

To show (v) under the condition $\int_{\mathbb{R}^2} (|u_0|^2 + |\nabla d_0|^2) \leq 8\pi$, we divide the argument into two cases:

(a) *There exist no finite time singularities.*

For, otherwise, (ii) implies that we can blow up near the first singular time T_1 to obtain a nontrivial harmonic map $\omega \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ and

$$8\pi \leq \int_{\mathbb{R}^2} |\nabla \omega|^2 \leq \lim_{t \uparrow T_1} \int_{\mathbb{R}^2} (|u|^2 + |\nabla d|^2)(t) \leq \int_{\mathbb{R}^2} (|u_0|^2 + |\nabla d_0|^2) \leq 8\pi.$$

This, combined with lemma 2.1, yields

$$\int_0^{T_1} \int_{\mathbb{R}^2} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) = 0$$

so that $u = \partial_t d \equiv 0$ in $\mathbb{R}^2 \times [0, T_1]$ and hence $d(t) = d_0 \in C^\infty \cap H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$, $0 \leq t \leq T_1$, is a harmonic map. This contradicts the fact that T_1 is a singular time.

(b) $\phi(t) \equiv \sup_{x \in \mathbb{R}^2, \tau \leq t} (|u| + |\nabla d|)(x, \tau)$ remains bounded as $t \uparrow +\infty$.

For, otherwise, there exist $t_k \uparrow +\infty$ and $x_k \in \mathbb{R}^2$ such that $\lambda_k = \phi(t_k) = (|u| + |\nabla d|)(x_k, t_k) \rightarrow +\infty$. Define $(u_k, d_k) : \mathbb{R}^2 \times [-t_k \lambda_k^2, 0] \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ by

$$u_k(x, t) = \lambda_k^{-1} u(x_k + \lambda_k^{-1} x, t_k + \lambda_k^{-2} t), \quad d_k(x, t) = d(x_k + \lambda_k^{-1} x, t_k + \lambda_k^{-2} t), \quad P_k(x, t) = \lambda_k^{-2} P(x_k + \lambda_k^{-1} x, t_k + \lambda_k^{-2} t).$$

Then (u_k, d_k, P_k) solves (1.8) on $\mathbb{R}^2 \times [-t_k \lambda_k^2, 0]$, and

$$1 = (|u_k| + |\nabla d_k|)(0, 0) \geq (|u_k| + |\nabla d_k|)(x, t), \quad \forall (x, t) \in \mathbb{R}^2 \times [-t_k \lambda_k^2, 0].$$

As in the proof of (ii), we can conclude that $(u_k, d_k) \rightarrow (0, d_\infty)$ in $C_{\text{loc}}^2(\mathbb{R}^2)$, where $d_\infty \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ is a nontrivial harmonic map with finite energy. As in (a), this implies that

$$\int_0^\infty \int_{\mathbb{R}^2} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) = 0$$

so that $u = \partial_t d \equiv 0$ in $\mathbb{R}^2 \times [0, +\infty)$ and hence $d(t) = d_0 \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$, $0 \leq t < +\infty$, is a harmonic map. This implies that $\phi(t)$ is constant for $0 < t < +\infty$ and we get a desired contradiction.

Since $\phi(t)$ is a bounded function of $t \in (0, +\infty)$, the higher order regularity Theorem 1.3 implies that

$$\|\nabla^l u(t)\|_{C^0(\mathbb{R}^2)} + \|\nabla^{l+1} d(t)\|_{C^0(\mathbb{R}^2)} \leq C(l), \quad \forall l \geq 1, \quad \forall t \geq 1.$$

Thus we can choose $t_k \uparrow \infty$ such that

$$\int_{\mathbb{R}^2} (|u|^2 + |\nabla u|^2)(t_k) \leq E_0, \quad \int_{\mathbb{R}^2} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2)(t_k) \rightarrow 0,$$

and

$$\|u(t_k)\|_{C^2(\mathbb{R}^2)} + \|d(t_k)\|_{C^2(\mathbb{R}^2)} \leq C.$$

Thus we may assume that there exist a harmonic map $d_\infty \in C^\infty \cap H_{e_0}^1(\mathbb{R}^2, \mathbb{S}^2)$ such that

$$(u(t_k), d(t_k)) \rightarrow (0, d_\infty) \text{ in } C_{\text{loc}}^2(\mathbb{R}^2, \mathbb{S}^2).$$

This proves (v) under the first condition.

To show (v) under the condition $(d_0)_3 \geq 0$. We argue as follows. First, we can approximate (u_0, d_0) by smooth initial data (u_0^k, d_0^k) such that the third component of d_0^k is non-negative, i.e., $(d_0^k)_3 \geq 0$. Then we can check that the short time smooth solutions (u^k, d^k) of (1.8), with the initial data (u_0^k, d_0^k) at $t = 0$, on $\mathbb{R}^2 \times [0, T_k]$, have bounded gradients:

$$\left\| |\nabla u^k| + |\nabla d^k| \right\|_{C^0(\mathbb{R}^2 \times [0, T_k])} \leq C(k) < +\infty.$$

Since $(d^k)_3$ satisfies the equation

$$\left(\partial_t - \frac{1}{|\lambda_1|} \Delta\right)(d^k)_3 + u \cdot \nabla (d^k)_3 = \left(\frac{1}{|\lambda_1|} |\nabla d^k|^2 + \frac{\lambda_2}{\lambda_1} ((\hat{d}^k)^T A_k \hat{d}^k)\right)(d^k)_3,$$

and the coefficient in front of $(d^k)_3$, $\left(\frac{1}{|\lambda_1|} |\nabla d^k|^2 + \frac{\lambda_2}{\lambda_1} ((\hat{d}^k)^T A_k \hat{d}^k)\right)$ is bounded. Hence we can apply the maximum principle (see [15]) to conclude that $(d^k)_3 \geq 0$ in $\mathbb{R}^2 \times [0, T_k]$. Sending k to infinity, we conclude that the global weak solution (u, d) to (1.8) and (1.14), obtained in the part (i), satisfies $d_3 \geq 0$. If (u, d) has any finite time singularity, then by performing the blow-up argument we would obtain a nontrivial harmonic map ω from \mathbb{S}^2 to \mathbb{S}^2 such that $\omega_3 \geq 0$, which is impossible. Hence (u, d) has no finite time singularity. If (u, d) has singularity at the time infinity, then we would also obtain a nontrivial harmonic map from \mathbb{S}^2 to the upper hemisphere, which is also impossible. Therefore, (u, d) has bounded C^2 -norm in $\mathbb{R}^2 \times (\delta, +\infty)$ for any $\delta > 0$. This proves (v) under the second condition. The proof of Theorem 1.4 is now complete. \square

Added Note. The third author presented the main results of this article in the workshop “Nonlinear analysis of continuum theories: statics and dynamics” at the University of Oxford, April 8-12, 2013. During the finalization of this paper, Wendong Wang sent the third author his preprint “GLOBAL EXISTENCE OF WEAK SOLUTION FOR THE 2-D ERICKSEN-LESLIE SYSTEM”, in which they also claimed an existence result, similar to part (i) of our Theorem 1.4, by a different argument. However, since they drawn the conclusion by a global energy inequality of second order before the first time of energy concentration, their proof is incomplete.

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