Heat flow of harmonic maps whose gradients belong to $L^n_x L^\infty_t$

Changyou Wang
Department of Mathematics, University of Kentucky
Lexington, KY 40506, USA

Abstract. *For any compact n-dimensional Riemannian manifold (M, g) without boundary, a compact Riemannian manifold $N \subset \mathbf{R}^k$ without boundary, and $0 < T \leq +\infty$, we prove that for $n \geq 4$, if $u : M \times (0, T] \to N$ is a weak solution to the heat flow of harmonic maps such that $\nabla u \in L^n_x L^\infty_t(M \times (0, T])$, then $u \in C^\infty(M \times (0, T], N)$. As a consequence, we show that for $n \geq 3$, if $0 < T < +\infty$ is the maximal time interval for the unique smooth solution $u \in C^\infty(M \times [0, T), N)$ of (1.1), then $\|\nabla u(t)\|_{L^n(M)}$ blows up as $t \uparrow T$.*

§1. Introduction

For $n \geq 1$, let (M, g) be a smooth, compact n-dimensional Riemannian manifold without boundary, and $N \subset \mathbf{R}^k$, $k \geq 2$, be a compact Riemannian manifold without boundary. For $0 < T \leq +\infty$, recall that a map $u \in C^2(M \times (0, T), N)$ is a solution to the heat flow of harmonic maps, if

$$(1.1) \qquad u_t - \Delta_g u = \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} A(u)\left(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\right) \quad \text{in } M \times (0, T),$$

where Δ_g is the Laplace-Beltrami operator of (M, g), $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of $N \subset \mathbf{R}^k$, and $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ is the inverse of $g = (g_{\alpha\beta})$. By the classical theorem of Eells-Sampson [ES], it is well-known that for any $\phi \in C^\infty(M, N)$, there is a maximal time interval $0 < T = T(\phi) \leq +\infty$ such that there exists a unique $u \in C^\infty(M \times [0, T), N)$ solving (1.1) along with the initial condition:

$$(1.2) \qquad u(x, 0) = \phi(x), \quad x \in M.$$

Moreover, T can be characterized by

$$(1.3) \qquad \limsup_{t \uparrow T} \|\nabla u(t)\|_{C(M)} = +\infty.$$

In this paper, we are interested in an optimal characterization of the maximal time interval T that is scaling invariant. In fact, we show the smoothness of weak solutions of the heat flow of harmonic maps whose gradients belong to $L^n_x L^\infty_t(M \times [0, T])$ for $n \geq 4$.

First, let's recall the notion of weak solutions of (1.1).

Definition 1.0. A map $u : M \times [0,T] \to N$ is a weak solution to (1.1), if

(1) $u_t \in L^2_x L^2_t(M \times [0,T])$, $\nabla u \in L^2_x L^\infty_t(M \times [0,T])$,

(2) u satisfies (1.1) in the distribution sense:

$$\int_0^T \int_M u_t \cdot \phi + \nabla u \cdot \nabla \phi = \int_0^T \int_M A(u)(\nabla u, \nabla u) \cdot \phi, \ \forall \ \phi \in C_0^\infty(M \times (0,T), \mathbf{R}^k).$$

Definition 1.1. For $1 \le p, q \le +\infty$, we say a function $f = f(x,t) : M \times [0,T] \to \mathbf{R}$ is in $L^p_x L^q_t(M \times [0,T])$, if

$$\begin{aligned} \|f\|_{L^p_x L^q_t(M\times[0,T])} &:= (\int_0^T (\int_M |f(x,t)|^p \, dx)^{\frac{q}{p}} \, dt)^{\frac{1}{q}}, \ 1 \le q < +\infty, \\ &:= \text{essup}_{t\in[0,T]} \|f(\cdot,t)\|_{L^p(M)}, \qquad q = +\infty \end{aligned}$$

is finite. If $p = q$, then we simply write $\|f\|_{L^p(M\times[0,T])}$ for $\|f\|_{L^p_x L^p_t(M\times[0,T])}$.

By a simple argument, we see that a scaling invariant norm for ∇u is $\nabla u \in L^p_x L^q_t(M \times [0,T])$ for some $p \in [n, +\infty)$ and $q \in [2, +\infty]$ satisfying

$$\frac{n}{p} + \frac{2}{q} = 1. \tag{1.4}$$

Recall that the scaling invariant space $L^p_x L^q_t$, with (p,q) satisfying (1.4), has played an important role in the regularity issue of Navier-Stokes equation for $v : \mathbf{R}^3 \times (0,+\infty) \to \mathbf{R}^3$ and $p : \mathbf{R}^3 \times (0,+\infty) \to \mathbf{R}$:

$$v_t - \Delta v + v \cdot \nabla v = \nabla p, \quad x \in \mathbf{R}^3, \ t \in (0,+\infty), \tag{1.5}$$

$$\text{div} v = 0, \quad x \in \mathbf{R}^3, \ t \in (0,+\infty), \tag{1.6}$$

$$v(x,0) = v_0(x), \ x \in \mathbf{R}^3. \tag{1.7}$$

Leray [Lj] first established the existence of a global weak solution for (1.5)-(1.7), now called *Leray-Hopf weak solution*, that satisfies an energy inequality:

$$\|v(t)\|^2_{L^2(\mathbf{R}^3)} + 2\int_0^t \int_{\mathbf{R}^3} |\nabla v|^2 \le \|v_0\|^2_{L^2(\mathbf{R}^3)}. \tag{1.8}$$

Although the regularity issue for Leray-Hopf weak solutions of (1.5)-(1.7) remains open, it is well-known that both uniqueness and smoothness for the class of weak solutions

v of (1.5)-(1.7), in which $v \in L^p_x L^q_t(\mathbf{R}^3 \times (0, +\infty))$ for some $p \in (3, +\infty]$ and $q \in [2, +\infty)$ satisfying *Serrin's condition* (1.4), have been established through works by Prodi [P], Serrin [Sj], and Ladyzhenskaya [Lo] in 1960's. On the other hand, for the end point case $p = 3, q = +\infty$, only until very recently Escauriaza-Seregin-Sverák [ESV1,2] have finally proved the smoothness for weak solutions $v \in L^3_x L^\infty_t(\mathbf{R}^3 \times (0, +\infty))$ of (1.5)-(1.7).

Motivated by these results for the Navier-Stokes equation, we consider the class of weak solutions $u : M \times [0, T] \to N$ of (1.1) with $\nabla u \in L^p_x L^q_t(M \times [0, T])$ for some $p \in [n, +\infty]$ and $q \in [2, +\infty]$ satisfying *Serrin's condition* (1.4).

In this context, our first result deals the end point case $p = n$ and $q = +\infty$.

Theorem 1.2. *For $n \geq 4$, if $u : M \times [0, T] \to N$ is a weak solution of (1.1) satisfying $\nabla u \in L^n_x L^\infty_t(M \times [0, T])$, then $u \in C^\infty(M \times (0, T], N)$.*

Remark 1.3. The example by Chang-Ding-Ye [CDY] on finite time singularity of (1.1) indicates that theorem 1.2 fails for $n = 2$. It is unclear to the author whether theorem 1.2 is true for $n = 3$, since we can't verify the local energy inequality (2.7) and the energy monotonicity inequality (2.10) for weak solutions u of (1.1), with $\nabla u \in L^3_x L^\infty_t(M \times [0, T])$.

As a consequence of both theorem 1.2 for $n \geq 4$ and some modifications of its proof for $n = 3$, we are able to prove

Corollary 1.4. *For $n \geq 3$ and $\phi \in C^\infty(M, N)$, assume that $T = T(\phi) \in (0, +\infty)$ is the maximal time interval for a smooth solution $u \in C^\infty(M \times [0, T), N)$ of (1.1)-(1.2). Then*

$$\limsup_{t \uparrow T} \|\nabla u(t)\|_{L^n(M)} = +\infty. \tag{1.9}$$

For non-end point cases $p \in (n, +\infty]$ and $q \in [2, +\infty)$ of (1.4), we have

Theorem 1.5. *For $n \geq 2$, let $u : M \times [0, T] \to N$ be a weak solution of (1.1), with $\nabla u \in L^p_x L^q_t(M \times [0, T])$ for some $p > n, q \geq 2$ satisfying (1.4). If either $n \geq 4$ or $2 \leq n < 4$ and $p \geq 4$, then $u \in C^\infty(M \times (0, T], N)$.*

The paper is organized as follows. In §2, we derive, for $n \geq 4$, both the energy inequality and the energy monotonicity inequality for solutions u of (1.1) with $\nabla u \in L^n_x L^\infty_t$. In §3, we establish a small energy regularity theorem for weak solution of (1.1) with $\nabla u \in L^n_x L^\infty_t$ and prove theorem 1.2, corollary 1.4. In §4, we prove theorem 1.5.

§2. Preliminary properties

In this section, we outline some preliminary properties for the class of weak solutions to (1.1) whose gradients are in $L^n_x L^\infty_t$. These include both the energy inequality and Struwe's energy monotonicity inequality [Sm].

First we have

Lemma 2.1. *For $n \ge 2$ and $0 < T \le +\infty$, suppose that $u : M \times [0,T] \to N$ is a weak solution of (1.1) with $\nabla u \in L^n_x L^\infty_t(M \times [0,T])$. Then we have that $u \in C([0,T], L^n(M))$, and*

$$\|\nabla u(t)\|_{L^n(M)} \le \|\nabla u\|_{L^n_x L^\infty_t(M\times[0,T])}, \ \forall\ t \in [0,T]. \tag{2.1}$$

Proof. For for any $0 \le t_1 < t_2 \le T$, since $u_t \in L^2(M \times [0,T])$, we have, by Hölder inequality,

$$\|u(\cdot,t_1) - u(\cdot,t_2)\|_{L^2(M)} \le \|u_t\|_{L^2(M\times[t_1,t_2])}\sqrt{|t_1-t_2|}. \tag{2.2}$$

This implies $u \in C([0,T], L^2(M))$. If $n \ge 3$, then we have, by interpolation inequalities,

$$\begin{aligned}\|u(\cdot,t_1) - u(\cdot,t_2)\|_{L^n(M)} &\le 2\|u(\cdot,t_1) - u(\cdot,t_2)\|^{\frac{2}{n}}_{L^2(M)}\|u\|^{\frac{n-2}{n}}_{L^\infty(M\times[0,T])}\\ &\le C\|u(\cdot,t_1) - u(\cdot,t_2)\|^{\frac{2}{n}}_{L^2(M)},\end{aligned} \tag{2.3}$$

as u is bounded on $M \times [0,T]$. This implies $u \in C([0,T], L^n(M))$. To see (2.1), observe that, according to the definition, there exists $E \subset [0,T]$ with $|E| = 0$ such that (2.1) is true for any $t \in [0,T] \setminus E$. For $t_0 \in E$, since $|E| = 0$, there exist $\{t_i\} \subset [0,T] \setminus E$ such that $t_i \to t_0$. Since $u(\cdot,t_i) \to u(\cdot,t_0)$ in $L^n(M)$, we have, by the lower semicontinuity,

$$\|\nabla u(t_0)\|_{L^n(M)} \le \liminf_{i\to\infty} \|\nabla u(t_i)\|_{L^n(M)} \le \|\nabla u\|_{L^n_x L^\infty_t(M\times[0,T])}.$$

The proof is complete. ■

Lemma 2.2. *For $0 < T \le +\infty$, let $u : M \times [0,T] \to N$ be a weak solution of (1.1) with $\nabla u \in L^n_x L^\infty_t(M \times [0,T])$. If $n \ge 4$, then*

$$\int_{M\times[s,t]} |u_t|^2\phi^2 + \int_M |\nabla u(t)|^2\phi^2 \le \int_M |\nabla u(s)|^2\phi^2 + 4\int_0^t \int_M |\nabla u|^2|\nabla\phi|^2 \tag{2.4}$$

holds for any $\phi \in C^\infty(M)$ *and* $0 \le s \le t \le T$. *In particular, for any* $0 < s \le t \le T$, *we have*

$$\int_{M\times[s,t]} |u_t|^2 + \int_M |\nabla u(t)|^2 \le \int_M |\nabla u(s)|^2. \tag{2.5}$$

Proof. Since $|u_t - \Delta u| = |A(u)(\nabla u, \nabla u)| \le C|\nabla u|^2 \in L^{\frac{n}{2}}(M \times [0,T])$, it follows from the theory of linear parabolic equations (cf. [Lg]) that for $n \ge 3$, $u_t, \nabla^2 u \in L^{\frac{n}{2}}(M\times[0,T])$. If $n \ge 4$, then we can multiply (1.1) by $u_t\phi^2$ and integrate it over M. Since $A(u)(\nabla u, \nabla u) \perp u_t\phi^2$, we have, by integration by parts,

$$2\int_M |u_t|^2\phi^2 + \frac{d}{dt}\int_M |\nabla u|^2\phi^2 = -4\int_M \phi u_t \nabla u \cdot \nabla\phi. \tag{2.6}$$

Integrating (2.6) over $[s,t]$ and applying Hölder inequality, we get (2.4). Notice that (2.5) follows from (2.4) by letting $\phi \equiv 1$. ■

Let $i_M > 0$ be the injectivity radius of M. For $\in M$, $t > 0$, and $0 < r < \min\{i_M, \sqrt{t}\}$, let $B_r(x) \subset M$ be the ball with center x and radius r, and $P_r(x,t) = B_r(x)\times[t-r^2,t]$ be the parabolic cylinder with center (x,t) and radius r. Now we have a local energy inequality.

Corollary 2.3. *Under the same conditions as in Lemma 2.2, we have, for any* $z_0 = (x_0,t_0) \in M\times(0,T]$ *and* $0 < r < \min\{i_M, \sqrt{t_0}\}$,

$$\|\nabla u\|^2_{L^2_x L^\infty_t(P_{\frac{r}{2}}(z_0))} + \int_{P_{\frac{r}{2}}(z_0)} |u_t|^2 \le Cr^{-2}\int_{P_r(z_0)} |\nabla u|^2. \tag{2.7}$$

Proof. Let $\psi \in C_0^\infty(B_r(x_0))$ be such that $0 \le \psi \le 1$, $\psi \equiv 1$ on $B_{\frac{r}{2}}(x_0)$, and $|\nabla\psi| \le 4r^{-1}$. Let $s_0 \in (t_0 - r^2, t_0 - \frac{r^2}{4})$ be such that

$$\int_{B_r(x_0)} |\nabla u(s_0)|^2 \le 2r^{-2}\int_{P_r(z_0)} |\nabla u|^2. \tag{2.8}$$

Applying (2.4) with $\phi = \psi$ and $[s,t] = [s_0,t_0]$, we get (2.7). ■

Now we derive the energy monotonicity inequality of Struwe's type for u. For this, we need some notations. For $z_0 = (x_0,t_0) \in M\times(0,T]$ and $0 < r < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$, let

$$S_r(z_0) = \{(x,t) \in M\times(0,+\infty) \mid t = t_0 - r^2\},$$

$$T_r(z_0) = \{(x,t) \in M \times (0,+\infty) \mid t_0 - 4r^2 \le t \le t_0 - r^2\},$$

and

$$G_{z_0}(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}} \exp(-\frac{|x-x_0|^2}{4(t_0-t)}),\ x \in \mathbf{R}^n,\ t < t_0 \tag{2.9}$$

be the fundamental solution to the backward heat equation on $\mathbf{R}^n \times (0,+\infty)$. If $z_0 = (0,0)$, then we simply write $G = G_{(0,0)}, T_r = T_r(0,0), B_r = B_r(0), \text{and} P_r(0,0) = P_r$. For small $\delta_0 \in (0, i_M)$, let $\eta_0 \in C_0^\infty(B_{\delta_0}(x_0))$ be such that $\eta_0 = 1$ on $B_{\frac{\delta_0}{2}}(x_0)$, $\eta_0 = 0$ outside $B_{\delta_0}(x_0)$, and $|\nabla \eta_0| \le \frac{2}{\delta_0}$.

For a weak solution $u : M \times (0,T] \to N$, define two normalized energy quantities as follows:

$$\Psi(u, z_0, R) = \int_{S_R(z_0)} \eta_0^2(x) |\nabla u|^2(x,t) G_{z_0}(x,t),$$
$$\Phi(u, z_0, R) = \int_{T_R(z_0)} \eta_0^2(x) |\nabla u|^2(x,t) G_{z_0}(x,t),$$

for $0 < R < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$.

Now we establish the energy monotonicity formula of Struwe's type ([Sm]) for the class of weak solutions under consideration.

Lemma 2.4. *Under the same conditions as in Lemma 2.2, there exist $c, C > 0$ depending only on M, g, n such that for any $z_0 = (x_0, t_0) \in M \times (0,T]$ and $0 < R_1 \le R_2 < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$, the following inequality holds:*

$$\begin{aligned} &\Phi(u, z_0, R_1) + c \int_{R_1}^{R_2} \frac{dr}{r} \int_{T_r(z_0)} \eta_0^2 \frac{|(x-x_0)\cdot \nabla u + 2(t-t_0) u_t|^2}{|t_0 - t|} G_{z_0} \\ &\le e^{C(R_2 - R_1)} \Phi(u, z_0, R_2) + C(R_2 - R_1) E_0, \end{aligned} \tag{2.10}$$

where $E_0 = \int_M |\nabla u(x,0)|^2$ is the Dirichlet energy of $u(\cdot, 0)$.

Proof. As in Lemma 2.2, we have $u_t, \nabla^2 u \in L^{\frac{n}{2}}(M \times [0,T]) \subset L^2(M \times [0,T])$ for $n \ge 4$. Hence, for any $\xi \in C_0^\infty(M \times [0,T])$ and $\theta \in C_0^\infty(M \times [0,T])$, we can multiply (1.1) by $\xi \cdot \nabla u + \theta u_t$ and integrate over $M \times [0,T]$. By integration by parts and direct calculations (see, for example, Feldman [F] Proposition 8), we get

$$\int_{M\times[0,T]} [u_t \xi \cdot \nabla u + \theta |u_t|^2 - \frac{1}{2} |\nabla u|^2 (\text{div}\xi + \theta_t) + \nabla u \otimes \nabla u : \nabla \xi + u_t \nabla u \cdot \nabla \theta] = 0. \tag{2.11}$$

As in [F] Proposition 8, for $0 < r < R < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$, let $\xi \equiv 0$ and $\theta = \eta_0^2 t G \beta_\epsilon(t)$, where $\beta_\epsilon \in C_0^\infty([t_0 - R^2, t_0 - r^2])$ be such that $0 \le \beta_\epsilon \le 1$, and $\beta_\epsilon \equiv 1$ on $[t_0 - R^2 + \epsilon, t_0 - r^2 - \epsilon]$. For simplicity, assume $z_0 = (0,0)$ and $M = \mathbf{R}^n$ (since the integration on M is taken in the support of η_0). Inserting such chosen ξ, θ into (2.11) and sending ϵ to zero, we get

$$\text{(2.12)}\ \Psi(u,(0,0),R) - \Psi(u,(0,0),r)$$
$$= -\int_{\mathbf{R}^n}\int_{-R^2}^{-r^2} [2t(u_t \nabla u \cdot \nabla \eta_0^2 G + \eta_0^2 u_t \nabla u \cdot \nabla G + |u_t|^2 \eta_0^2 G) - \eta_0^2 |\nabla u|^2 (tG)_t].$$

Since $\nabla G = \frac{x}{2t} G$ and $(tG)_t = -\frac{n-2}{2} G - \frac{|x|^2}{4t} G$, we have

$$\text{(2.13)}\ \Psi(u,(0,0),R) - \Psi(u,(0,0),r)$$
$$= -\int_{\mathbf{R}^n}\int_{-R^2}^{-r^2} G[2tu_t \nabla u \nabla \eta_0^2 + \eta_0^2(u_t x \nabla u + 2t|u_t|^2 + (\frac{n-2}{2} + \frac{|x|^2}{4t})|\nabla u|^2)].$$

Now, inserting $\xi = \eta_0^2 x G, \theta \equiv 0$ into (2.11) and integrating over $\mathbf{R}^n \times [-R^2, -r^2]$, and using $\mathrm{div}(xG) = (\frac{|x|^2}{2t} + n)G$ and $\nabla(xG) = G(I + \frac{x \otimes x}{2t})$, we get

$$\text{(2.14)}\quad \int_{\mathbf{R}^n}\int_{-R^2}^{-r^2} [\eta_0^2 x \cdot \nabla u u_t - \frac{1}{2} x \cdot \nabla \eta_0^2 |\nabla u|^2$$
$$- \eta_0^2 (\frac{|x|^2}{4t} + \frac{n-2}{2})|\nabla u|^2 + \eta_0^2 \frac{|x \cdot \nabla u|^2}{2t} + x \cdot \nabla u \nabla \eta_0^2 \cdot \nabla u] G = 0.$$

Adding (2.13) and (2.14), we obtain

$$\text{(2.15)}\qquad \begin{aligned} &\Psi(u,(0,0),R) - \Psi(u,(0,0),r) \\ &\ge \int_{\mathbf{R}^n}\int_{-R^2}^{-r^2} \eta_0^2 \frac{|x \cdot \nabla u + 2tu_t|^2}{2|t|} G \\ &+ \int_{\mathbf{R}^n}\int_{-R^2}^{-r^2} [\frac{1}{2} x \cdot \nabla \eta_0^2 |\nabla u|^2 - (x \cdot \nabla u + 2tu_t) \nabla \eta_0^2 \cdot \nabla u] G \\ &\ge \int_{\mathbf{R}^n}\int_{-R^2}^{-r^2} \eta_0^2 \frac{|x \cdot \nabla u + 2tu_t|^2}{4|t|} G \\ &- C \int_{\mathbf{R}^n}\int_{-R^2}^{-r^2} (|x||\nabla \eta_0| + |t||\nabla \eta_0|^2)|\nabla u|^2 G. \end{aligned}$$

Notice that

$$\text{(2.16)}\qquad \Phi(u,(0,0),R) = 2\int_R^{2R} \frac{\Psi(u,(0,0),r)}{r}\, dr.$$

As in [F] Proposition 10 or [Sm], (2.10) follows from ((2.15) and (2.16). ■

As an immediate consequence of (2.10) and (2.15), we have

Corollary 2.5. *Under the same conditions as in Lemma 2.2. There exists $C > 0$ depending only on M, g, n such that for any $z_0 = (x_0, t_0) \in M \times (0, T]$ and $0 < r < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$,*

$$r^{-n} \int_{P_r(z)} |\nabla u|^2 \leq C r_0^{-n} \int_{P_{r_0}(z_0)} |\nabla u|^2 \tag{2.17}$$

holds for any $z \in P_{\frac{r_0}{4}}(z_0)$ and $0 < r < \frac{r_0}{4}$.

Proof. Using (2.10) and (2.15) and choosing a suitable cut-off function $\eta_0 \in C_0^\infty(B_{r_0}(x_0))$, (2.17) can be proved exactly in the same way as Chen-Li-Lin [CLL] Lemma 2.2. We omit the details here. ■

We end this section with a remark on Corollary 2.3 and Corollary 2.5 for short time smooth solutions to (1.1).

Remark 2.6. For $n \geq 2$ and $0 < T < +\infty$. Suppose that $u \in C^\infty(M \times (0, T), N)$ is a smooth solution of (1.1) and T is the singular time of u. Then both (2.7) and (2.17) remain true for all $0 < t_0 \leq T$.

Proof. It is easy to see that the arguments to prove Lemma 2.2 and Lemma 2.4 are valid for all smooth solutions $u \in C^\infty(M \times (0, T), N)$ to (1.1). Therefore, the conclusions of Corollary 2.3 and 2.5 hold. ■

§3. Apriori estimates and proof of Theorem 1.2, Corollary 1.4

This section is devoted to the proof of Theorem 1.2. The first step is to establish apriori estimates under a small energy condition, which yields that there are at most finitely many singular points at any time t. The second step is to exclude the existence of nontrivial, self-similar solutions $\phi(x, t) = \phi(\frac{x}{\sqrt{-t}}) : \mathbf{R}_-^{n+1} \to N$ of (1.1) with $\nabla \phi \in L_x^n L_t^\infty(\mathbf{R}_-^{n+1})$ and $\phi(\cdot, 0) =$ constant. Henceforth, we denote $\mathbf{R}_-^{n+1} = \mathbf{R}^n \times (-\infty, 0]$. We would like to remark, by the work of Lin-Wang [LW], that the existence of such nontrivial, self-similar solutions (or quasi-harmonic spheres) is an obstruction to smoothness of suitable weak solutions to (1.1).

Lemma 3.1. *For $n \geq 4$, let $\Lambda > 0$ be any given number. Then there exist $\epsilon_0 > 0$, $\theta_0 \in (0, \frac{1}{2})$, and $C_0 > 0$ depending only on M, g, n, Λ such that if u is a weak solution of*

(1.1) and satisfies, for $z_0 = (x_0, t_0) \in M \times (0, T]$ *and* $0 < r_0 < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$,

$$\|\nabla u\|_{L^n_x L^\infty_t (P_{r_0}(z_0))} \le \Lambda \quad \textit{and} \quad r_0^{-n} \int_{P_{r_0}(z_0)} |\nabla u|^2 \le \epsilon_0^2, \tag{3.1}$$

then

$$(\theta_0 r_0)^{-n} \int_{P_{\theta_0 r_0}(z_0)} |\nabla u|^2 \le \frac{1}{2} r_0^{-n} \int_{P_{r_0}(z_0)} |\nabla u|^2. \tag{3.2}$$

Proof. Since this is a local result, we may assume, for simplicity, $(M, g) = (\mathbf{R}^n, dx^2)$. By translation and dilation, we further assume $z_0 = (0, 0)$ and $r_0 = 1$.

Since $n \ge 4$, it follows from Corollary 2.3 and 2.5 that we have

$$r^{-n} \int_{P_r(z)} (|\nabla u|^2 + r^2 |u_t|^2) \le C \int_{P_1} |\nabla u|^2, \ \ \forall z \in P_{\frac{1}{4}} \ \text{ and } 0 < r \le \frac{1}{4}. \tag{3.3}$$

Denote the average of u over $P_r(z)$ by $u_{P_r(z)} = \frac{1}{|P_r(z)|} \int_{P_r(z)} u$. By Poincaré inequality, (3.3) implies

$$\begin{aligned} [u]_{\mathrm{BMO}(P_{\frac{1}{2}})} &:= \sup\{r^{-(n+2)} \int_{P_r(z)} |u - u_{P_r(z)}| : P_r(z) \subset P_{\frac{1}{2}}\} \\ &\le C \sup_{P_r(z) \subset P_{\frac{1}{2}}} \{r^{-n} \int_{P_r(z)} (|\nabla u|^2 + r^2 |u_t|^2)\}^{\frac{1}{2}} \\ &\le C (\int_{P_1} |\nabla u|^2)^{\frac{1}{2}}. \end{aligned} \tag{3.4}$$

It follows from John-Nirenberg's inequality ([JN]) that for any $p > 2$, there exists $C(n, p) > 0$ such that for any $0 < r \le \frac{1}{4}$, we have

$$\{r^{-(n+2)} \int_{P_r} |u - u_{P_r}|^p\}^{\frac{1}{p}} \le C(n, p) \{r^{-(n+2)} \int_{P_r} |u - u_{P_r}|^2\}^{\frac{1}{2}}. \tag{3.5}$$

For $r \in (0, \frac{1}{2})$ to be chosen later, let $v : P_r \to \mathbf{R}^k$ solve

$$v_t - \Delta v = 0 \ \text{ in } P_r, \tag{3.6}$$

$$v = u \ \text{ on } \partial_p P_r, \tag{3.7}$$

where $\partial_p P_r = (\partial B_r \times [-r^2, 0]) \cup (B_r \times \{-r^2\})$ is the parabolic boundary of P_r.

Subtracting (1.1) by (3.6), multiplying the resulting equation by $u-v$, and integrating it over P_r, we obtain

$$(3.8)\qquad 2\int_{B_r} |\nabla(u-v)|^2 \le C\int_{B_r} |\nabla u|^2|u-v|^2$$
$$\le C(\int_{B_r} |\nabla u|^2)^{\frac{1}{2}}(\int_{B_r} |\nabla u|^n)^{\frac{1}{n}}(\int_{B_r} |u-v|^{2^*})^{\frac{1}{2^*}},$$

where $2^* = \frac{2n}{n-2}$. Integrating (3.8) over $t \in [-r^2, 0]$ and applying Hölder inequality, we then have

$$(3.9)\qquad r^{-n}\int_{P_r} |\nabla(u-v)|^2$$
$$\le Cr^{-n}\int_{-r^2}^{0}(\int_{B_r} |\nabla u|^2)^{\frac{1}{2}}(\int_{B_r} |\nabla u|^n)^{\frac{1}{n}}(\int_{B_r} |u-v|^{2^*})^{\frac{1}{2^*}}$$
$$\le C\|\nabla u\|_{L^n_x L^\infty_t(P_r)}(r^{-n}\int_{P_r} |\nabla u|^2)^{\frac{1}{2}}\{r^{-n}\int_{-r^2}^{0}(\int_{B_r} |u-v|^{2^*})^{\frac{2}{2^*}}\}^{\frac{1}{2}}$$
$$\le C\Lambda\{r^{-n}\int_{P_r} |\nabla u|^2\}^{\frac{1}{2}}\{r^{-(n+2)}\int_{P_r} |u-v|^{2^*}\}^{\frac{1}{2^*}}.$$

By the theory of linear parabolic equations (cf. [Lg]), we have

$$(3.10)\qquad \int_{P_r} |u-v|^p \le C(n,p)\int_{P_r} |u-u_{P_r}|^p,\ \forall\, p \in (1,+\infty),$$

and

$$(3.11)\qquad (\theta r)^{-n}\int_{P_{\theta r}} |\nabla v|^2 \le C\theta^2 r^{-n}\int_{P_r} |\nabla u|^2,\ \forall\, \theta \in (0, \frac{1}{2}).$$

Putting (3.5) together with (3.10) ($p = 2^*$), we have

$$(3.12)\qquad \{r^{-(n+2)}\int_{P_r} |u-v|^{2^*}\}^{\frac{1}{2^*}} \le C\{r^{-(n+2)}\int_{P_r} |u-v|^2\}^{\frac{1}{2}}.$$

By (3.11), (3.12), (2.17) and Hölder inequality, (3.9) gives

$$(\theta r)^{-n}\int_{P_{\theta r}} |\nabla u|^2 \le 2[(\theta r)^{-n}\int_{P_{\theta r}} |\nabla v|^2 + (\theta r)^{-n}\int_{P_{\theta r}} |\nabla(u-v)|^2]$$
$$\le C[\theta^2 r^{-n}\int_{P_r} |\nabla u|^2 + \theta^{-n}\Lambda\{r^{-n}\int_{P_r} |\nabla u|^2\}^{\frac{1}{2}}\{r^{-(n+2)}\int_{P_r} |u-u_{P_r}|^2\}^{\frac{1}{2}}]$$
$$(3.13)\qquad \le (C\theta^2 + \frac{1}{4})\int_{P_1} |\nabla u|^2 + C\theta^{-2n}\Lambda^2 r^{-(n+2)}\int_{P_r} |u-u_{P_r}|^2.$$

To proceed with the proof, we need the following claim.

Claim. *There exist $\epsilon_0 > 0$, $r_1 \in (0, \frac{1}{2})$, $C > 0$ depending on n such that if $u : P_1 \to N$ is a weak solution of (1.1), and*

$$\int_{P_1} |\nabla u|^2 \le \epsilon_0^2,$$

then

$$r^{-(n+2)} \int_{P_r} |u - u_{P_r}|^2 \le C r^2 \int_{P_1} |\nabla u|^2, \ \forall\ 0 < r \le r_1. \tag{3.14}$$

To prove this claim, we argue by contradiction. Suppose that (3.14) were false. Then for any $r_1 \in (0, \frac{1}{2})$ and $L > 0$, there exist a sequence of weak solutions $\{u_i\}$ of (1.1) such that

$$\int_{P_1} |\nabla u_i|^2 = \epsilon_i^2 \to 0, \tag{3.15}$$

but

$$r_1^{-(n+2)} \int_{P_{r_1}} |u_i - (u_i)_{P_{r_1}}|^2 \ge L r_1^2 \int_{P_1} |\nabla u_i|^2. \tag{3.16}$$

Define $v_i = \frac{u_i - (u_i)_{P_{\frac{1}{2}}}}{\epsilon_i} : P_1 \to \mathbf{R}^k$. Then we have

$$(v_i)_{P_{\frac{1}{2}}} = 0 \ \text{ and } \int_{P_1} |\nabla v_i|^2 = 1, \tag{3.17}$$

$$r_1^{-(n+2)} \int_{P_{r_1}} |v_i - (v_i)_{P_{r_1}}|^2 \ge L, \tag{3.18}$$

and

$$(v_i)_t - \Delta v_i = \epsilon_i A(u_i)(\nabla v_i, \nabla v_i), \ \text{ in } P_1. \tag{3.19}$$

On the other hand, by Corollary 2.3, we have

$$\int_{P_{\frac{1}{2}}} |(v_i)_t|^2 \le C \int_{P_1} |\nabla v_i|^2 \le L. \tag{3.20}$$

After taking possible subsequences, we may assume $v_i \to v$ weakly in $H^1(P_{\frac{1}{2}}, \mathbf{R}^k)$ and strongly in $L^2(P_{\frac{1}{2}}, \mathbf{R}^k)$. (3.18) and (3.19) imply that v satisfies $v_{P_{\frac{1}{2}}} = 0, \int_{P_{\frac{1}{2}}} |\nabla v|^2 \le 1$, and

$$v_t - \Delta v = 0 \text{ in } P_{\frac{1}{2}}.$$

Hence, by (3.11), we have

$$r_1^{-(n+2)} \int_{P_{r_1}} |v - v_{P_{r_1}}|^2 \le C r_1^2 \int_{P_{\frac{1}{2}}} |\nabla v|^2 \le C r_1^2. \tag{3.21}$$

This contradicts (3.18), provided that $L > 0$ is sufficiently large. Hence (3.14) holds.

Inserting (3.14) into (3.13), we get

$$(\theta r_1)^{-n} \int_{P_{\theta r_1}} |\nabla u|^2 \le [C\theta^2 + \frac{1}{4} + C\theta^{-2n}\Lambda^2 r_1^2] \int_{P_1} |\nabla u|^2. \tag{3.22}$$

Therefore, if we choose sufficiently small $\theta = \theta_1 \in (0, \frac{1}{2})$ and $r_1 = r_1(\Lambda, \theta_1) \in (0, \frac{1}{2})$, then

$$(\theta_1 r_1)^{-n} \int_{P_{\theta_1 r_1}} |\nabla u|^2 \le \frac{1}{2} \int_{P_1} |\nabla u|^2. \tag{3.23}$$

Hence (3.14) holds with $\theta_0 = \theta_1 r_1$. The proof of Lemma 3.1 is complete. ■

Proposition 3.2. *For $n \ge 4$, assume that $u : M \times (0, T] \to N$ is a weak solution of (1.1) with $\nabla u \in L^n_x L^\infty_t (M \times [0, T])$. Then there exists $\epsilon_0 > 0$ depending on M, g, n and $\|\nabla u\|_{L^n_x L^\infty_t (M \times [0,T])}$ such that if, for $z_0 = (x_0, t_0) \in M \times (0, T]$ and $0 < r_0 < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$,*

$$r_0^{-n} \int_{P_{r_0}(z_0)} |\nabla u|^2 \le \epsilon_0^2,$$

then $u \in C^\infty(P_{\frac{r_0}{2}}(z_0), N)$, and

$$\|u\|_{C^l(P_{\frac{r_0}{2}}(z_0))} \le C(n, l, r_0, \epsilon_0, \|\nabla u\|_{L^n_x L^\infty_t (M \times [0,T])}). \tag{3.24}$$

Proof. Set $\Lambda = \|\nabla u\|_{L^n_x L^\infty_t (M \times [0,T])}$. Let $\epsilon_0 = \epsilon_0(n, \Lambda) > 0$ be given by Lemma 3.1, since

$$(\frac{r_0}{2})^{-n} \int_{P_{\frac{r_0}{2}}(z)} |\nabla u|^2 \le 2^n \epsilon_0^2, \ \forall\, z \in P_{\frac{r_0}{2}}(z_0), \tag{3.25}$$

Lemma 3.1 implies that there is $\theta_0 \in (0, \frac{1}{4})$ such that

$$(\theta_0 r_0)^{-n} \int_{P_{\theta_0 r_0}(z)} |\nabla u|^2 \leq \frac{1}{2} r_0^{-n} \int_{P_{r_0}(z)} |\nabla u|^2, \ \forall\, z \in P_{\frac{r_0}{2}}(z_0). \tag{3.26}$$

By iterating (3.26) l-times, we have

$$(\theta_0^l r_0)^{-n} \int_{P_{\theta_0^l r_0}(z)} |\nabla u|^2 \leq (\frac{1}{2})^l r_0^{-n} \int_{P_{r_0}(z)} |\nabla u|^2, \ \forall\, z \in P_{\frac{r_0}{2}}(z_0). \tag{3.27}$$

It follows from (3.27) and Corollary 2.3 that there exists $\alpha_0 \in (0,1)$ such that

$$r^{-n} \int_{P_r(z)} (|\nabla u|^2 + r^2 |u_t|^2) \leq C(\frac{r}{r_0})^{2\alpha_0} r_0^{-n} \int_{P_{r_0}(z_0)} |\nabla u|^2, \tag{3.28}$$

holds for all $z \in P_{\frac{r_0}{2}}(z_0)$ and $0 < r \leq \frac{r_0}{2}$. Hence, by the Morrey's decay Lemma (cf. [F]), we have $u \in C^{\alpha_0}(P_{\frac{r_0}{2}}(z_0), N)$. By the standard method (cf. [F]), the higher order regularity of u and (2.24) follow. This completes the proof. ■

For any weak solution $u : M \times (0,T] \to N$ of (1.1), define

$$\Sigma = \{z_0 = (x_0, t_0) \in M \times (0,T] \mid u \text{ is discontinuous at } z_0\},$$

and

$$\Sigma(t_0) = \Sigma \cap \{t_0\}, \quad \text{for } t_0 \in (0,T].$$

Lemma 3.3. *For $n \geq 4$, let $u : M \times (0,T] \to N$ be a weak solution of (1.1) with $\nabla u \in L_x^n L_t^\infty(M \times [0,T])$. Then, for any $t_0 \in (0,T]$, $\Sigma(t_0)$ is finite and*

$$\mathit{card}(\Sigma(t_0)) \leq \epsilon_0^{-n} \limsup_{\rho \downarrow 0} \frac{1}{\rho^2} \int_{t_0-\rho^2}^{t_0} \int_M |\nabla u(t)|^n \leq \epsilon_0^{-n} \|\nabla u\|^n_{L_x^n L_t^\infty(M\times[0,T])}. \tag{3.29}$$

Proof. By Proposition 3.2, we have that $x_0 \in \Sigma(t_0)$ iff there exists $\epsilon_0 > 0$ such that

$$r^{-n} \int_{P_r(x_0,t_0)} |\nabla u|^2 \geq \epsilon_0^2, \ \forall\, r > 0. \tag{3.30}$$

By Hölder inequality, (3.30) implies

$$r^{-2} \int_{P_r(x_0,t_0)} |\nabla u|^n \geq \epsilon_0^n, \ \forall\, r > 0. \tag{3.31}$$

Now, for any finite subset $\{x_1, \cdots, x_l\} \subset \Sigma(t_0)$, let $r_0 > 0$ be so small that $\{B_r(x_i)\}_{i=1}^l$ are mutually disjoint for any $0 < r \leq r_0$. By (3.31), we have

$$r^{-2} \int_{P_r(x_i,t_0)} |\nabla u(t)|^n \geq \epsilon_0^n, \ 1 \leq i \leq l. \tag{3.32}$$

Therefore we have, for any $0 < r \leq r_0$,

$$\begin{aligned} l\epsilon_0^n &\leq r^{-2} \sum_{i=1}^l \int_{P_r(x_i,t_0)} |\nabla u|^n \leq r^{-2} \int_{t_0-r^2}^{t_0} \int_{\cup_{i=1}^l B_r(x_i)} |\nabla u|^n \\ &\leq r^{-2} \int_{t_0-r^2}^{t_0} \int_M |\nabla u|^n. \end{aligned} \tag{3.33}$$

This implies (3.29) and the proof is complete. ■

Proof of Theorem 1.2:

We prove theorem 1.2 by contradiction. For simplicity, assume $M = \mathbf{R}^n$. Suppose that $\Sigma(t_0) \neq \emptyset$. Then, by Proposition 3.2, there exists $x_0 \in \Sigma(t_0) \subset \mathbf{R}^n$ such that

$$r^{-n} \int_{P_r(x_0,t_0)} |\nabla u|^2 \geq \epsilon_0^2, \ \forall \ r > 0. \tag{3.34}$$

For $r_i \downarrow 0$, define $v_i(x,t) = u(x_0 + r_i x, t_0 + r_i^2 t) : \mathbf{R}^n \times (-r_i^{-2} t_0, 0] \to N$. Then it is easy to see that v_i is a weak solution of (1.1),

$$\|\nabla v_i(t)\|_{L_x^n L_t^\infty(\mathbf{R}^n \times [-r_i^{-2} t_0, 0])} = \|\nabla u(t)\|_{L_x^n L_t^\infty(\mathbf{R}^n \times [0,t_0])} < +\infty, \tag{3.35}$$

and

$$R^{-n} \int_{P_R} |\nabla v_i|^2 = (Rr_i)^{-n} \int_{P_{Rr_i}(x_0,t_0)} |\nabla u|^2 \geq \epsilon_0^2, \ \forall \ R > 0. \tag{3.36}$$

By (3.35) and Hölder inequality, we have

$$\sup_i \int_{P_R} |\nabla v_i|^2 \leq R^n \|\nabla u\|_{L_x^n L_t^\infty(\mathbf{R}^n \times [-R^2,0])}^{2n} \leq CR^n, \ \forall \ R > 0. \tag{3.37}$$

Moreover, by Corollary 2.3, we have

$$\sup_i \int_{P_R} |(v_i)_t|^2 \leq CR^{-2} \sup_i \int_{P_{2R}} |\nabla v_i|^2 \leq CR^{n-2}, \ \forall \ R > 0. \tag{3.38}$$

It follows from (3.37) and (3.38) that $\{v_i\} \subset H^1_{\rm loc}(\mathbf{R}^{n+1}_-, N)$ is a bounded sequence. Hence we may assume that there exists $v : \mathbf{R}^{n+1}_- \to N$ such that $\nabla v_i \to \nabla v$ weakly in $L^2_{\rm loc}(\mathbf{R}^{n+1}_-)$, and $v_i \to v$ strongly in $L^2_{\rm loc}(\mathbf{R}^{n+1}_-)$. Since

$$(v_i)_t - \Delta v_i = A(v_i)(\nabla v_i, \nabla v_i) \text{ in } \mathbf{R}^n \times (-r_i^{-2} t_0, 0], \tag{3.39}$$

and

$$|A(v_i)(\nabla v_i, \nabla v_i)| \le C|\nabla v_i|^2 \text{ is bounded in } L^1_{\rm loc}(\mathbf{R}^{n+1}_-), \tag{3.40}$$

the convergence theorem of Chen-Hong-Hungerbühler [CHH] implies

$$\nabla v_i \to \nabla v \text{ for a.e. } z \in \mathbf{R}^{n+1}_-. \tag{3.41}$$

This, combined with (3.35), implies

$$\nabla v_i \to \nabla v \text{ strongly in } L^2_{\rm loc}(\mathbf{R}^{n+1}_-). \tag{3.42}$$

Therefore, $v : \mathbf{R}^{n+1}_- \to N$ is a weak solution of (1.1), with $\nabla v \in L^n_x L^\infty_t(\mathbf{R}^{n+1}_-)$, and

$$R^{-n} \int_{P_R} |\nabla v|^2 \ge \epsilon_0^2, \ \forall\ R > 0. \tag{3.43}$$

Applying the monotonicity inequality (2.10) with $\eta_0 \equiv 1$, we have that for any $z_0 = (x_0, t_0) \in \mathbf{R}^n \times (0, T]$,

$$\Theta(u, z_0) = \lim_{r \downarrow 0} \Phi_{z_0, r}(u)$$

exists and is finite. For any $R > 0$, observe

$$\begin{aligned} \Phi_{(0,0),R}(v) &= \int_{T_R} |\nabla v|^2 G = \lim_{i \to \infty} \int_{T_R} |\nabla v_i|^2 G \\ &= \lim_{i \to \infty} \int_{T_{Rr_i}(t_0)} |\nabla u|^2 G_{(x_0,t_0)}(x,t) \\ &= \lim_{r \downarrow 0} \Phi_{(x_0,t_0),r}(u). \end{aligned} \tag{3.44}$$

Since v also satisfies (2.10), (3.44) implies that for any $R > 0$,

$$\begin{aligned} 0 &= \Phi_{(0,0),R}(v) - \Theta(v, (0,0)) = \lim_{r \downarrow 0} (\Phi_{(0,0),R}(v) - \Phi_{(0,0),r}(v)) \\ &\ge \lim_{r \downarrow 0} \int_r^R \frac{dr}{r} \int_{T_r} \frac{|x \cdot \nabla v + 2t v_t|^2}{2|t|} G. \end{aligned}$$

Hence we have

$$\int_{T_r} \frac{|x\cdot\nabla v+2tv_t|^2}{|t|}G=0,\ \forall r>0,$$

or equivalently,

$$(x\cdot\nabla v+2tv_t)(x,t)=0 \text{ for a.e. } (x,t)\in\mathbf{R}^{n+1}_-. \tag{3.45}$$

We divide (3.45) into two cases.

Case 1. $v_t(x,t)=0$ for a.e. $(x,t)\in\mathbf{R}^{n+1}_-$. By (3.45), we have

$$v(x,t)=\phi(x)=\phi(\frac{x}{|x|}) \text{ for a.e. } (x,t)\in\mathbf{R}^{n+1}_-.$$

Since $\nabla v\in L^n_xL^\infty_t(\mathbf{R}^{n+1}_-)$, we have

$$\int_{\mathbf{R}^n}|\nabla(\phi(\frac{x}{|x|}))|^n=\int_0^{+\infty}\frac{dr}{r}\int_{S^{n-1}}|\nabla_{S^{n-1}}\phi|^n\,dH^{n-1}<+\infty.$$

This forces ϕ to be a constant, which contradicts (3.43).

Case 2. v is t-dependent. By (3.45), we have (cf. [Sm] [LW]) that there exists a map $\phi:\mathbf{R}^n\to N$ such that

$$v(x,t)=\phi(\frac{x}{\sqrt{-t}}),\ \text{for a.e. } (x,t)\in\mathbf{R}^{n+1}_-. \tag{3.46}$$

Moreover,

$$\|\nabla\phi\|_{L^n(\mathbf{R}^n)}=\|\nabla v\|_{L^n_xL^\infty_t(\mathbf{R}^{n+1}_-)}<+\infty,$$

and ϕ satisfies the so-called quasi-harmonic map equation (cf. [LW]):

$$\Delta\phi-\frac{1}{2}x\cdot\nabla\phi+A(\phi)(\nabla\phi,\nabla\phi)=0,\ x\in\mathbf{R}^n. \tag{3.47}$$

Since the singular set of ϕ coincides with the singular set of $v(\cdot,-1)$, Lemma 3.1 implies that there exists a finite subset $\{x_i\}_{i=1}^l\subset\mathbf{R}^n$ such that $\phi\in C^\infty(\mathbf{R}^n\setminus\{x_i\}_{i=1}^l,N)$.

Now we want to show that ϕ is a constant map. To do it, first we need

Claim 2. *There exist $p_0\in N$ such that $v(x,0)=p_0$ for a.e. $x\in\mathbf{R}^n$.*

In fact, by Lemma 2.1, we have $v_i(\cdot,0)\to v(\cdot,0)$ in $L^n_{\text{loc}}(\mathbf{R}^n)$. Hence, for any $R>0$, we have

$$\begin{aligned}\int_{B_R}|v(x,0)-v_R|^n&\le 2^n\lim_{i\to\infty}(\int_{B_R}|v_i(x,0)-v(x,0)|^n+\int_{B_R}|v_i(x,0)-(v_i)_R|^n)\\&\le CR^n\lim_{i\to\infty}\int_{B_R}|\nabla v_i|^n(x,0)\\&=CR^n\lim_{i\to\infty}\int_{B_{Rr_i}(x_0)}|\nabla u|^n(x,t_0)=0,\end{aligned}$$

where $v_R = \frac{1}{|B_R|}\int_{B_R} v(x,0)\,dx$, $(v_i)_R = \frac{1}{|B_R|}\int_{B_R} v_i(x,0)\,dx$, and we have used the Poincaré inequality and $\nabla u(\cdot,t_0) \in L^n(\mathbf{R}^n)$ in the last two steps.

Since

$$\int_{-1}^0 \int_{\mathbf{R}^n} |\nabla v|^n \le \|\nabla v\|^n_{L^n_x L^\infty_t(\mathbf{R}^n\times[-1,0])} < +\infty,$$

there exists a sufficiently large $R_0 = R_0(\epsilon_0) > 0$ such that

$$\int_{-1}^0 \int_{\mathbf{R}^n\setminus B_{R_0-1}} |\nabla v|^n \le \epsilon_0^n, \tag{3.48}$$

where $\epsilon_0 > 0$ is given by Lemma 3.1. Hence Lemma 3.1 implies that for any $k \ge 1$,

$$|\nabla^k v|(x,t) \le C(\epsilon_0, k),\ \forall (x,t) \in (\mathbf{R}^n \setminus B_{R_0}) \times [-1,0]. \tag{3.49}$$

Set $\omega = \nabla v$. Then ω satisfies:

$$|\omega_t - \Delta\omega| = |\nabla(A(v)(\nabla v, \nabla v))| \le C(|\omega| + |\nabla\omega|),\ \text{in } (\mathbf{R}^n \setminus B_{R_0}) \times [-1,0], \tag{3.50}$$

$$\omega(x,0) = 0,\ x \in \mathbf{R}^n \setminus B_{R_0}. \tag{3.51}$$

Therefore, the unique continuation theorem of [ESS1] implies $\omega \equiv 0$ on $(\mathbf{R}^n\setminus B_{R_0})\times[-1,0]$. In particular, $\phi(x) = p_0$ for $x \in \mathbf{R}^n \setminus B_{R_0}$. On the other hand, if we define $\Omega = B_{2R_0} \setminus \{x_i\}_{i=1}^l$, then $\Omega \subset \mathbf{R}^n$ is a connected open set and $\phi \in C^\infty(\Omega, N)$ solves (3.47). Hence, by the standard unique continuation theorem on 2nd elliptic equations, we conclude $\phi = p_0$ in B_{R_0}, and hence $\phi \equiv p_0$ on $\mathbf{R}^n$.

This clearly contradicts (3.43). Hence the proof of Theorem 1.2 is complete. ■

Proof of Corollary 1.4:

We argue by contradiction. Suppose that $0 < T < +\infty$ is the first singular time for u, and $\nabla u \in L^n_x L^\infty_t(M \times [0,T])$. If $n \ge 4$, then it contradicts theorem 1.2. If $n = 3$, then it follows from Remark 2.6 that u satisfies both (2.7) and (2.17). Moreover, one can check that the proof for both Lemma 3.1 and theorem 1.2 work as long as $n \ge 3$ and u satisfies (2.7), (2.17). In particular, $u \in C^\infty(M \times (0,T], N)$ for $n = 3$. Hence T is not the first singular time for u and we get the desired contradiction. ■

§4. Proof of Theorem 1.5

It is similar to that of Lemma 3.1 and Proposition 3.2. By Hölder inequality, we have that for any $z_0 = (x_0, t_0) \in M \times (0, T]$ and $0 < r < \sqrt{t_0}$,

$$
\begin{aligned}
(r^{-n} \int_{P_r(z_0)} |\nabla u|^2)^{\frac{1}{2}} &\leq r^{1-(\frac{n}{p}+\frac{2}{q})} \|\nabla u\|_{L_x^p L_t^q(P_r(z_0))}, \quad \text{for } n < p < +\infty,\ 2 < q < +\infty, \\
&\leq (\int_{t_0 - r^2}^{t_0} \|\nabla u(t)\|^2_{L^\infty(B_r(x_0))}\, dt)^{\frac{1}{2}}, \quad \text{for } p = +\infty,\ q = 2.
\end{aligned}
\tag{4.1}
$$

Hence, for any pair $(p, q) \in (n, +\infty] \times [2, +\infty)$ satisfying (1.4), we have

$$
\lim_{r \downarrow 0} (r^{-n} \int_{P_r(z_0)} |\nabla u|^2)^{\frac{1}{2}} = \lim_{r \downarrow 0} \|\nabla u\|_{L_x^p L_t^q(P_r(z_0))} = 0. \tag{4.2}
$$

Therefore, for any small $\epsilon_0 > 0$ there exists $0 < r_0 < \frac{\sqrt{t_0}}{2}$ such that

$$
r^{-n} \int_{P_r(z_0)} |\nabla u|^2 \leq \|\nabla u\|^2_{L_x^p L_t^q(P_r(z_0))} \leq \epsilon_0^2,\ \forall\ 0 < r \leq 2r_0. \tag{4.3}
$$

Now we need

$$
r^{2-n} \int_{P_r(z_0)} |u_t|^2 \leq C\epsilon_0^2,\ \forall\ 0 < r \leq r_0. \tag{4.4}
$$

For $p > n \geq 4$ or $n = 2, 3$ and $p \geq 4$, since $\nabla u \in L_x^p L_t^q(M \times [0, T])$, it follows that for a.e. $t \in [0, T]$, $|\nabla u(t)| \in L^{\frac{p}{2}}(M) \subset L^2(M)$, $|\Delta u(t)| \leq |u_t(t)| + C|\nabla u(t)|^2 \in L^2(M)$. Hence $\nabla^2 u(t) \in L^2(M)$ for a.e. $t \in [0, T]$, and hence Lemma 2.2 is applicable and u satisfies (2.7). Hence (4.4) follows.

Based on (4.3) and (4.4), we can adapt the proof of Lemma 3.1 and Proposition 3.2 to show $u \in C^\infty(P_{\frac{r_0}{2}}(z_0), N)$ as follows. First, (4.3) and (4.4) imply

$$
[u]_{\mathrm{BMO}(P_{\frac{r_0}{2}}(z_0))} \leq C\{r_0^{-n} \int_{P_{r_0}(z_0)} (|\nabla u|^2 + r_0^2 |u_t|^2)\}^{\frac{1}{2}} \leq C\epsilon_0. \tag{4.5}
$$

As in §3, we have, for any $1 < p < +\infty$ and $P_r(z) \subset P_{\frac{r_0}{2}}(z_0)$,

$$
\{r^{-(n+2)} \int_{P_r(z)} |u - u_{z,r}|^p\}^{\frac{1}{p}} \leq C(n, p)\{r^{-(n+2)} \int_{P_r(z)} |u - u_{z,r}|^2\}^{\frac{1}{2}}. \tag{4.6}
$$

For any fixed $P_r(z) \subset P_{\frac{r_0}{2}}(z_0)$, let $v : P_r(z) \to \mathbf{R}^k$ solve

$$
v_t - \Delta v = 0 \text{ in } P_r(z);\ v = u \text{ on } \partial_p P_r(z). \tag{4.7}
$$

We divide the proof into two cases.

Case 1. $n < p < +\infty$ and $2 < q < +\infty$. Notice $q = \frac{2p}{p-n}$. As in (3.8) and (3.9), we have

$$\begin{aligned}
&\int_{P_r(z)} |\nabla(u-v)|^2 \\
&\leq C\int_{t-r^2}^{t} \|\nabla u(s)|_{L^2(B_r(x))}\|\nabla u(s)\|_{L^p(B_r(x))}\|(u-v)(s)\|_{L^{\frac{2p}{p-2}}(B_r(x))} \\
&\leq C\|\nabla u\|_{L^2(P_r(z))}\|\nabla u\|_{L^p_xL^q_t(P_r(z))}\|u-v\|_{L_x^{\frac{2p}{p-2}}L_t^{\frac{2q}{q-2}}(P_r(z))} \\
&\leq Cr^{\frac{n+2}{2}-\frac{n+2}{l}-1}\|\nabla u\|_{L^2(P_r(z))}\|\nabla u\|_{L^p_xL^q_t(P_r(z))}\|u-v\|_{L^l(P_r(z))} \\
&\leq Cr^{-1}\|\nabla u\|_{L^p_xL^q_t(P_r(z))}\|\nabla u\|_{L^2(P_r(z))}\|u-v\|_{L^2(P_r(z))} \\
&\leq C\|\nabla u\|_{L^p_xL^q_t(P_r(z))}\|\nabla u\|_{L^2(P_r(z))}\|\nabla(u-v)\|_{L^2(P_r(z))}, \qquad (4.8)
\end{aligned}$$

where $l = \max\{\frac{2p}{p-2}, \frac{2q}{q-2}\}$, and we have used (4.6) and Poincaré inequality:

$$\int_{P_r(z)} |u-v|^2 \leq Cr^2\int_{P_r(z)} |\nabla(u-v)|^2 \qquad (4.9)$$

in the last step.

Case 2. $p = +\infty$ and $q = 2$. Lemma 2.1 and 2.2 imply

$$\sup_{s\in[t-r^2,t]}\int_{B_r(x)} |\nabla u(s)|^2 \leq Cr^{-2}\int_{P_{2r}(z)} |\nabla u|^2. \qquad (4.10)$$

Similar to (4.8), we have, by (4.9) and (4.10),

$$\begin{aligned}
\int_{P_r(z)} |\nabla(u-v)|^2 &\leq C\int_{t-r^2}^{t} \|\nabla u(s)\|_{L^2(B_r(x))}\|\nabla u(s)\|_{L^\infty(B_r(x))}\|(u-v)(s)\|_{L^2(B_r(x))} \\
&\leq C\|\nabla u\|_{L^2_xL^\infty_t(P_r(z))}\|\nabla u\|_{L^\infty_xL^2_t(P_r(z))}\|u-v\|_{L^2(P_r(z))} \\
&\leq Cr^{-1}\|\nabla u\|_{L^2(P_{2r}(z))}\|\nabla u\|_{L^\infty_xL^2_t(P_r(z))}\|u-v\|_{L^2(P_r(z))} \\
&\leq C\|\nabla u\|_{L^2(P_{2r}(z))}\|\nabla u\|_{L^\infty_xL^2_t(P_r(z))}\|\nabla(u-v)\|_{L^2(P_r(z))}. \qquad (4.11)
\end{aligned}$$

Combining (4.8) with (4.11), we have

$$\begin{aligned}
r^{-n}\int_{P_r(z)} |\nabla(u-v)|^2 &\leq C\|\nabla u\|^2_{L^p_xL^q_t(P_r(z))}(2r)^{-n}\int_{P_{2r}(z)} |\nabla u|^2 \qquad (4.12) \\
&\leq C\epsilon_0^2(2r)^{-n}\int_{P_{2r}(z)} |\nabla u|^2.
\end{aligned}$$

This, combined with (3.11), implies that there is $\theta_0 \in (0, \frac{1}{2})$ such that

$$(\theta_0 r)^{-n} \int_{P_{\theta_0 r}(z)} |\nabla u|^2 \leq C(\theta_0^{-n}\epsilon_0^2 + \theta_0^2) r^{-n} \int_{P_r(z)} |\nabla u|^2$$

$$\leq \frac{1}{2} r^{-n} \int_{P_r(z)} |\nabla u|^2, \tag{4.13}$$

provided that θ_0 and ϵ_0 are chosen to be sufficiently small. Applying (4.13) repeatedly and using Proposition 3.2, we conclude that $u \in C^\infty(P_{\frac{r_0}{2}}(z_0), N)$. Hence the proof of theorem 1.5 is complete. ■

Acknowledgement. The author thanks Professor F.H.Lin for his suggestions on this work. This work is partially supported by NSF grant 0400718.

REFERENCES

[CDY] K. C. Chang, W. Y. Ding, R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces.* J. Differential Geom. 36 (1992), no. 2, 507–515.

[CHH] Y. M. Chen, M. C. Hong, N. Hungerbhler, *Heat flow of p-harmonic maps with values into spheres.* Math. Z. 215 (1994), no. 1, 25–35.

[CLL] Y. Chen, J. Li, F. H. Lin, *Partial Regularity for Weak Heat Flows into Spheres.* Comm. Pure Appl. Math. XLVIII, 429–448 (1995).

[ES] J. Eells, J. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. 86 1964 109–160.

[ESS1] L. Escauriaza, G. Seregin, V. Sverák, *Backward uniqueness for parabolic equations.* Arch. Ration. Mech. Anal. 169 (2003), no. 2, 147–157.

[ESS2] L. Escauriaza, G. Seregin, V. Sverák, $L_{3,\infty}$ *solutions of the Navier-Stokes equations and backward uniqueness.* Russian Math. Surveys 58, No. 2 (2003), 211-250.

[F] M. Feldman, *Partial Regularity for Harmonic Maps of Evolution into Spheres.* Comm. PDE. 19, 761–790 (1994).

[JN] F. John, L. Nirenberg, *On functions of bounded mean oscillation.* Comm. Pure Appl. Math. 14 1961 415–426.

[Lj] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace.* Acta Math. 63 (1934), 193-248.

[Lg] G. Lieberman, Second order parabolic differential equations. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. xii+439 pp.

[LW] F. H. Lin, C. Y. Wang, *Harmonic and quasi-harmonic spheres.* Comm. Anal. Geom. 7 (1999), no. 2, 397–429.

[Lo] O. Ladyzhenskaya, *Uniqueness and smoothness of generalized solutions of Navier-Stokes equations.* Zap. Nauv cn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 5 1967 169–185.

[P] G. Prodi, *Un teorema di unicit per le equazioni di Navier-Stokes.* Ann. Mat. Pura Appl. (4) 48 1959 173–182.

[Sj] J. Serrin, *The initial value problem for the Navier-Stokes equations.* 1963 Nonlinear Problems (Proc. Sympos., Madison, Wis. pp. 69–98 Univ. of Wisconsin Press, Madison, Wis.

[Sm] M. Struwe, *On the evolution of harmonic maps in higher dimensions.* J. Differential Geom. 28 (1988), no. 3, 485–502.