MA 523/Spring 2017
Homework 2
(Due Thursday, February 9 in class or before 3pm in MATH 714)

1. Modify the proof of the mean value formulas to show that for $n \geq 3$, it holds that

$$u(0) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g\, d\sigma + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \Big(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\Big) f\, dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

2. We say $v \in C^2(\overline{U})$ is subharmonic if

$$-\Delta v \leq 0 \text{ in } U.$$

   (a) Prove for subharmonic v that

$$v(x) \leq \frac{1}{\alpha(n)r^n} \int_{B(x,r)} v\, dy \text{ for all } B(x,r) \subset U.$$

   (b) Prove that $\max_{\overline{U}} v = \max_{\partial U} v$.

   (c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v = \phi(u)$. Prove v is subharmonic.

   (d) Prove $v = |Du|^2$ is subharmonic, whenever u is harmonic.

3. Prove that there exists a constant C, depending only on n, such that

$$\max_{B(0,1)} |u| \leq C\big(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|\big)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

4. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2}\frac{r + |x|}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B(0,r)$.

5. If $u \in C^2(\mathbb{R}^n)$ is a harmonic function, which satisfies the growth condition:

$$|u(x)| \leq C(1 + |x|^k), \ \forall\, x \in \mathbb{R}^n,$$

for some constant $C > 0$ and a positive integer k, prove that u is a polynomial of degree at most k.