MA 523/Spring 2017

Homework 3

(Due Thursday, February 23 in class or before 3pm in MATH 714)

1. Let u be the solution of
$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{R}^n_+ \\ u = g \text{ on } \partial\mathbb{R}^n_+ \end{cases}$$
given by Poisson's formula for $\mathbb{R}^n_+$. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show Du is not bounded near $x = 0$.

2. Let U^+ denote the open half-ball $B(0,1) \cap \mathbb{R}^n_+$. Assume $u \in C(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set
$$v(x) = \begin{cases} u(x) & if\ x_n \geq 0 \\ -u(x_1, \cdots, x_{n-1}, -x_n) & if\ x_n < 0 \end{cases}$$
for $x \in U = B(0,1)$. Prove v is harmonic in $B(0,1)$.

3. Show that for $n = 2$ the function
$$v = \frac{1}{8\pi} r^2 \log r,\ \ r = |x - \xi|$$
is a fundamental solution for the operator Δ^2.

4. Find all solutions with spherical symmetry of the biharmonic equation $\Delta^2 u = 0$ in n dimensions.

5. Let $L = \Delta + c$ in $n = 3$ dimensions, where c is a positive constant.
(a) Find all solutions of $Lu = 0$ with spherical symmetry.
(b) Prove that
$$K(x, \xi) = -\frac{\cos(\sqrt{c}r)}{4\pi r},\ \ r = |x - \xi|$$
is a fundamental solution for L with pole ξ.
(c) Show that a solution u of $Lu = 0$ in $B(\xi, \rho)$ for $\sin(\sqrt{c}\rho) \neq 0$ has the modified mean value property
$$u(\xi) = \frac{\sqrt{c}\rho}{\sin(\sqrt{c}\rho)} \frac{1}{4\pi\rho^2} \int_{\partial B(\xi,\rho)} u(x)\, d\sigma(x).$$