MA 523/Spring 2017

Homework 4

(Due Thursday, March 9 in class or before 3pm in MATH 714)

1. Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.
   (i) Show $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves the heat equaltion for each $\lambda \in \mathbb{R}$.
   (ii) Use (i) to show $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well

2. Assume $n = 1$ and $u(x,t) = v\left(\frac{x^2}{t}\right)$.
   (a) Show $u_t = u_{xx}$ if and only if
   $$4zv''(z) + (2+z)v'(z) = 0 (z > 0). \tag{1}$$
   (b) Show that the general solution of (1) is
   $$v(z) = c\int_0^z e^{-\frac{s}{4}} s^{-\frac{1}{2}}\, ds + d.$$
   (c) Differentiate $v(\frac{x^2}{t})$ with respect to x and select the constant c properly, so as to obtain the fundamental solution Φ for $n = 1$.

3. Write down an explicit formula for a solution of
   $$\begin{cases} u_t - \Delta u + cu = f & in\ \mathbb{R}^n \times (0,\infty) \\ u = g & on\ \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{2}$$
   where $c \in \mathbb{R}$.

4. Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula
   $$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s)\, ds$$
   for a solution of the initial/boundary-value problem
   $$\begin{cases} u_t - u_{xx} = 0 & in\ \mathbb{R}_+ \times (0,\infty) \\ u = 0 & on\ \mathbb{R}_+ \times \{t = 0\} \\ u = g & on\ \{x = 0\} \times [0, \infty). \end{cases}$$
   (Hint: Let $v(x,t) = u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.)

5. We say $v \in C_1^2(\mathbb{R}^n \times (0,T))$ is a subsolution of the heat equation if
   $$v_t - \Delta v \leq 0 \text{ in } \mathbb{R}^n \times (0,T).$$
   Show that
   (i) if u is a solution of the heat equation and $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and convex, then $v = \phi(u)$ is a subsolution to the heat equation.
   (ii) if u solves the heat equation, then $v := |Du|^2 + u_t^2$ is a subsolution of the heat equation.

6. Let $u_1(x,t),\cdots,u_n(s,t)$ be n solutions of $u_t = u_{ss}$. Prove that

$$u(x,t) = u(x_1,\cdots,x_n,t) = \Pi_{k=1}^n u_k(x_k,t)$$

solves $u_t = \Delta u$ in $\mathbb{R}^n \times (0,+\infty)$.

7. Let $n = 1$ and μ be a positive constant. Let $u(x,t)$ be a positive solution of class C^2 of

$$u_t = \mu u_{xx}, \text{ in } \mathbb{R} \times (0,\infty).$$

Show that $\theta = -\frac{2\mu u_x}{u}$ solves the Burger equation:

$$\theta_t + \theta\theta_x = \mu\theta_{xx}.$$

8. Define for $x, y, t \in \mathbb{R}$, $t \neq 0$

$$K(x,y,t) = (4\pi|t|)^{-\frac{1}{2}} \exp(-\frac{(x-y)^2}{4t}).$$

Show that

$$K(x,0,s+t) = \int K(x,y,t)K(y,0,s)\,dy$$

holds
(a) when $s > 0, t > 0$
(b) when $0 < t < -s$.