

# WELL-POSEDNESS OF NEMATIC LIQUID CRYSTAL FLOW IN $L^3_{\text{uloc}}(\mathbb{R}^3)$

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ABSTRACT. In this paper, we establish the local well-posedness for the Cauchy problem of the simplified version of hydrodynamic flow of nematic liquid crystals (1.1) in  $\mathbb{R}^3$  for any initial data  $(u_0, d_0)$  having small  $L^3_{\text{uloc}}$ -norm of  $(u_0, \nabla d_0)$ . Here  $L^3_{\text{uloc}}(\mathbb{R}^3)$  is the space of uniformly locally  $L^3$ -integrable functions. For any initial data  $(u_0, d_0)$  with small  $\|(u_0, \nabla d_0)\|_{L^3(\mathbb{R}^3)}$ , we show that there exists a unique, global solution to (1.1) which is smooth for  $t > 0$  and has monotone decreasing  $L^3$ -energy for  $t \geq 0$ .

## 1. INTRODUCTION

In this paper, we consider the Cauchy problem for the following hydrodynamic system modeling the flow of nematic liquid crystal materials in  $\mathbb{R}^3$ : for  $0 < T \leq \infty$  and  $(u, P, d) : \mathbb{R}^3 \times [0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$ , the system is given by

$$\begin{cases} u_t + u \cdot \nabla u - \nu \Delta u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d), & \text{in } \mathbb{R}^3 \times (0, T), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3 \times (0, T), \\ d_t + u \cdot \nabla d = \gamma(\Delta d + |\nabla d|^2 d), & \text{in } \mathbb{R}^3 \times (0, T), \\ (u, d) = (u_0, d_0), & \text{on } \mathbb{R}^3 \times \{0\}, \end{cases} \quad (1.1)$$

for a given initial data  $(u_0, d_0) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times S^2$  with  $\nabla \cdot u_0 = 0$ . Here  $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  represents the velocity field of the fluid,  $d : \mathbb{R}^3 \rightarrow S^2$  – the unit sphere in  $\mathbb{R}^3$  – is a unit vector field representing the macroscopic molecular orientation of the nematic liquid crystal material,  $P : \mathbb{R}^3 \rightarrow \mathbb{R}$  represents the pressure function. The constants  $\nu, \lambda$ , and  $\gamma$  are positive constants that represent the viscosity of the fluid, the competition between kinetic and potential energy, and the microscopic elastic relaxation time for the molecular orientation field.  $\nabla \cdot$  denotes the divergence operator in  $\mathbb{R}^3$ , and  $\nabla d \odot \nabla d$  denotes the symmetric  $3 \times 3$  matrix:

$$(\nabla d \odot \nabla d)_{ij} = \langle \nabla_i d, \nabla_j d \rangle, \quad 1 \leq i, j \leq 3.$$

Throughout this paper, we denote  $\langle v, w \rangle$  or  $v \cdot w$  as the inner product in  $\mathbb{R}^3$  for  $v, w \in \mathbb{R}^3$ .

The system (1.1) is a simplified version of the famous Ericksen-Leslie model for the hydrodynamics of nematic liquid crystals developed by Ericksen and Leslie during the period of 1958 through 1968 [6, 15, 4]. This system reduces to the Oseen-Frank model in the static theory of liquid crystals. It is a macroscopic continuum description of the time evolution of the materials under the influence of flow field  $u$  and the macroscopic description of the microscopic orientation field  $d$  of rod-like liquid crystals. The current form of system (1.1) was first proposed by Lin [17] back in the late 1980's. From the mathematical point of view, (1.1) is a system coupling the non-homogeneous incompressible Navier-Stokes equation and the transported heat flow of harmonic maps to  $S^2$ . Lin-Liu [19, 20] initiated the mathematical analysis of (1.1) by considering its Ginzburg-Landau approximation or the so-called orientation with variable degrees in the terminology of Ericksen. Namely, the Dirichlet energy  $\int \frac{1}{2} |\nabla d|^2$  for  $d : \mathbb{R}^3 \rightarrow S^2$  is replaced by the Ginzburg-Landau energy  $\int \frac{1}{2} |\nabla d|^2 + \frac{1}{4\epsilon^2} (1 - |d|^2)^2$  ( $\epsilon > 0$ ) for  $d : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . Hence (1.1)<sub>3</sub> is replaced by

$$\partial_t d + u \cdot \nabla d = \gamma(\Delta d + \frac{1}{\epsilon^2} (1 - |d|^2) d). \quad (1.2)$$

Lin-Liu proved in [19, 20] (i) the existence of a unique, global smooth solution in dimension two and in dimension three under large viscosity  $\nu$ ; and (ii) the existence of suitable weak solutions and their partial

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regularity in dimension three, analogous to the celebrated regularity theorem by Caffarelli-Kohn-Nirenberg [3] for the three-dimensional incompressible Navier-Stokes equation.

As already pointed out by [19, 20], it is a very challenging problem to study the convergence of solutions  $(u_\epsilon, P_\epsilon, d_\epsilon)$  to (1.1)<sub>1</sub>-(1.1)<sub>2</sub>-(1.2) when  $\epsilon \downarrow 0$ . In particular, the existence of global Leray-Hopf type weak solutions to the initial and boundary value problem of (1.1) has only been established recently by Lin-Lin-Wang [21] in dimension two, see also Hong [9] and Xu-Zhang [29] and Hong-Xin [12] for related works.

Because of the super-critical nonlinear term  $\nabla \cdot (\nabla d \odot \nabla)$  in (1.1)<sub>1</sub>, it has been an outstanding open problem whether there exists a global Leray-Hopf type weak solution to (1.1) in  $\mathbb{R}^3$  for any initial data  $(u_0, d_0) \in L^2(\mathbb{R}^3, \mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3, S^2)$  with  $\nabla \cdot u_0 = 0$ . It is standard that in  $\mathbb{R}^3$  the local existence of a unique, strong solution to (1.1) can be obtained for any initial data  $u_0 \in W^{s,2}(\mathbb{R}^3)$  and  $d_0 \in W^{s+1,2}(\mathbb{R}^3, S^2)$  for  $s > 3$  with  $\nabla \cdot u_0 = 0$ , see for example [28]. A blow-up criterion for local strong solutions to (1.1), similar to the Beale-Kato-Majda criterion for the Navier-Stokes equation (see [1]), was obtained by Huang-Wang [11]. For small initial data in certain Besov spaces, Li-Wang [23] obtained the global existence of strong solutions to (1.1). We would like to mention that Wang [27] has recently obtained the global (or local) well-posedness of (1.1) for initial data  $(u_0, d_0)$  belonging to possibly the largest space  $\text{BMO}^{-1} \times \text{BMO}$  with  $\nabla \cdot u_0 = 0$ , which is an invariant space under parabolic scaling associated with (1.1), with small norms.

In this paper, we are mainly interested in the local well-posedness of (1.1) for any initial data  $(u_0, d_0)$  such that  $(u_0, \nabla d_0) \in L^3_{\text{uloc}}(\mathbb{R}^3)$ . Henceforth  $L^3_{\text{uloc}}(\mathbb{R}^3)$  denotes the space of uniformly locally  $L^3$ -integrable functions. It turns out that  $L^3_{\text{uloc}}(\mathbb{R}^3)$  is also invariant under parabolic scaling associated with (1.1).

Now we give the definition of  $L^3_{\text{uloc}}(\mathbb{R}^3)$ . The readers can consult the monograph by Lemarié-Rieusset [16] for applications of the space  $L^3_{\text{uloc}}(\mathbb{R}^3)$  to the Navier-Stokes equation.

**Definition 1.1.** *A function  $f \in L^3_{\text{loc}}(\mathbb{R}^3)$  belongs to the space  $L^3_{\text{uloc}}(\mathbb{R}^3)$  consisting of uniformly locally  $L^3$ -integrable functions, if there exists  $0 < R < +\infty$  such that*

$$\|f\|_{L^3_R(\mathbb{R}^3)} := \sup_{x \in \mathbb{R}^3} \left( \int_{B_R(x)} |f|^3 \right)^{\frac{1}{3}} < +\infty. \quad (1.3)$$

It is clear that

- $L^3(\mathbb{R}^3) \subset L^3_{\text{uloc}}(\mathbb{R}^3)$ .
- If  $f \in L^3_{\text{uloc}}(\mathbb{R}^3)$ , then  $\|f\|_{L^3_R(\mathbb{R}^3)}$  is finite for any  $0 < R < +\infty$ . For any two  $0 < R_1 \leq R_2 < \infty$ , it holds

$$\|f\|_{L^3_{R_1}(\mathbb{R}^3)} \leq \|f\|_{L^3_{R_2}(\mathbb{R}^3)} \lesssim \left( \frac{R_2}{R_1} \right) \|f\|_{L^3_{R_1}(\mathbb{R}^3)}, \quad \forall f \in L^3_{\text{uloc}}(\mathbb{R}^3). \quad (1.4)$$

- $L^3_{\text{uloc}}(\mathbb{R}^3) \subset \bigcap_{0 < R < \infty} \text{BMO}_R^{-1}(\mathbb{R}^3)$  (see [13] or [27]). Moreover, for any  $0 < R < \infty$ , it holds

$$\|f\|_{\text{BMO}_R^{-1}(\mathbb{R}^3)} \lesssim \|f\|_{L^3_R(\mathbb{R}^3)}, \quad \forall f \in L^3_{\text{uloc}}(\mathbb{R}^3). \quad (1.5)$$

Throughout this paper, we write  $A \lesssim B$  if there exists a universal constant  $C > 0$  such that  $A \leq CB$ . Here are a few more notations and conventions that we will use through this paper. For two matrices  $M, N$  of order 3, we use  $M : N = \sum_{1 \leq i, j \leq 3} M^{ij} N^{ij}$  to denote their scalar product. For two vectors  $u, v \in \mathbb{R}^3$ , we

let  $u \otimes v$  denote their tensor product:  $(u \otimes v)_{ij} = u^i v^j$ ,  $1 \leq i, j \leq 3$ . For  $0 < s < +\infty$  and  $1 \leq p \leq \infty$ , we denote by  $W^{s,p}(\mathbb{R}^3)$  and  $\dot{W}^{s,p}(\mathbb{R}^3)$  as the Sobolev space and the homogeneous Sobolev spaces respectively. For  $0 \leq a < b < +\infty$ , denote

$$C_b^\infty(\mathbb{R}^3 \times [a, b]) = \bigcap_{m \geq 0} \left\{ f \in C^m(\mathbb{R}^3 \times [a, b]) : \|f\|_{C^m(\mathbb{R}^3 \times [a, b])} < +\infty \right\},$$

$$L^\infty([a, b], L^3_{\text{uloc}}(\mathbb{R}^3)) = \left\{ f \in L^\infty([a, b], L^3_1(\mathbb{R}^3)) \right\},$$

and

$$C_*^0([a, b], L^3_{\text{uloc}}(\mathbb{R}^3)) = \left\{ f \in C([a, b], L^3_1(\mathbb{R}^3)) \cap L^\infty([a, b], L^3_1(\mathbb{R}^3)) : \text{as } t \downarrow 0, f(t) \rightarrow f(a) \text{ in } L^3_{\text{loc}}(\mathbb{R}^3) \right\}.$$

Repeated indices are summed unless specified otherwise. Upper indices denote components and lower indices denote derivatives.

Now we state our main theorem.

**Theorem 1.2.** *There exist  $\epsilon_0 > 0$  and  $\tau_0 > 0$  such that if  $u_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , with  $\nabla \cdot u_0 = 0$ , and  $d_0 : \mathbb{R}^3 \rightarrow S^2$  satisfies  $(d_0 - e_0) \in L^3(\mathbb{R}^3)$  for some  $e_0 \in S^2$ , and*

$$\| |(u_0, \nabla d_0)| \|_{L^3_R(\mathbb{R}^3)} := \sup_{x \in \mathbb{R}^3} \left( \int_{B_R(x)} |u_0|^3 + |\nabla d_0|^3 \right)^{\frac{1}{3}} \leq \epsilon_0 \quad (1.6)$$

for some  $0 < R < \infty$ , then there exist  $T_0 \geq \tau_0 R^2$  and a unique solution  $(u, d) : \mathbb{R}^3 \times [0, T_0) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  of (1.1) such that the following properties hold:

(i) For  $t \downarrow 0$ ,  $(u(t), d(t)) \rightarrow (u_0, d_0)$  and  $\nabla d(t) \rightarrow \nabla d_0$  in  $L^3_{\text{loc}}(\mathbb{R}^3)$ .

(ii)

$$(u, d) \in \bigcap_{0 < \delta < T_0} C_b^\infty(\mathbb{R}^3 \times [\delta, T_0 - \delta], \mathbb{R}^3 \times S^2), \quad (u, \nabla d) \in \bigcap_{0 < T' < T_0} C_*^0([0, T'], L^3_{\text{uloc}}(\mathbb{R}^3)).$$

(iii)

$$\| |(u(t), \nabla d(t)) | \|_{L^\infty([0, \tau_0 R^2], L^3_R(\mathbb{R}^3))} \leq C \epsilon_0. \quad (1.7)$$

(iv) If  $T_0 < +\infty$  is the maximum time interval then it must hold

$$\limsup_{t \uparrow T_0} \| |(u(t), \nabla d(t)) | \|_{L^3_r(\mathbb{R}^3)} > \epsilon_0, \quad \forall 0 < r < \infty. \quad (1.8)$$

The ideas to prove Theorem 1.2 are motivated by those employed by [21]. There are five main ingredients, which include

- approximate  $(u_0, d_0)$  by smooth  $(u_0^k, d_0^k)$  (see Lemma 5.1 below) and obtain  $0 < T_k < +\infty$  and a sequence of smooth solutions  $(u^k, P^k, d^k)$  of (1.1) in  $\mathbb{R}^3 \times [0, T_k]$ , under the initial data  $(u_0^k, d_0^k)$ ;
- utilizing the local  $L^3$ -energy inequality (3.1), obtain uniform lower bounds of  $T_k$ ;
- apply the  $\epsilon_0$ -regularity Theorem 4.4 to obtain a priori derivative estimates of  $(u^k, d^k)$  and then take limit to obtain the local existence of  $L^3_{\text{uloc}}$ -solutions to (1.1);
- apply Theorem 4.4 again to characterize the finite maximal time interval; and
- adapt the proof of [27] to show the uniqueness.

For a solution  $(u, P, d)$  to (1.1), denote its  $L^3$ -energy by

$$E_3(u, \nabla d)(t) = \int_{\mathbb{R}^3} (|u(t)|^3 + |\nabla d(t)|^3), \quad t \geq 0.$$

Concerning the global well-posedness of (1.1), we have

**Theorem 1.3.** *There exists an  $\epsilon_0 > 0$  such that if  $(u_0, d_0) \in L^3(\mathbb{R}^3, \mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3, S^2)$ , with  $\nabla \cdot u_0 = 0$ , satisfies*

$$E_3(u_0, \nabla d_0) \leq \epsilon_0^3, \quad (1.9)$$

then there exists a unique global solution  $(u, d) : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  of (1.1) such that  $(u, d) \in C^\infty(\mathbb{R}^3 \times (0, +\infty)) \cap C([0, \infty), L^3(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3))$ ,  $E_3(u, \nabla d)(t)$  is monotone decreasing for  $t \geq 0$ , and

$$\| \nabla^m u(t) \|_{L^\infty(\mathbb{R}^3)} + \| \nabla^{m+1} d(t) \|_{L^\infty(\mathbb{R}^3)} \leq \frac{C \epsilon_0}{t^{\frac{m}{2}}}, \quad \forall t > 0, m \geq 0. \quad (1.10)$$

We mention here that the first conclusion of Theorem 1.3 has been proven by [5], which is based on refinement of the argument by Wang [27]. Since the exact values of  $\nu, \lambda, \gamma$  don't play a role in this paper, we henceforth assume

$$\nu = \lambda = \gamma = 1.$$

The paper is written as follows. In §2, we derive an inequality for the global  $L^3$ -energy of smooth solutions of (1.1). In §3, we derive an inequality for the local  $L^3$ -energy of smooth solutions of (1.1) and prove Theorem 1.3. In §4, we will prove an  $\epsilon_0$ -regularity for suitable weak solutions to (1.1). In particular, a priori derivative estimates hold for smooth solutions to (1.1) under a smallness condition. In §5, we will prove Theorem 1.2.

2. INEQUALITY ON THE GLOBAL  $L^3$ -ENERGY AND PROOF OF THEOREM 1.3

In this section, we will derive an inequality for the  $L^3$ -energy  $E_3(u, \nabla d)(t)$  for any smooth solution  $(u, d) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$ , for  $0 < T \leq \infty$ , of the system (1.1) for nematic liquid crystals.

**Lemma 2.1.** *There exists  $C > 0$  such that for  $0 < T \leq \infty$  if  $(u, d) \in C^\infty(\mathbb{R}^3 \times [0, T], \mathbb{R}^3 \times \mathbb{S}^2) \cap C([0, T], L^3(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3))$  and  $P \in L^\infty([0, T], L^{\frac{3}{2}}(\mathbb{R}^3))$  solves (1.1), then it holds*

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} (|u|^3 + |\nabla d|^3) + \left[ 1 - C \|u\|_{L^3(\mathbb{R}^3)}^2 \right] \int_{\mathbb{R}^3} |u| |\nabla u|^2 \\ & + \left[ 1 - C (\|u\|_{L^3(\mathbb{R}^3)} + \|u\|_{L^3(\mathbb{R}^3)} \|\nabla d\|_{L^3(\mathbb{R}^3)} + \|\nabla d\|_{L^3(\mathbb{R}^3)}^2) \right] \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \leq 0. \end{aligned} \quad (2.1)$$

*Proof.* Taking spatial derivatives of (1.1)<sub>3</sub>, multiplying the resulting equation by  $|\nabla d| \nabla d$ , and integrating over  $\mathbb{R}^3$ , we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{3} |\nabla d|^3 = \int_{\mathbb{R}^3} \nabla(\Delta d) : |\nabla d| \nabla d - \int_{\mathbb{R}^3} \nabla(u \cdot \nabla d) : |\nabla d| \nabla d - \int_{\mathbb{R}^3} \nabla(|\nabla d|^2 d) : |\nabla d| \nabla d. \quad (2.2)$$

For terms on the right hand side of (2.2), by integration by parts we have

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla(\Delta d) : |\nabla d| \nabla d &= - \int_{\mathbb{R}^3 \cap \{|\nabla d| > 0\}} \nabla^2 d : \nabla(|\nabla d| \nabla d) \\ &= - \int_{\mathbb{R}^3 \cap \{|\nabla d| > 0\}} (|\nabla d| |\nabla^2 d|^2 + \frac{|\nabla^2 d \cdot \nabla d|^2}{|\nabla d|}) \\ &\leq - \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2, \\ \int_{\mathbb{R}^3} \nabla(u \cdot \nabla d) : |\nabla d| \nabla d &= - \int_{\mathbb{R}^3} (u \cdot \nabla d) \cdot ((\nabla|\nabla d|) \cdot \nabla d + |\nabla d| \Delta d) \\ &\lesssim \int_{\mathbb{R}^3} |u| |\nabla d|^2 |\nabla^2 d|, \end{aligned}$$

and, using  $|d| = 1$ ,

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla(|\nabla d|^2 d) : |\nabla d| \nabla d &= \int_{\mathbb{R}^3} (\nabla|\nabla d|^2) \cdot |\nabla d| \nabla \left( \frac{|d|^2}{2} \right) + \int_{\mathbb{R}^3} |\nabla d|^2 \nabla d : |\nabla d|^2 \nabla d \\ &= \int_{\mathbb{R}^3} |\nabla d|^5. \end{aligned}$$

Putting these estimates into (2.2) yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla d|^3 + \int_{\mathbb{R}^3} |\nabla(|\nabla d|^{\frac{3}{2}})|^2 \lesssim \int_{\mathbb{R}^3} |\nabla d|^5 + |u| |\nabla d|^2 |\nabla^2 d|, \quad (2.3)$$

where we have used the following variant of the Kato inequality

$$|\nabla|\nabla d|^{\frac{3}{2}}| = \frac{3}{2} |\nabla d|^{\frac{1}{2}} |\nabla|\nabla d|| \leq \frac{3}{2} |\nabla d|^{\frac{1}{2}} |\nabla^2 d|.$$

Observe that by the Sobolev inequality and the Kato inequality above, we have

$$\int_{\mathbb{R}^3} |\nabla d|^9 = \int_{\mathbb{R}^3} (|\nabla d|^{\frac{3}{2}})^6 \lesssim \left( \int_{\mathbb{R}^3} |\nabla|\nabla d|^{\frac{3}{2}}|^2 \right)^3 \lesssim \left( \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \right)^3. \quad (2.4)$$

Hence, by the Hölder inequality and (2.4), we have

$$\|\nabla d\|_{L^5(\mathbb{R}^3)}^5 \leq \|\nabla d\|_{L^3(\mathbb{R}^3)}^2 \|\nabla d\|_{L^9(\mathbb{R}^3)}^3 \lesssim \left( \int_{\mathbb{R}^3} |\nabla d|^3 \right)^{2/3} \left( \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \right).$$

For the second term on the right-hand side of (2.3), by the Hölder inequality and (2.4) we have

$$\begin{aligned} \int_{\mathbb{R}^3} |u| |\nabla d|^2 |\nabla^2 d| &\leq \|u\|_{L^3(\mathbb{R}^3)} \|\nabla d\|_{L^6(\mathbb{R}^3)}^{\frac{3}{2}} \|\nabla d\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 d\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|u\|_{L^3(\mathbb{R}^3)} \|\nabla |\nabla d|^{\frac{3}{2}}\|_{L^2(\mathbb{R}^3)} \|\nabla d\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 d\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|u\|_{L^3(\mathbb{R}^3)} \|\nabla d\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 d\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}}. \end{aligned}$$

Inserting these two estimates into (2.3) yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla d|^3 + \left[1 - C \left( \|\nabla d\|_{L^3(\mathbb{R}^3)}^2 + \|u\|_{L^3(\mathbb{R}^3)} \right)\right] \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \leq 0. \quad (2.5)$$

Next we estimate the  $L^3$ -norm of  $u$ . Multiplying (1.1)<sub>1</sub> by  $|u|u$  and integrating over  $\mathbb{R}^3$  gives

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{3} |u|^3 \\ &= \int_{\mathbb{R}^3} \Delta u \cdot |u|u - \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot |u|u - \int_{\mathbb{R}^3} \nabla P \cdot |u|u - \int_{\mathbb{R}^3} (\nabla \cdot (\nabla d \odot \nabla d)) \cdot |u|u. \end{aligned} \quad (2.6)$$

For the terms on the right hand side of (2.6), by integration by parts we have

$$\begin{aligned} \int_{\mathbb{R}^3} (\Delta u) \cdot |u|u &= - \int_{\mathbb{R}^3} |\nabla u|^2 |u| + |u| |\nabla |u||^2, \\ \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot |u|u &= \int_{\mathbb{R}^3} u \cdot \nabla \left( \frac{|u|^3}{3} \right) = 0, \\ \int_{\mathbb{R}^3} \nabla P \cdot |u|u &= - \int_{\mathbb{R}^3} P u \cdot \nabla |u| + P |u| (\nabla \cdot u) = - \int_{\mathbb{R}^3} P u \cdot \nabla |u|, \end{aligned}$$

and

$$\begin{aligned} - \int_{\mathbb{R}^3} (\nabla \cdot (\nabla d \odot \nabla d)) \cdot |u|u &= \int_{\mathbb{R}^3} (\nabla d \odot \nabla d) : \nabla (|u|u) \\ &= \int_{\mathbb{R}^3} (\nabla d \odot \nabla d) : \nabla |u| \otimes u + |u| (\nabla d \odot \nabla d) : \nabla u \\ &\lesssim \int_{\mathbb{R}^3} |\nabla d|^2 |u| |\nabla u|. \end{aligned}$$

Substituting these estimates into (2.6), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^3 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 |u| \lesssim \int_{\mathbb{R}^3} |P| |u| |\nabla |u|| + \int_{\mathbb{R}^3} |\nabla d|^2 |u| |\nabla u|. \quad (2.7)$$

Using the Kato inequality  $|\nabla |u|| \leq |\nabla u|$ , the Cauchy inequality and the Hölder inequality in (2.7), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^3 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 &\leq C \int_{\mathbb{R}^3} |u| (|P|^2 + |\nabla d|^4) + \frac{1}{2} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \\ &\leq C (\|P\|_{L^3(\mathbb{R}^3)}^2 + \|\nabla d\|_{L^6(\mathbb{R}^3)}^4) \|u\|_{L^3(\mathbb{R}^3)} + \frac{1}{2} \int_{\mathbb{R}^3} |u| |\nabla u|^2. \end{aligned}$$

Therefore we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^3 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 \lesssim (\|P\|_{L^3(\mathbb{R}^3)}^2 + \|\nabla d\|_{L^6(\mathbb{R}^3)}^4) \|u\|_{L^3(\mathbb{R}^3)}. \quad (2.8)$$

We need to estimate  $\|P\|_{L^3(\mathbb{R}^3)}$ . To do so, we take divergence of (1.1)<sub>1</sub> to obtain

$$-\Delta P = \nabla \cdot \nabla \cdot (u \otimes u + \nabla d \odot \nabla d). \quad (2.9)$$

Set

$$g^{jk} := u^j u^k + \nabla_j d \cdot \nabla_k d, \quad 1 \leq j, k \leq 3.$$

Then we have

$$P = \Delta^{-1} (\nabla_{jk}^2 g^{jk}) = -\mathbf{R}_j \mathbf{R}_k (g^{jk}). \quad (2.10)$$

Henceforth  $\mathbf{R}_j = (-\Delta)^{-\frac{1}{2}} \nabla_j$  denotes the  $j^{\text{th}}$ -Riesz transform on  $\mathbb{R}^3$  for  $1 \leq j \leq 3$ .

Since  $\mathbf{R}_j : L^q(\mathbb{R}^3) \rightarrow L^q(\mathbb{R}^3)$  is bounded for  $1 < q < \infty$  (see Stein [25]), we have

$$\|P\|_{L^3(\mathbb{R}^3)} = \|\mathbf{R}_j \mathbf{R}_k(g^{jk})\|_{L^3(\mathbb{R}^3)} \lesssim \|g^{jk}\|_{L^3(\mathbb{R}^3)} \lesssim \|u\|_{L^6(\mathbb{R}^3)}^2 + \|\nabla d\|_{L^6(\mathbb{R}^3)}^2. \quad (2.11)$$

Inserting (2.11) into (2.8) yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^3 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 \lesssim (\|u\|_{L^6(\mathbb{R}^3)}^4 + \|\nabla d\|_{L^6(\mathbb{R}^3)}^4) \|u\|_{L^3(\mathbb{R}^3)}. \quad (2.12)$$

Using the Hölder inequality, the Sobolev inequality, and  $|\nabla |u|^{\frac{3}{2}}| \lesssim |\nabla u| |u|^{\frac{1}{2}}$ , we have

$$\|u\|_{L^6(\mathbb{R}^3)}^4 \leq \|u\|_{L^3(\mathbb{R}^3)} \|u\|_{L^9(\mathbb{R}^3)}^3 \lesssim \|u\|_{L^3(\mathbb{R}^3)} \|\nabla |u|^{\frac{3}{2}}\|_{L^2(\mathbb{R}^3)}^2 \leq \|u\|_{L^3(\mathbb{R}^3)} \int_{\mathbb{R}^3} |u| |\nabla u|^2.$$

Similarly we have

$$\|\nabla d\|_{L^6(\mathbb{R}^3)}^4 \lesssim \|\nabla d\|_{L^3(\mathbb{R}^3)} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2.$$

Substituting these two estimates into (2.12), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^3 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 \lesssim \|u\|_{L^3(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} |u| |\nabla u|^2 + \|u\|_{L^3(\mathbb{R}^3)} \|\nabla d\|_{L^3(\mathbb{R}^3)} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2. \quad (2.13)$$

Combining (2.5) and (2.13) yields (2.1).  $\square$

**Corollary 2.2.** *There exists  $\epsilon_0 > 0$  such that for  $0 < T \leq \infty$ , if  $(u, d) \in C^\infty(\mathbb{R}^3 \times [0, T], \mathbb{R}^3 \times S^2) \cap L^\infty([0, T], L^3(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3))$  is a solution to (1.1) satisfying*

$$E_3(u_0, \nabla d_0) \leq \epsilon_0^3, \quad (2.14)$$

*then  $E_3(u(t), \nabla d(t))$  is monotone decreasing for  $0 \leq t < T$ .*

*Proof.* Denote

$$E_3(t) := E_3(u(t), \nabla d(t)), \quad t \geq 0.$$

Let  $T_{\max} \in [0, T)$  be defined by

$$T_{\max} = \max \left\{ t \in [0, T) : E_3(s) \leq 2\epsilon_0^3, \forall 0 \leq s \leq t \right\}.$$

By continuity and (2.14), we have that  $0 < T_{\max} \leq T$ , and

$$E_3(t) \leq 2\epsilon_0^3, \quad 0 \leq t < T_{\max}, \quad E_3(T_{\max}) = 2\epsilon_0^3. \quad (2.15)$$

Suppose  $T_{\max} < T$ . Choose  $\epsilon_0 > 0$  so small that

$$1 - C\epsilon_0^2 \geq 0 \text{ and } 1 - C(\epsilon_0 + 2\epsilon_0^2) \geq 0.$$

Then (2.15) and (2.1) imply that

$$\frac{d}{dt} E_3(t) \leq \frac{d}{dt} E_3(t) + [1 - C\epsilon_0^2] \int_{\mathbb{R}^3} |u| |\nabla u|^2 + [1 - C(\epsilon_0 + 2\epsilon_0^2)] \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \leq 0$$

holds for  $0 \leq t \leq T_{\max}$ . Hence  $E_3(t)$  is decreasing in  $[0, T_{\max}]$  and

$$E_3(T_{\max}) \leq E_3(0) \leq \epsilon_0^3 < 2\epsilon_0^3.$$

This contradicts the definition of  $T_{\max}$ . Thus  $T_{\max} = T$  and  $E_3(t)$  is monotone decreasing in  $[0, T)$ .  $\square$

**Proof of Theorem 1.3:** Since  $C^\infty(\mathbb{R}^3, S^2)$  is dense in  $\dot{W}^{1,3}(\mathbb{R}^3, S^2)$  (see [24]), it is not hard to show that there exist  $\{(u_0^k, d_0^k)\} \subset C^\infty(\mathbb{R}^3, \mathbb{R}^3) \times C^\infty(\mathbb{R}^3, S^2)$  such that

$$\nabla \cdot u_0^k = 0 \text{ in } \mathbb{R}^3, \quad \lim_{k \rightarrow \infty} (\|u_0^k - u_0\|_{L^3(\mathbb{R}^3)} + \|\nabla(d_0^k - d_0)\|_{L^3(\mathbb{R}^3)}) = 0.$$

Consider the system (1.1) under the initial condition  $(u, d)|_{t=0} = (u_0^k, d_0^k)$ . It is standard that there exist  $T_k > 0$  and smooth solutions  $(u_k, d_k) \in C^\infty(\mathbb{R}^3 \times [0, T_k], \mathbb{R}^3 \times S^2) \cap C([0, T_k], L^3(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3))$  to (1.1).

Since  $E_3(u_0, \nabla d_0) \leq \epsilon_0^3$ , we may assume that  $E_3(u_0^k, \nabla d_0^k) \leq 2\epsilon_0^3$  for all  $k \geq 1$ . Hence by Corollary 2.2, we conclude that

$$\sup_{0 \leq t \leq T_k} E_3(u^k(t), \nabla d^k(t)) \leq E_3(u_0^k, \nabla d_0^k) \leq 2\epsilon_0^3, \quad \forall k \geq 1.$$

For the corresponding pressure functions  $P^k$ , since

$$\Delta P^k = -\nabla \cdot \nabla \cdot (u^k \otimes u^k + \nabla d^k \odot \nabla d^k) \text{ in } \mathbb{R}^3,$$

we have

$$\sup_{0 \leq t \leq T_k} \|P^k\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim \sup_{0 \leq t \leq T_k} (\|u^k\|_{L^3(\mathbb{R}^3)}^2 + \|\nabla d^k\|_{L^3(\mathbb{R}^3)}^2) \leq C\epsilon_0^2.$$

Let  $T_k$  be the maximal time interval for  $(u_k, d_k)$ . If  $0 < T_k < +\infty$ , then by Theorem 4.4 in §4 below we conclude that  $(u_k, d_k) \in C_b^\infty(\mathbb{R}^3 \times [0, T_k], \mathbb{R}^3 \times S^2)$ . Hence  $(u_k(T_k), d_k(T_k)) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3 \times S^2) \cap L^3(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3)$ , and

$$E_3((u_k(T_k), \nabla d_k(T_k))) \leq 2\epsilon_0^3$$

so that we can extend the smooth solutions  $(u_k, d_k)$  beyond the time  $T_k$ , which would contradict the maximality of  $T_k$ . Therefore  $T_k = \infty$  and the smooth solution  $(u_k, d_k)$  exists globally. Moreover,  $E_3(u_k(t), \nabla d_k(t))$  is monotone decreasing and less than  $2\epsilon_0^3$ . By Theorem 4.4, we have the derivative estimates:

$$\|\nabla^m u_k(t)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla^{m+1} d_k(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C\epsilon_0}{t^{\frac{m}{2}}}, \quad \forall t > 0, \quad m \geq 1. \quad (2.16)$$

After taking possible subsequences, we may assume that there exists  $(u, d) \in C^\infty(\mathbb{R}^3 \times (0, +\infty), \mathbb{R}^3 \times S^2) \cap C([0, +\infty), L^3(\mathbb{R}^3) \cap \dot{W}^{1,3}(\mathbb{R}^3))$  such that as  $k \rightarrow \infty$ ,

(1)  $(u_k, d_k) \rightarrow (u, d)$  in  $C_{\text{loc}}^m(\mathbb{R}^3 \times (0, +\infty))$  for any  $m \geq 1$ .

(2)  $(u_k, \nabla d_k) \rightarrow (u, \nabla d)$  weak\* in  $L^\infty([0, +\infty), L^3(\mathbb{R}^3))$ .

Thus  $(u, d) \in C^\infty(\mathbb{R}^3 \times (0, +\infty), \mathbb{R}^3 \times S^2)$  solves (1.1)<sub>1</sub>, (1.1)<sub>2</sub>, and (1.1)<sub>3</sub>, and the estimate (1.10) holds.

Using the equation (1.1), we can get that for any  $0 < T < +\infty$ ,

$$\sup_{k \geq 1} \left\| (\partial_t u^k, \partial_t d^k) \right\|_{L^{\frac{3}{2}}([0, T], W^{-1, \frac{3}{2}}(\mathbb{R}^3))} \leq C(T) < +\infty.$$

This implies that  $(u, d) \in C([0, T], L^3(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3))$  and  $(u, d)|_{t=0} = (u_0, d_0)$ . Applying Corollary 2.2 again, we conclude that  $E_3(u(t), \nabla d(t))$  is monotone decreasing for  $t \geq 0$ . The part of uniqueness can be proved as in the step 6 of the proof of Theorem 1.2 in §5, which is omitted here. The proof is complete.  $\square$

We would like to mention applications of Theorem 1.3 to the heat flow of harmonic maps and the Navier-Stokes equation.

1) If  $u \equiv 0$ , then (1.1)<sub>3</sub> reduces to the heat flow of harmonic maps to  $S^2$  for  $d : \mathbb{R}^3 \times (0, +\infty) \rightarrow S^2$ :

$$\begin{cases} \partial_t d = \Delta d + |\nabla d|^2 d & \text{in } \mathbb{R}^3 \times (0, +\infty) \\ d = d_0 & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases} \quad (2.17)$$

2) If  $d$  is a constant unit vector, then (1.1)<sub>1</sub> and (1.1)<sub>2</sub> reduce to the Navier-Stokes equation:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = 0 & \text{in } \mathbb{R}^3 \times (0, +\infty) \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^3 \times (0, +\infty) \\ u = u_0 & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases} \quad (2.18)$$

The following properties follow directly from Theorem 1.3. We would like to point out the observation of monotone decreasing property of the  $L^3$ -energy seems new.

**Remark 2.3.** 1) There exists  $\epsilon_0 > 0$  such that if  $d_0 : \mathbb{R}^3 \rightarrow S^2$  satisfies  $\int_{\mathbb{R}^3} |\nabla d_0|^3 \leq \epsilon_0^3$ , then there is a unique global solution  $d : \mathbb{R}^3 \times [0, +\infty) \rightarrow S^2$  of (2.17) such that  $d \in C([0, +\infty), \dot{W}^{1,3}(\mathbb{R}^3, S^2)) \cap C^\infty(\mathbb{R}^3 \times (0, +\infty), S^2)$ , and  $\int_{\mathbb{R}^3} |\nabla d(t)|^3$  is monotone decreasing for  $t \geq 0$ .

2) There exists  $\epsilon_0 > 0$  such that if  $u_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , with  $\nabla \cdot u_0 = 0$ , satisfies  $\int_{\mathbb{R}^3} |u_0|^3 \leq \epsilon_0^3$ , then there is a unique, global solution  $u : \mathbb{R}^3 \times [0, +\infty) \rightarrow \mathbb{R}^3$  of (2.18) such that  $u \in C([0, +\infty), L^3(\mathbb{R}^3)) \cap C^\infty(\mathbb{R}^3 \times (0, +\infty), \mathbb{R}^3)$ , and  $\int_{\mathbb{R}^3} |u(t)|^3$  is monotone decreasing for  $t \geq 0$ .

3. INEQUALITY OF THE LOCAL  $L^3$ -ENERGY

In this section, we will derive an inequality of the local  $L^3$ -energy for smooth solutions  $(u, d) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3 \times S^2$  for  $0 < T \leq \infty$ , of the system (1.1). More precisely, we have

**Lemma 3.1.** *There exists  $C > 0$  such that for  $0 < T \leq \infty$ , if  $(u, d) \in C^\infty(\mathbb{R}^3 \times [0, T], \mathbb{R}^3 \times S^2) \cap C([0, T], L^2(\mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3))$  is a smooth solution of the system (1.1), then*

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} (|u|^3 + |\nabla d|^3) \phi^2 + \int_{\mathbb{R}^3} \left( |\nabla(|u|^{\frac{3}{2}} \phi)|^2 + |\nabla(|\nabla d|^{\frac{3}{2}} \phi)|^2 \right) \\ & \leq C \int_{\mathbb{R}^3} (|u|^3 + |\nabla d|^3) |\nabla \phi|^2 + CR^{-2} \sup_{y \in \mathbb{R}^3} \left( \int_{B_R(y)} |u|^3 + |\nabla d|^3 \right)^{\frac{5}{3}} \\ & + C \left( \int_{\text{spt} \phi} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} \left( |\nabla(|u|^{\frac{3}{2}} \phi)|^2 + |\nabla(|\nabla d|^{\frac{3}{2}} \phi)|^2 \right), \end{aligned} \quad (3.1)$$

holds for any  $\phi \in C_0^\infty(\mathbb{R}^3)$ , with  $0 \leq \phi \leq 1$ ,  $\text{spt} \phi = B_R(x_0)^1$  for some  $R > 0$  and  $x_0 \in \mathbb{R}^3$ , and  $|\nabla \phi| \leq 4R^{-1}$ .

*Proof.* We divide the proof into three steps.

**Step 1.** Estimation of the local  $L^3$ -energy of  $\nabla d$ . Differentiating (1.1)<sub>3</sub> with respect to  $x$ , integrating against  $\phi^2 |\nabla d| \nabla d$  over  $\mathbb{R}^3$ , and applying integration by parts, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla d|^3 \phi^2 + 3 \int_{\mathbb{R}^3} \nabla^2 d : \nabla(\phi^2 |\nabla d| \nabla d) \\ & \leq 3 \int_{\mathbb{R}^3} |\nabla d|^5 \phi^2 + 3 \int_{\mathbb{R}^3} (u \cdot \nabla d) \cdot \nabla \cdot (\phi^2 |\nabla d| \nabla d), \end{aligned} \quad (3.2)$$

where we have used  $|d| = 1$  and the following identity to obtain the first term on the right hand side:

$$\nabla(|\nabla d|^2 d) \cdot |\nabla d| \nabla d = \frac{1}{2} \nabla(|\nabla d|^2) |\nabla d| \nabla(|d|^2) + |\nabla d|^3 \nabla d \cdot \nabla d = |\nabla d|^5.$$

For the second term on the left hand side of (3.2), direct calculations using  $|\nabla |\nabla d|| \leq |\nabla^2 d|$  and the Hölder inequality imply

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla^2 d : \nabla(\phi^2 |\nabla d| \nabla d) &= \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \phi^2 + \int_{\mathbb{R}^3 \cap \{|\nabla d| > 0\}} (|\nabla d|^2 \nabla |\nabla d| \cdot \nabla \phi^2 + |\nabla d| |\nabla |\nabla d||^2 \phi^2) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \phi^2 - C \int_{\mathbb{R}^3} |\nabla d|^3 |\nabla \phi|^2. \end{aligned}$$

For the second term on the right hand side of (3.2), by the Cauchy inequality we have

$$\begin{aligned} \int_{\mathbb{R}^3} (u \cdot \nabla d) \cdot \nabla \cdot (\phi^2 |\nabla d| \nabla d) &\leq 2 \int_{\mathbb{R}^3} |u| |\nabla d|^2 |\nabla^2 d| \phi^2 + |u| |\nabla d|^3 \phi |\nabla \phi| \\ &\leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \phi^2 + C \left( \int_{\text{spt} \phi} |u|^3 \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} |\nabla d|^9 \phi^6 \right)^{\frac{1}{3}} \\ &\quad + C \left( \int_{\text{spt} \phi} |u|^3 \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} |\nabla d|^9 \phi^6 \right)^{\frac{1}{6}} \left( \int_{\mathbb{R}^3} |\nabla d|^3 |\nabla \phi|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \phi^2 + C \int_{\mathbb{R}^3} |\nabla d|^3 |\nabla \phi|^2 \\ &\quad + C \left( \int_{\text{spt} \phi} |u|^3 \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} |\nabla d|^9 \phi^6 \right)^{\frac{1}{3}}. \end{aligned}$$

By the Hölder inequality and the Sobolev inequality, we have

$$\left( \int_{\mathbb{R}^3} |\nabla d|^9 \phi^6 \right)^{\frac{1}{3}} \lesssim \int_{\mathbb{R}^3} |\nabla(|\nabla d|^{\frac{3}{2}} \phi)|^2, \quad \int_{\mathbb{R}^3} |\nabla d|^5 \phi^2 \lesssim \left( \int_{\text{spt} \phi} |\nabla d|^3 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} |\nabla(|\nabla d|^{\frac{3}{2}} \phi)|^2.$$

<sup>1</sup>Here  $\text{spt} \phi$  denotes the support of  $\phi$ .

Putting these estimates into (3.2) yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \phi^2 |\nabla d|^3 + \int_{\mathbb{R}^3} |\nabla^2 d|^2 |\nabla d| \phi^2 \\ & \lesssim \int_{\mathbb{R}^3} |\nabla d|^3 |\nabla \phi|^2 + \left( \int_{\text{spt} \phi} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} \left| \nabla (|\nabla d|^{\frac{3}{2}} \phi) \right|^2. \end{aligned} \quad (3.3)$$

**Step 2.** Estimation of the local  $L^3$ -energy of  $u$ . Multiplying (1.1)<sub>1</sub> by  $\phi^2 |u|u$  and integrating over  $\mathbb{R}^3$  yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} |u|^3 \phi^2 + 3 \int_{\mathbb{R}^3} \nabla u \cdot \nabla (\phi^2 |u|u) \\ & \lesssim \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d| |u|^2 \phi^2 + \int_{\mathbb{R}^3} |\nabla u| |u|^3 \phi^2 + \int_{\mathbb{R}^3} |P - c| |\nabla (\phi^2 |u|u)| \end{aligned} \quad (3.4)$$

where  $c \in \mathbb{R}$  is a constant to be chosen later.

By the Cauchy inequality, the Hölder inequality, and the Sobolev inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \nabla u \cdot \nabla (\phi^2 |u|u) \geq \frac{1}{2} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \phi^2 - 4 \int_{\mathbb{R}^3} |u|^3 |\nabla \phi|^2, \\ & \int_{\mathbb{R}^3} |\nabla u| |u|^3 \phi^2 \leq \frac{1}{4} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \phi^2 + C \left( \int_{\text{spt} \phi} |u|^3 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} \left| \nabla (|u|^{\frac{3}{2}} \phi) \right|^2, \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d| |u|^2 \phi^2 \leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \phi^2 + C \int_{\mathbb{R}^3} |\nabla d| |u|^4 \phi^2 \\ & \leq \frac{1}{8} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \phi^2 + C \left( \int_{\text{spt} \phi} |\nabla d|^3 \right)^{\frac{1}{3}} \left( \int_{\text{spt} \phi} |u|^3 \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} |u|^9 \phi^6 \right)^{\frac{1}{3}}. \end{aligned}$$

For the last term on the right hand side of (3.4) we have

$$\int_{\mathbb{R}^3} |P - c| |\nabla \cdot (|u|u\phi^2)| \leq \frac{1}{8} \int_{\mathbb{R}^3} |u| |\nabla u|^2 \phi^2 + C \int_{\mathbb{R}^3} |P - c|^2 |u| \phi^2 + C \int_{\mathbb{R}^3} |u|^3 |\nabla \phi|^2.$$

Putting these inequalities into (3.4) we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} |u|^3 \phi^2 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 \phi^2 \\ & \leq C \int_{\mathbb{R}^3} |u|^3 |\nabla \phi|^2 + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla d| |\nabla^2 d|^2 \phi^2 + C \int_{\mathbb{R}^3} |P - c|^2 |u| \phi^2 \\ & \quad + C \left( \int_{\text{spt} \phi} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} (|\nabla (|u|^{\frac{3}{2}} \phi)|^2 + |\nabla (|\nabla d|^{\frac{3}{2}} \phi)|^2). \end{aligned} \quad (3.5)$$

Combining (3.3) with (3.5) yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} (|u|^3 + |\nabla d|^3) \phi^2 + \int_{\mathbb{R}^3} (|\nabla (|u|^{\frac{3}{2}} \phi)|^2 + |\nabla (|\nabla d|^{\frac{3}{2}} \phi)|^2) \\ & \leq C \int_{\mathbb{R}^3} (|u|^3 + |\nabla d|^3) |\nabla \phi|^2 + C \int_{\mathbb{R}^3} |u| |P - c|^2 \phi^2 \\ & \quad + C \left( \int_{\text{spt} \phi} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}} \int_{\mathbb{R}^3} (|\nabla (|u|^{\frac{3}{2}} \phi)|^2 + |\nabla (|\nabla d|^{\frac{3}{2}} \phi)|^2). \end{aligned} \quad (3.6)$$

**Step 3.** Estimation of the pressure function  $P$ . By the Hölder inequality, we have

$$\int_{\mathbb{R}^3} |u| |P - c|^2 \phi^2 \leq \left( \int_{\text{spt} \phi} |u|^3 \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} |P - c|^3 \phi^3 \right)^{\frac{2}{3}}.$$

We see that (3.1) follows from (3.6) and the estimate (3.7) of Lemma 3.2 below. The proof is complete.  $\square$

**Lemma 3.2.** *Under the same assumptions as in Lemma 3.1, assume that  $\phi \in C_0^\infty(\mathbb{R}^3)$  satisfies  $0 \leq \phi \leq 1$ ,  $\text{spt } \phi = B_R(x_0)$  for some  $x_0 \in \mathbb{R}^3$ , and  $|\nabla \phi| \leq 2R^{-1}$ . Then there exists  $C > 0$  such that for any  $t \in (0, T)$  there is  $c(t) \in \mathbb{R}$  so that the following estimate holds*

$$\begin{aligned} \left( \int_{\mathbb{R}^3} |P(t) - c(t)|^3 \phi^3 \right)^{\frac{1}{3}} &\leq C \left( \int_{\text{spt} \phi} |u(t)|^3 + |\nabla d(t)|^3 \right)^{\frac{1}{6}} \left( \int_{\mathbb{R}^3} (|\nabla(|u(t)|^{\frac{3}{2}} \phi)|^2 + |\nabla(|\nabla d(t)|^{\frac{3}{2}} \phi)|^2) \right)^{\frac{1}{2}} \\ &\quad + CR^{-1} \sup_{y \in \mathbb{R}^3} \left( \int_{B_R(y)} |u(t)|^3 + |\nabla d(t)|^3 \right)^{\frac{2}{3}}. \end{aligned} \quad (3.7)$$

*Proof.* For simplicity, we write  $(u, P, d)$  and  $c$  for  $(u(t), P(t), d(t))$  and  $c(t)$  respectively. Since

$$-\Delta P = \nabla_{jk}^2(g^{jk}), \quad g^{jk} := u^j u^k + \nabla_j d \cdot \nabla_k d,$$

we have

$$P = -\mathbf{R}_j \mathbf{R}_k(g^{jk})$$

where  $\mathbf{R}_j$  is the  $j$ -th Riesz transform on  $\mathbb{R}^3$ . Hence we have

$$\begin{aligned} (P - c)\phi &= -\mathbf{R}_j \mathbf{R}_k(g^{jk})\phi - c\phi \\ &= -\mathbf{R}_j \mathbf{R}_k(g^{jk}\phi) - [\phi, \mathbf{R}_j \mathbf{R}_k](g^{jk}) - c\phi \end{aligned} \quad (3.8)$$

where  $[\phi, \mathbf{R}_j \mathbf{R}_k]$  is the commutator between  $\phi$  and  $\mathbf{R}_j \mathbf{R}_k$  given by

$$[\phi, \mathbf{R}_j \mathbf{R}_k](f) = \phi \cdot \mathbf{R}_j \mathbf{R}_k(f) - \mathbf{R}_j \mathbf{R}_k(\phi f), \quad f \in C_0^\infty(\mathbb{R}^3).$$

We now estimate  $[\phi, \mathbf{R}_j \mathbf{R}_k](g^{jk})$  as follows.

$$\begin{aligned} &[\phi, \mathbf{R}_j \mathbf{R}_k](g^{jk})(x) \\ &= \phi(x) \mathbf{R}_j \mathbf{R}_k(g^{jk})(x) - \mathbf{R}_j \mathbf{R}_k(\phi g^{jk})(x) \\ &= \phi(x) \int_{\mathbb{R}^3} \frac{(x^j - y^j)(x^k - y^k)}{|x - y|^5} g^{jk}(y) - \int_{\mathbb{R}^3} \frac{(x^j - y^j)(x^k - y^k)}{|x - y|^5} \phi(y) g^{jk}(y) \\ &= \int_{\mathbb{R}^3} \frac{(\phi(x) - \phi(y))(x^j - y^j)(x^k - y^k)}{|x - y|^5} g^{jk}(y). \end{aligned}$$

For any  $x \in \text{spt } \phi = B_R(x_0)$ , we have

$$\begin{aligned} &[\phi, \mathbf{R}_j \mathbf{R}_k](g^{jk})(x) + c\phi(x) \\ &= \int_{B_{2R}(x_0)} \frac{(\phi(x) - \phi(y))(x^j - y^j)(x^k - y^k)}{|x - y|^5} g^{jk}(y) dy + c\phi(x) \\ &\quad + \phi(x) \left[ \int_{\mathbb{R}^3 \setminus B_{2R}(x_0)} \frac{(x^j - y^j)(x^k - y^k)}{|x - y|^5} g^{jk}(y) dy + c \right] \\ &= I(x) + II(x). \end{aligned}$$

For  $I(x)$ , we have that

$$\begin{aligned} |I(x)| &\leq \int_{B_{2R}(x_0)} \frac{|\phi(x) - \phi(y)| |x^j - y^j| |x^k - y^k|}{|x - y|^5} |g^{jk}(y)| \\ &\leq CR^{-1} \int_{\mathbb{R}^3} \frac{\chi_{B_{2R}(x_0)}(y) (|u|^2 + |\nabla d|^2)(y)}{|x - y|^2} \\ &= CR^{-1} \mathbf{I}_1((|u|^2 + |\nabla d|^2) \chi_{B_{2R}(x_0)})(x), \end{aligned}$$

where  $\chi_{B_{2R}(x_0)}$  is the characteristic function of  $B_{2R}(x_0)$ , and  $\mathbf{I}_1$  is the Riesz potential on  $\mathbb{R}^3$  of order 1 given by

$$\mathbf{I}_1(f)(x) = \int_{\mathbb{R}^3} \frac{|f(y)|}{|x - y|^2}, \quad x \in \mathbb{R}^3, \quad \forall f \in L^1(\mathbb{R}^3).$$

Recall that by the Hardy-Littlewood-Sobolev inequality,  $\mathbf{I}_1 : L^{\frac{3}{2}}(\mathbb{R}^3) \rightarrow L^3(\mathbb{R}^3)$  satisfies

$$\|\mathbf{I}_1(f)\|_{L^3(\mathbb{R}^3)} \lesssim \|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}. \quad (3.9)$$

Hence we have

$$\begin{aligned}
\|I\|_{L^3(\mathbb{R}^3)} &\lesssim R^{-1} \left\| \mathbf{I}_1 \left( (|u|^2 + |\nabla d|^2) \chi_{B_{2R}(x_0)} \right) \right\|_{L^3(\mathbb{R}^3)} \\
&\lesssim R^{-1} \left\| (|u|^2 + |\nabla d|^2) \chi_{B_{2R}(x_0)} \right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\
&\lesssim R^{-1} \left( \int_{B_{2R}(x_0)} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}} \\
&\lesssim R^{-1} \sup_{y \in \mathbb{R}^3} \left( \int_{B_R(y)} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}}.
\end{aligned} \tag{3.10}$$

To estimate  $II$ , choose

$$c = - \int_{\mathbb{R}^3 \setminus B_{2R}(x_0)} \frac{(x_0 - y)^j (x_0 - y)^k}{|x_0 - y|^5} g^{jk}(y).$$

Note that

$$|c| \lesssim R^{-3} \sum_{j,k} \int_{\mathbb{R}^3} |g^{jk}| \lesssim R^{-3} \left( \|u\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla d\|_{L^2(\mathbb{R}^3)}^2 \right) < +\infty.$$

Therefore we have

$$\begin{aligned}
|II(x)| &= \left| \phi(x) \int_{\mathbb{R}^3 \setminus B_{2R}(x_0)} \left( \frac{(x^j - y^j)(x^k - y^k)}{|x - y|^5} - \frac{(x_0 - y)^j (x_0 - y)^k}{|x_0 - y|^5} \right) g^{jk}(y) \right| \\
&\lesssim R |\phi(x)| \int_{\mathbb{R}^3 \setminus B_{2R}(x_0)} \frac{1}{|x - y|^4} (|u|^2 + |\nabla d|^2)(y),
\end{aligned}$$

where we have used the following inequality (see [25]):

$$\left| \frac{(x^j - y^j)(x^k - y^k)}{|x - y|^5} - \frac{(x_0 - y)^j (x_0 - y)^k}{|x_0 - y|^5} \right| \lesssim \frac{|x_0 - x|}{|x_0 - y|^4}, \text{ for } x \in B_R(x_0) \text{ and } y \in \mathbb{R}^3 \setminus B_{2R}(x_0).$$

Thus we have

$$\begin{aligned}
|II|(x) &\lesssim R \int_{\mathbb{R}^3 \setminus B_{2R}(x_0)} \frac{1}{|x_0 - y|^4} (|u|^2 + |\nabla d|^2)(y) \\
&\lesssim R \sum_{k=2}^{\infty} \frac{1}{(kR)^4} \int_{B_{(k+1)R}(x_0) \setminus B_{kR}(x_0)} (|u|^2 + |\nabla d|^2) \\
&\lesssim \frac{1}{R^3} \left[ \sum_{k=2}^{\infty} \frac{1}{k^2} \right] \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} (|u|^2 + |\nabla d|^2) \\
&\lesssim R^{-2} \sup_{y \in \mathbb{R}^3} \left( \int_{B_R(y)} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}}.
\end{aligned}$$

Integrating  $II$  over  $B_R(x_0)$  we get

$$\|II\|_{L^3(\mathbb{R}^3)} \lesssim R^{-1} \sup_{y \in \mathbb{R}^3} \left( \int_{B_R(y)} |u|^3 + |\nabla d|^3 \right)^{\frac{2}{3}}. \tag{3.11}$$

Additionally, we have

$$\begin{aligned}
\|\mathbf{R}_j \mathbf{R}_k (g^{jk} \phi)\|_{L^3(\mathbb{R}^3)} &\lesssim \left( \int_{\mathbb{R}^3} (|u|^6 + |\nabla d|^6) \phi^3 \right)^{\frac{1}{3}} \\
&\lesssim \left( \int_{\text{spt} \phi} |u|^3 + |\nabla d|^3 \right)^{\frac{1}{6}} \left( \int_{\mathbb{R}^3} (|u|^9 + |\nabla d|^9) \phi^6 \right)^{\frac{1}{6}} \\
&\lesssim \left( \int_{\text{spt} \phi} |u|^3 + |\nabla d|^3 \right)^{\frac{1}{6}} \left( \int_{\mathbb{R}^3} (|\nabla(|u|^{\frac{3}{2}} \phi)|^2 + |\nabla(|\nabla d|^{\frac{3}{2}} \phi)|^2) \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.12}$$

Combining the estimates (3.10) and (3.11) with (3.12) yields (3.7). This completes the proof of Lemma 3.2.  $\square$

## 4. REGULARITY OF SUITABLE WEAK SOLUTIONS

In this section, we will derive a priori estimates for smooth solutions to the system (1.1) under a smallness condition for the  $L^3$ -norm of  $(u, \nabla d)$ . Since the method is flexible enough, it also yields the smoothness for a subclass of suitable weak solutions to (1.1). We present the result in the context of suitable weak solutions to (1.1). The notion of suitable weak solutions was first introduced by Caffarelli-Kohn-Nirenberg [3] in the context of incompressible Navier-Stokes equations. Here we adapt this notion to (1.1), similar to the definition given by Lin [18] on the Navier-Stokes equation.

Let  $0 < T \leq \infty$  and  $\Omega \subset \mathbb{R}^3$  be a bounded smooth domain.

**Definition 4.1.** *A triple of functions  $(u, P, d) : \Omega \times (0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  is called a suitable weak solution to the system (1.1) in  $\Omega \times (0, T)$  if the following properties hold:*

- (1)  $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times (0, T))$ ,  $P \in L^{\frac{3}{2}}(\Omega \times (0, T))$  and  $d \in L_t^2 H_x^2(\Omega \times (0, T))$ ;
- (2)  $(u, P, d)$  satisfies the system (1.1) in the sense of distributions; and
- (3)  $(u, P, d)$  satisfies the local energy inequality (4.1).

Now we would like to point out that the class of smooth solutions belongs to the class of suitable weak solutions to the system (1.1). Let  $\mathbb{I}_3$  denote the identity matrix of order 3.

**Lemma 4.2.** *Suppose that  $(u, d) \in C^\infty(\Omega \times (0, T), \mathbb{R}^3 \times \mathbb{R} \times S^2)$  is a solution of (1.1) in  $\Omega \times (0, T)$ . Then for any nonnegative  $\phi \in C_0^\infty(\Omega \times (0, T))$ , it holds that*

$$\begin{aligned}
2 \int_{\Omega \times (0, T)} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) \phi &\leq \int_{\Omega \times (0, T)} (|u|^2 + |\nabla d|^2) (\phi_t + \Delta \phi) \\
&+ \int_{\Omega \times (0, T)} (|u|^2 + |\nabla d|^2 + 2P) u \cdot \nabla \phi \\
&+ 2 \int_{\Omega \times (0, T)} (\nabla d \odot \nabla d - |\nabla d|^2 \mathbb{I}_3) : \nabla^2 \phi \\
&+ 2 \int_{\Omega \times (0, T)} \nabla d \odot \nabla d : u \otimes \nabla \phi.
\end{aligned} \tag{4.1}$$

*Proof.* Multiplying (1.1)<sub>1</sub> by  $u\phi$  and integrating the resulting equation over  $\Omega \times (0, T)$  yields

$$\begin{aligned}
&\int_{\Omega \times (0, T)} u_t \cdot u\phi + \int_{\Omega \times (0, T)} (u \cdot \nabla u) \cdot u\phi - \int_{\Omega \times (0, T)} \Delta u \cdot u\phi + \int_{\Omega \times (0, T)} \nabla P \cdot u\phi \\
&= \int_{\Omega \times (0, T)} \nabla d \odot \nabla d : \nabla(u\phi).
\end{aligned} \tag{4.2}$$

Applying integration by parts, the terms on the left hand side of (4.2) can be estimated by

$$\begin{aligned}
\int_{\Omega \times (0, T)} u_t \cdot u\phi &= - \int_{\Omega \times (0, T)} \frac{1}{2} |u|^2 \phi_t, \\
\int_{\Omega \times (0, T)} (u \cdot \nabla u) \cdot u\phi &= - \int_{\Omega \times (0, T)} \frac{1}{2} |u|^2 u \cdot \nabla \phi, \\
\int_{\Omega \times (0, T)} \Delta u \cdot u\phi &= - \int_{\Omega \times (0, T)} |\nabla u|^2 \phi + \int_{\Omega \times (0, T)} \frac{1}{2} |u|^2 \Delta \phi, \\
\int_{\Omega \times (0, T)} \nabla P \cdot u\phi &= - \int_{\Omega \times (0, T)} P(u \cdot \nabla \phi).
\end{aligned}$$

For the term on the right hand side of (4.2), we have

$$\int_{\Omega \times (0, T)} \nabla d \odot \nabla d : \nabla \cdot (u\phi) = \int_{\Omega \times (0, T)} \nabla d \odot \nabla d : [(\nabla u)\phi + u \otimes \nabla \phi].$$

Putting these identities into (4.2) yields

$$\begin{aligned} & \int_{\Omega \times (0, T)} -\frac{1}{2}|u|^2(\phi_t + \Delta\phi) - \left(\frac{1}{2}|u|^2 + P\right)(u \cdot \nabla\phi) + \int_{\Omega \times (0, T)} |\nabla u|^2\phi \\ &= \int_{\Omega \times (0, T)} (\nabla d \odot \nabla d) : [\phi \nabla u + u \otimes \nabla\phi]. \end{aligned} \quad (4.3)$$

Differentiating (1.1)<sub>3</sub> with respect to  $x$  and integrating against  $(\nabla d)\phi$ , we have

$$\int_{\Omega \times (0, T)} (\nabla d)_t : (\nabla d)\phi + \int_{\Omega \times (0, T)} \nabla(u \cdot \nabla d) : (\nabla d)\phi = \int_{\Omega \times (0, T)} \nabla[\Delta d + |\nabla d|^2 d] : (\nabla d)\phi \quad (4.4)$$

For the first term on the left hand side of (4.4), we have

$$\int_{\Omega \times (0, T)} (\nabla d)_t : (\nabla d)\phi = - \int_{\Omega \times (0, T)} \frac{1}{2} |\nabla d|^2 \phi_t.$$

Using (1.1)<sub>2</sub>, we can simplify the second term on the left hand side of (4.4) into

$$\begin{aligned} \int_{\Omega \times (0, T)} \nabla(u \cdot \nabla d) : (\nabla d)\phi &= \int_{\Omega \times (0, T)} u^\alpha_j d_j \cdot d_\alpha \phi + \int_{\Omega \times (0, T)} u^j d_{j\alpha} \cdot d_\alpha \phi \\ &= \int_{\Omega \times (0, T)} \nabla u : \nabla d \odot \nabla d \phi + \int_{\Omega \times (0, T)} \frac{1}{2} u \cdot \nabla(|\nabla d|^2)\phi \\ &= \int_{\Omega \times (0, T)} \nabla u : \nabla d \odot \nabla d \phi - \int_{\Omega \times (0, T)} \frac{1}{2} (u \cdot \nabla\phi) |\nabla d|^2. \end{aligned}$$

For the term on the right hand side of (4.4), differentiating  $|d| = 1$  gives

$$\nabla d \cdot d = 0 \quad \text{and} \quad \Delta d \cdot d + |\nabla d|^2 = 0.$$

Thus, by integration by parts we have

$$\begin{aligned} \int_{\Omega \times (0, T)} \nabla[\Delta d + |\nabla d|^2 d] \cdot \nabla d \phi &= - \int_{\Omega \times (0, T)} [\Delta d + |\nabla d|^2 d] \cdot [\Delta d \phi + \nabla d \cdot \nabla\phi] \\ &= - \int_{\Omega \times (0, T)} |\Delta d + |\nabla d|^2 d|^2 \phi - \int_{\Omega \times (0, T)} \Delta d \cdot (\nabla d \cdot \nabla\phi). \end{aligned}$$

By integration by parts we have

$$\begin{aligned} - \int_{\Omega \times (0, T)} \Delta d (\nabla d \cdot \nabla\phi) &= \int_{\Omega \times (0, T)} (\nabla d \odot \nabla d) : \nabla^2 \phi - \int_{\Omega \times (0, T)} \frac{1}{2} |\nabla d|^2 \Delta \phi \\ &= \int_{\Omega \times (0, T)} (\nabla d \odot \nabla d - |\nabla d|^2 \mathbb{I}_3) : \nabla^2 \phi + \int_{\Omega \times (0, T)} \frac{1}{2} |\nabla d|^2 \Delta \phi. \end{aligned}$$

Inserting these identities into (4.4) yields

$$\begin{aligned} & \int_{\Omega \times (0, T)} \left[ -\frac{1}{2} |\nabla d|^2 (\phi_t + \Delta\phi) - \frac{1}{2} |\nabla d|^2 (u \cdot \nabla\phi) \right] + \int_{\Omega \times (0, T)} \nabla u : \nabla d \odot \nabla d \phi \\ &= \int_{\Omega \times (0, T)} (\nabla d \odot \nabla d - |\nabla d|^2 \mathbb{I}_3) : \nabla^2 \phi - \int_{\Omega \times (0, T)} |\Delta d + |\nabla d|^2 d|^2 \phi. \end{aligned} \quad (4.5)$$

Combining (4.3) with (4.5) yields (4.1).  $\square$

**Corollary 4.3.** *Suppose that  $(u, P, d) : \Omega \times (0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  is a suitable weak solution of the system (1.1) in  $\Omega \times (0, T)$ . Then for any nonnegative  $\phi \in C^\infty(\Omega \times (0, T))$  and  $0 < t < T$ , it holds*

$$\begin{aligned} & \int_{\Omega \times \{t\}} (|u|^2 + |\nabla d|^2)\phi + 2 \int_{\Omega \times (0, t)} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2)\phi \\ & \leq \int_{\Omega \times (0, t)} (|u|^2 + |\nabla d|^2)(\phi_t + \Delta\phi) + \int_{\Omega \times (0, t)} (|u|^2 + |\nabla d|^2 + 2P)u \cdot \nabla\phi \\ & + 2 \int_{\Omega \times (0, t)} (\nabla d \odot \nabla d - |\nabla d|^2 \mathbb{I}_3) : \nabla^2 \phi + 2 \int_{\Omega \times (0, t)} \nabla d \odot \nabla d : u \otimes \nabla\phi. \end{aligned} \quad (4.6)$$

*Proof.* For  $\epsilon > 0$ , let  $\eta_\epsilon \in C^\infty([0, t])$  be such that  $0 \leq \eta \leq 1$ ,  $\eta = 1$  in  $[0, t - 2\epsilon]$ , and  $\eta = 0$  in  $[t - \epsilon, t]$ . (4.6) follows by first applying (4.1), with  $\phi$  replaced by  $\eta_\epsilon(t)\phi(x, t)$ , and then taking  $\epsilon \rightarrow 0$ .  $\square$

Let  $\mathcal{C}(3) > 0$  denote the best Sobolev constant of  $\mathbb{R}^3$ :

$$\mathcal{C}(3) := \inf \left\{ \frac{\|\nabla f\|_{L^2(\mathbb{R}^3)}}{\|f\|_{L^6(\mathbb{R}^3)}} : 0 \neq f \in C_0^\infty(\mathbb{R}^3) \right\}, \quad (4.7)$$

and  $\mathcal{D}(3) > 0$  denote the constant in the following  $W^{2,2}$ -estimate:

$$\|\nabla^2 f\|_{L^2(B_1)} \leq \mathcal{D}(3)\|\Delta f\|_{L^2(B_1)} + C\|\nabla f\|_{W^{\frac{1}{2},2}(\partial B_1)}, \quad \forall f \in W^{2,2}(B_1). \quad (4.8)$$

For  $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times (0, T)$  and  $r_0 > 0$ , denote

$$B_{r_0}(x_0) = \{x \in \mathbb{R}^3 : |x - x_0| < r_0\}, \quad P_{r_0}(z_0) = B_{r_0}(x_0) \times (t_0 - r_0^2, t_0].$$

Now we are ready to prove the following  $\epsilon_0$ -regularity theorem.

**Theorem 4.4.** *For any  $\delta > 0$ , there exists  $\epsilon_0 > 0$  such that  $(u, P, d) : \Omega \times (0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  is a suitable weak solution to (1.1), and satisfies, for  $z_0 = (x_0, t_0) \in \Omega \times (0, T)$  and  $P_{r_0}(z_0) \subset \Omega \times (0, T)$ ,*

$$\left( r_0^{-2} \int_{P_{r_0}(z_0)} |u|^3 \right)^{\frac{1}{3}} + \left( r_0^{-2} \int_{P_{r_0}(z_0)} |P|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( r_0^{-2} \int_{P_{r_0}(z_0)} |\nabla d|^3 \right)^{\frac{1}{3}} \leq \epsilon_0, \quad (4.9)$$

and

$$\|\nabla d\|_{L_t^\infty L_x^3(P_{r_0}(z_0))} < \frac{1 - \delta}{\mathcal{C}(3)\mathcal{D}(3)}, \quad (4.10)$$

then  $(u, d) \in C^\infty(P_{\frac{r_0}{4}}(z_0), \mathbb{R}^3 \times S^2)$ , and the following estimate holds:

$$\|(u, d)\|_{C^m(P_{\frac{r_0}{4}}(z_0))} \leq C(m, r_0, \epsilon_0), \quad \forall m \geq 0. \quad (4.11)$$

The crucial ingredient to prove Theorem 4.4 is the following decay lemma, which is analogous to that of the Navier-Stokes equations by [18] and [7].

**Lemma 4.5.** *For any  $\delta > 0$ , there exist  $\epsilon_0 > 0$  and  $\theta_0 \in (0, \frac{1}{2})$  such that if  $(u, P, d) : \Omega \times (0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  is a suitable weak solution of (1.1), and satisfies, for  $z_0 = (x_0, t_0) \in \Omega \times (0, T)$  and  $P_{r_0}(z_0) \subset \Omega \times (0, T)$ , both (4.9) and (4.10), then it holds that*

$$\begin{aligned} & \left[ \left( (\theta_0 r_0)^{-2} \int_{P_{\theta_0 r_0}(z_0)} |u|^3 \right)^{\frac{1}{3}} + \left( (\theta_0 r_0)^{-2} \int_{P_{\theta_0 r_0}(z_0)} |P|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( (\theta_0 r_0)^{-2} \int_{P_{\theta_0 r_0}(z_0)} |\nabla d|^3 \right)^{\frac{1}{3}} \right] \\ & \leq \frac{1}{2} \left[ \left( r_0^{-2} \int_{P_{r_0}(z_0)} |u|^3 \right)^{\frac{1}{3}} + \left( r_0^{-2} \int_{P_{r_0}(z_0)} |P|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( r_0^{-2} \int_{P_{r_0}(z_0)} |\nabla d|^3 \right)^{\frac{1}{3}} \right]. \end{aligned} \quad (4.12)$$

*Proof.* By the invariance of (1.1) under translations and parabolic dilations, it suffices to consider the case that  $z_0 = (0, 0)$  and  $r_0 = 1$ . We will prove the Lemma by contradiction. Suppose that the conclusion were false. Then there would exist  $\delta_0 > 0$  such that for any  $\theta \in (0, 1)$  there are a sequence of suitable weak solutions  $(u_i, P_i, d_i)$  of (1.1) in  $P_1$ , that satisfy

$$\left( \int_{P_1} |u_i|^3 \right)^{\frac{1}{3}} + \left( \int_{P_1} |P_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \int_{P_1} |\nabla d_i|^3 \right)^{\frac{1}{3}} = \epsilon_i \rightarrow 0, \quad (4.13)$$

$$\|\nabla d_i\|_{L_t^\infty L_x^3(P_1)} \leq \frac{1 - \delta_0}{\mathcal{C}(3)\mathcal{D}(3)}, \quad (4.14)$$

and

$$\begin{aligned} & \left[ \left( \theta^{-2} \int_{P_\theta} |u_i|^3 \right)^{\frac{1}{3}} + \left( \theta^{-2} \int_{P_\theta} |P_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \theta^{-2} \int_{P_\theta} |\nabla d_i|^3 \right)^{\frac{1}{3}} \right] \\ & > \frac{1}{2} \left[ \left( \int_{P_1} |u_i|^3 \right)^{\frac{1}{3}} + \left( \int_{P_1} |P_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \int_{P_1} |\nabla d_i|^3 \right)^{\frac{1}{3}} \right]. \end{aligned} \quad (4.15)$$

Now we define the blow-up sequence  $(v_i, Q_i, e_i) : P_1 \rightarrow \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$  by

$$v_i(z) = \frac{u_i(z)}{\epsilon_i}, \quad Q_i(z) = \frac{P_i(z)}{\epsilon_i}, \quad e_i(z) = \frac{d_i(z) - (d_i)_1}{\epsilon_i}, \quad z \in P_1,$$

where  $(d_i)_1 = \frac{1}{|P_1|} \int_{P_1} d_i$  is the average of  $d_i$  over  $P_1$ .

Then  $(v_i, Q_i, e_i)$  satisfy the following equations in  $P_1$ :

$$\begin{cases} \partial_t v_i - \Delta v_i + \nabla Q_i = -\epsilon_i [v_i \cdot \nabla v_i + \nabla \cdot (\nabla e_i \odot \nabla e_i)], \\ \nabla \cdot v_i = 0, \\ \partial_t e_i - \Delta e_i = \epsilon_i [|\nabla e_i|^2 d_i - v_i \cdot \nabla e_i]. \end{cases} \quad (4.16)$$

It follows from (4.13) and (4.15) that for any  $\theta \in (0, \frac{1}{2})$ ,

$$\left( \int_{P_1} |v_i|^3 \right)^{\frac{1}{3}} + \left( \int_{P_1} |Q_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \int_{P_1} |\nabla e_i|^3 \right)^{\frac{1}{3}} = 1, \quad (4.17)$$

and

$$\left( \theta^{-2} \int_{P_\theta} |v_i|^3 \right)^{\frac{1}{3}} + \left( \theta^{-2} \int_{P_\theta} |Q_i|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \theta^{-2} \int_{P_\theta} |\nabla e_i|^3 \right)^{\frac{1}{3}} > \frac{1}{2}. \quad (4.18)$$

Applying the  $W^{\frac{3}{2}, 1}$ -estimate to the equation (4.16)<sub>3</sub>, we have that  $\nabla^2 e_i \in L^{\frac{3}{2}}(P_{\frac{7}{8}})$  and

$$\|\nabla^2 e_i\|_{L^{\frac{3}{2}}(P_{\frac{7}{8}})} \lesssim \left( \|v_i\|_{L^3(P_1)}^2 + \|\nabla e_i\|_{L^3(P_1)}^2 \right) \leq C. \quad (4.19)$$

By the Fubini Theorem and (4.19), we may assume that

$$\int_{\partial B_{\frac{3}{4}} \times [-(\frac{3}{4})^2, 0]} |\nabla^2 e_i|^{\frac{3}{2}} \leq C \int_{P_{\frac{7}{8}}} |\nabla^2 e_i|^{\frac{3}{2}} \leq C. \quad (4.20)$$

Since  $(u_i, Q_i, d_i)$  satisfies (4.6) in  $P_1$ , by choosing suitable test functions  $\phi$  we have that

$$\begin{aligned} & \sup_{-(\frac{3}{4})^2 \leq t \leq 0} \int_{B_{\frac{3}{4}}} (|u_i|^2 + |\nabla d_i|^2) + \int_{P_{\frac{3}{4}}} (|\nabla u_i|^2 + |\Delta d_i + |\nabla d_i|^2 d_i|^2) \\ & \leq C \int_{P_1} (|u_i|^2 + |\nabla d_i|^2) + (|Q_i| + |u_i|^2 + |\nabla d_i|^2) |u_i|. \end{aligned} \quad (4.21)$$

Rescaling (4.21), applying (4.17), and using the Hölder inequality, we have

$$\begin{aligned} & \sup_{-(\frac{3}{4})^2 \leq t \leq 0} \int_{B_{\frac{3}{4}}} (|v_i|^2 + |\nabla e_i|^2) + \int_{P_{\frac{3}{4}}} (|\nabla v_i|^2 + |\Delta e_i + \epsilon_i |\nabla e_i|^2 d_i|^2) \\ & \leq C \int_{P_1} (|v_i|^2 + |\nabla e_i|^2) + (|Q_i| + \epsilon_i |v_i|^2 + \epsilon_i |\nabla e_i|^2) |v_i| \leq C. \end{aligned} \quad (4.22)$$

By the  $W^{2,2}$ -estimate (4.8) and the Sobolev inequality, we have

$$\begin{aligned} \int_{B_{\frac{3}{4}}} |\nabla^2 e_i|^2 & \leq \mathcal{D}^2(3) \int_{B_{\frac{3}{4}}} |\Delta e_i|^2 + C \|\nabla e_i\|_{W^{\frac{1}{2}, 2}(\partial B_{\frac{3}{4}})}^2 \\ & \leq \mathcal{D}^2(3) \int_{B_{\frac{3}{4}}} |\Delta e_i|^2 + C \|\nabla^2 e_i\|_{W^{2, \frac{3}{2}}(\partial B_{\frac{3}{4}})}^2, \end{aligned}$$

so that, by integrating over  $t \in [-(\frac{3}{4})^2, 0]$  and applying (4.20), it holds that

$$\begin{aligned} \int_{P_{\frac{3}{4}}} |\nabla^2 e_i|^2 & \leq \mathcal{D}^2(3) \int_{P_{\frac{3}{4}}} |\Delta e_i|^2 + C \int_{-(\frac{3}{4})^2}^0 \|\nabla^2 e_i\|_{W^{2, \frac{3}{2}}(\partial B_{\frac{3}{4}})}^2 \\ & \leq C + \mathcal{D}^2(3) \int_{P_{\frac{3}{4}}} |\Delta e_i|^2. \end{aligned} \quad (4.23)$$

By the point-wise identity  $|\Delta e_i|^2 = |\Delta e_i + \epsilon_i |\nabla e_i|^2 d_i|^2 + \epsilon_i^2 |\nabla e_i|^4$ , we have

$$\int_{P_{\frac{3}{4}}} |\Delta e_i|^2 = \int_{P_{\frac{3}{4}}} |\Delta e_i + \epsilon_i |\nabla e_i|^2 d_i|^2 + \epsilon_i^2 \int_{P_{\frac{3}{4}}} |\nabla e_i|^4. \quad (4.24)$$

By the Hölder inequality, the Young inequality, and the Sobolev inequality, we have

$$\begin{aligned} \|\nabla e_i\|_{L^4(B_{\frac{3}{4}})}^4 &\leq \|\nabla e_i\|_{L^3(B_{\frac{3}{4}})}^2 \|\nabla e_i\|_{L^6(B_{\frac{3}{4}})}^2 \\ &\leq \|\nabla e_i\|_{L^3(B_{\frac{3}{4}})}^2 \left( \|\nabla e_i - (\nabla e_i)_{\frac{3}{4}}\|_{L^6(B_{\frac{3}{4}})}^2 + \|(\nabla e_i)_{\frac{3}{4}}\|_{L^6(B_{\frac{3}{4}})}^2 \right)^2 \\ &\leq (1 + \delta_0)^2 \mathcal{C}^2(3) \|\nabla e_i\|_{L^3(B_{\frac{3}{4}})}^2 \|\nabla^2 e_i\|_{L^2(B_{\frac{3}{4}})}^2 + C(\delta_0) \|\nabla e_i\|_{L^3(B_{\frac{3}{4}})}^2 \|\nabla e_i\|_{L^2(B_{\frac{3}{4}})}^2, \end{aligned} \quad (4.25)$$

where  $(\nabla e_i)_{\frac{3}{4}}$  is the average of  $\nabla e_i$  over  $B_{\frac{3}{4}}$ . Integrating (4.25) over  $t \in [-(\frac{3}{4})^2, 0]$  yields

$$\begin{aligned} \epsilon_i^2 \int_{P_{\frac{3}{4}}} |\nabla e_i|^4 &\leq (1 + \delta_0)^2 \mathcal{C}^2(3) \|\nabla d_i\|_{L_t^\infty L_x^3(P_{\frac{3}{4}})}^2 \int_{P_{\frac{3}{4}}} |\nabla^2 e_i|^2 \\ &\quad + C(\delta_0) \left( \sup_{-(\frac{3}{4})^2 \leq t \leq 0} \int_{B_{\frac{3}{4}}} |\nabla d_i|^2 \right) \|\nabla e_i\|_{L^3(P_{\frac{3}{4}})}^2 \\ &\leq C(\delta_0) + (1 + \delta_0)^2 \mathcal{C}^2(3) \|\nabla d_i\|_{L_t^\infty L_x^3(P_{\frac{3}{4}})}^2 \int_{P_{\frac{3}{4}}} |\nabla^2 e_i|^2. \end{aligned} \quad (4.26)$$

Inserting the estimate (4.26) first into (4.24) and then (4.23), we obtain

$$\begin{aligned} &\left[ 1 - (1 + \delta_0)^2 \mathcal{C}^2(3) \mathcal{D}^2(3) \|\nabla d_i\|_{L_t^\infty L_x^3(P_1)}^2 \right] \int_{P_{\frac{3}{4}}} |\nabla^2 e_i|^2 \\ &\leq C(\delta_0) + C \int_{P_{\frac{3}{4}}} |\Delta e_i + \epsilon_i |\nabla e_i|^2 d_i|^2 \leq C(\delta_0). \end{aligned} \quad (4.27)$$

Therefore, by applying (4.22) to (4.27), we have

$$\int_{P_{\frac{3}{4}}} |\nabla^2 e_i|^2 \leq C(\delta_0). \quad (4.28)$$

Combining the estimates (4.22) and (4.28), we obtain

$$\int_{P_{\frac{1}{2}}} |Q_i|^{\frac{3}{2}} + \sup_{t \in [-\frac{1}{4}, 0]} \int_{B_{\frac{1}{2}}} (|v_i|^2 + |\nabla e_i|^2) + \int_{P_{\frac{1}{2}}} (|\nabla v_i|^2 + |\nabla^2 e_i|^2) \leq C. \quad (4.29)$$

We may assume, after taking possible subsequences, that

$$\begin{cases} Q_i \rightarrow Q \text{ weakly in } L^{\frac{3}{2}}(P_{\frac{1}{2}}), \\ v_i \rightarrow v \text{ strongly in } L^2(P_{\frac{1}{2}}), \nabla v_i \rightarrow \nabla v \text{ weakly in } L^2(P_{\frac{1}{2}}), \\ e_i \rightarrow e \text{ and } \nabla e_i \rightarrow \nabla e \text{ strongly in } L^2(P_{\frac{1}{2}}), \nabla^2 e_i \rightarrow \nabla^2 e \text{ weakly in } L^2(P_{\frac{1}{2}}). \end{cases}$$

Sending  $i$  to  $\infty$  in the equation (4.16) yields that  $(v, Q, e)$  satisfies in  $P_{\frac{1}{2}}$

$$\begin{cases} \partial_t v - \Delta v + \nabla Q = 0, \\ \nabla \cdot v = 0, \\ \partial_t e - \Delta e = 0. \end{cases} \quad (4.30)$$

Using the Sobolev inequality and interpolations, we see that (4.29) gives

$$\int_{P_{\frac{1}{2}}} |v|^3 + |Q|^{\frac{3}{2}} + |\nabla e|^3 \leq C. \quad (4.31)$$

Hence, by the standard estimates on the linear Stokes equation and the heat equation, we have that for any  $\theta \in (0, \frac{1}{2})$ , it holds

$$\theta^{-2} \int_{P_\theta} (|v|^3 + |\nabla e|^3) \leq C\theta^3 \int_{P_{\frac{1}{2}}} (|v|^3 + |\nabla e|^3) \leq C\theta^3, \quad \theta^{-2} \int_{P_\theta} |Q|^{\frac{3}{2}} \leq C\theta \int_{P_{\frac{1}{2}}} |Q|^{\frac{3}{2}} \leq C\theta. \quad (4.32)$$

In order to reach a contradiction, we need to show that  $(v_i, Q_i, e_i)$  converges to  $(v, Q, e)$  strongly in  $L^3(P_{\frac{2}{5}})$ . To do so, we recall the following Lemma (see [26]).

**Lemma 4.6.** *Let  $X_0 \subset X \subset X_1$  be Banach spaces such that  $X_0$  is compactly embedded in  $X$ ,  $X$  is continuously embedded in  $X_1$ , and  $X_0, X_1$  are reflexive. Then for  $1 < \alpha_0, \alpha_1 < \infty$ ,*

$$\left\{ u \in L^{\alpha_0}(0, T; X_0) : \partial_t u \in L^{\alpha_1}(0, T; X_1) \right\} \text{ is compactly embedded in } L^{\alpha_0}(0, T; X).$$

Now we have the following claims.

*Claim 1.*  $v_i \rightarrow v$  strongly in  $L^2(P_{\frac{2}{5}})$ . From (4.29) and interpolation inequalities, we have

$$\begin{cases} \|v_i\|_{L^{\frac{10}{3}}(P_{\frac{1}{2}})} + \|v_i\|_{L_t^\infty L_x^2(P_{\frac{1}{2}})} + \|\nabla v_i\|_{L^2(P_{\frac{1}{2}})} \leq C, \\ \|\nabla e_i\|_{L^{\frac{10}{3}}(P_{\frac{1}{2}})} + \|\nabla e_i\|_{L_t^\infty L_x^2(P_{\frac{1}{2}})} + \|\nabla^2 e_i\|_{L^2(P_{\frac{1}{2}})} \leq C. \end{cases}$$

So by the Hölder inequality, we have

$$\begin{cases} \int_{P_{\frac{1}{2}}} |v_i \cdot \nabla v_i|^{\frac{5}{4}} \leq \left( \int_{P_{\frac{1}{2}}} |v_i|^{\frac{10}{3}} \right)^{\frac{3}{8}} \left( \int_{P_{\frac{1}{2}}} |\nabla v_i|^2 \right)^{\frac{5}{8}} \leq C, \\ \int_{P_{\frac{1}{2}}} |\nabla \cdot (\nabla e_i \odot \nabla e_i)|^{\frac{5}{4}} \leq \left( \int_{P_{\frac{1}{2}}} |\nabla^2 e_i|^2 \right)^{\frac{5}{8}} \left( \int_{P_{\frac{1}{2}}} |\nabla e_i|^{\frac{10}{3}} \right)^{\frac{3}{8}} \leq C. \end{cases}$$

These inequalities imply

$$\left\| \epsilon_i [v_i \cdot \nabla v_i + \nabla \cdot (\nabla e_i \odot \nabla e_i)] \right\|_{L^{\frac{5}{4}}(P_{\frac{1}{2}})} \leq C. \quad (4.33)$$

By (4.33) and the  $W_{\alpha}^{2,1}$ -estimate of the linear Stokes equation, we have

$$\left\| \partial_t v_i \right\|_{L^{\frac{5}{4}}(P_{\frac{2}{5}})} \leq C. \quad (4.34)$$

Hence  $\{v_i\}$  is bounded in

$$\mathbf{X}_1 = \left\{ u \in L_t^2 H_x^1(P_{\frac{2}{5}}) : \partial_t u \in L_t^{\frac{5}{4}} L_x^{\frac{5}{4}}(P_{\frac{2}{5}}) \right\}.$$

Since  $\mathbf{X}_1$  is compactly embedded in  $L_t^2 L_x^2(P_{\frac{2}{5}})$  by Lemma 4.6, we conclude that  $v_i \rightarrow v$  strongly in  $L^2(P_{\frac{2}{5}})$ .

*Claim 2.*  $\nabla e_i \rightarrow \nabla e$  strongly in  $L^2(P_{\frac{2}{5}})$ . Using (4) and the Hölder inequality we have

$$\|v_i \cdot \nabla e_i\|_{L^{\frac{20}{11}}(P_{\frac{1}{2}})} \leq \|v_i\|_{L^{\frac{10}{3}}(P_{\frac{1}{2}})} \|\nabla e_i\|_{L^4(P_{\frac{1}{2}})} \leq C,$$

so that

$$\left\| |\nabla e_i|^2 d_i + v_i \cdot \nabla e_i \right\|_{L^{\frac{20}{11}}(P_{\frac{1}{2}})} \leq C. \quad (4.35)$$

Hence the  $W_{\alpha}^{2,1}$ -estimate for the heat equation implies

$$\left\| \partial_t \nabla e_i \right\|_{L_t^{\frac{20}{9}} W_x^{-1, \frac{20}{9}}(P_{\frac{2}{5}})} \leq C. \quad (4.36)$$

By (4) and (4.36), we have  $\{\nabla e_i\}$  is bounded in

$$\mathbf{X}_2 = \left\{ u \in L_t^2 H_x^1(P_{\frac{2}{5}}) : \partial_t u \in L_t^{\frac{20}{9}} W_x^{-1, \frac{20}{9}}(P_{\frac{2}{5}}) \right\},$$

and so by Lemma 4.6, we have that  $\nabla e_i \rightarrow \nabla e$  strongly in  $L^2(P_{\frac{2}{5}})$ . It is easy to see that by interpolations, the claims imply that

$$v_i \rightarrow v, \quad \nabla e_i \rightarrow \nabla e \text{ strongly in } L^3(P_{\frac{2}{5}}). \quad (4.37)$$

From (4.37) and (4.32), we conclude that for any  $\theta \in (0, \frac{1}{4})$  and  $i$  sufficiently large,

$$\theta^{-2} \int_{P_\theta} |v_i|^3 + |\nabla e_i|^3 \leq \theta^{-2} \int_{P_\theta} |v|^3 + |\nabla e|^3 + o(1) \leq C\theta^3. \quad (4.38)$$

Finally using the estimate (4.40) below, with  $\tau = \theta$  and  $r = \frac{1}{2}$ , we have that for any  $0 < \theta < \frac{1}{4}$ ,

$$\theta^{-2} \int_{P_\theta} |P_i|^{\frac{3}{2}} \leq C \left[ \theta^{-2} \int_{P_{\frac{1}{2}}} (|u_i|^3 + |\nabla d_i|^3) + \theta \int_{P_{\frac{1}{2}}} |P_i|^{\frac{3}{2}} \right].$$

After scaling, this implies that for any  $0 < \theta < \frac{1}{4}$ ,

$$\theta^{-2} \int_{P_\theta} |Q_i|^{\frac{3}{2}} \leq C \left[ \theta^{-2} \epsilon_i^{\frac{3}{2}} \int_{P_{\frac{1}{2}}} (|v_i|^3 + |\nabla e_i|^3) + \theta \int_{P_{\frac{1}{2}}} |Q_i|^{\frac{3}{2}} \right] \leq C(\epsilon_i^{\frac{3}{2}} \theta^{-2} + \theta). \quad (4.39)$$

Combining (4.38) and (4.39), we have that for sufficiently large  $i = i(\theta)$ ,

$$\theta^{-2} \int_{P_\theta} (|v_i|^3 + |\nabla e_i|^3 + |Q_i|^{\frac{3}{2}}) \leq C\theta.$$

This contradicts (4.18), if we choose  $\theta \in (0, \frac{1}{4})$  sufficiently small.  $\square$

The next Lemma gives the estimate of pressure function, which is needed in the proof of Lemma 4.5.

**Lemma 4.7.** *Suppose that  $(u, P, d)$  is a suitable weak solution of (1.1) on  $P_1$ . Then for any  $0 < r \leq 1$  and  $\tau \in (0, \frac{r}{2})$ , it holds that*

$$\frac{1}{\tau^2} \int_{P_\tau} |P|^{\frac{3}{2}} \leq C \left[ \left(\frac{r}{\tau}\right)^2 \frac{1}{r^2} \int_{P_r} (|u - u_r(t)|^3 + |\nabla d|^3) + \left(\frac{\tau}{r}\right) \frac{1}{r^2} \int_{P_r} |P|^{\frac{3}{2}} \right], \quad (4.40)$$

where  $u_r(t) = \frac{1}{|B_r|} \int_{B_r} u(x, t)$  for  $-r^2 \leq t \leq 0$ . In particular, it holds that

$$\begin{aligned} \frac{1}{\tau^2} \int_{P_\tau} |P|^{\frac{3}{2}} &\leq C \left(\frac{r}{\tau}\right)^2 \left( \sup_{-r^2 \leq t \leq 0} \frac{1}{r} \int_{B_r} |u|^2 \right)^{\frac{3}{4}} \left( \frac{1}{r} \int_{P_r} |\nabla u|^2 \right)^{\frac{3}{4}} \\ &\quad + C \left[ \left(\frac{r}{\tau}\right)^2 \frac{1}{r^2} \int_{P_r} |\nabla d|^3 + \left(\frac{\tau}{r}\right) \frac{1}{r^2} \int_{P_r} |P|^{\frac{3}{2}} \right]. \end{aligned} \quad (4.41)$$

*Proof.* By scaling, it suffices to consider the case  $r = 1$ . Using the equation (1.1)<sub>2</sub>, we have

$$\begin{aligned} \operatorname{div} \operatorname{div} [(u - u_1(t)) \otimes (u - u_1(t))] &= \nabla_j \nabla_i ((u - u_1(t))^i (u - u_1(t))^j) \\ &= \nabla_j ((u - u_1(t))^i \nabla_i (u - u_1(t))^j) = \nabla_j ((u - u_1(t))^i \nabla_i u^j) \\ &= \nabla_j (u - u_1(t))^i \nabla_i u^j + (u - u_1(t))^i \nabla_i \nabla_j u^j \\ &= (\nabla_j u^i) (\nabla_i u^j) = \nabla_j \nabla_i (u^i u^j) = \operatorname{div} \operatorname{div} (u \otimes u). \end{aligned}$$

Taking the divergence of (1.1)<sub>1</sub>, this yields

$$\Delta P = -\operatorname{div} \operatorname{div} [(u - u_1(t)) \otimes (u - u_1(t)) + \nabla d \odot \nabla d]. \quad (4.42)$$

Let  $\eta \in C_0^\infty(\mathbb{R}^3)$  be a cut-off function of  $B_{\frac{1}{2}}$ , i.e.  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  on  $B_{\frac{1}{2}}$ ,  $\eta \equiv 0$  outside  $B_1$ , and  $|\nabla \eta| \leq C$ . Define  $\tilde{P}$  by

$$\tilde{P}(x, t) = - \int_{\mathbb{R}^3} \nabla_y^2 G(x - y) : \eta^2(y) ((u - u_1(t)) \otimes (u - u_1(t)) + \nabla d \odot \nabla d)(y, t),$$

where  $G$  is the fundamental solution of the Laplace equation on  $\mathbb{R}^3$ . We have

$$\Delta \tilde{P} = \operatorname{div} \operatorname{div} ((u - u_1(t)) \otimes (u - u_1(t)) + \nabla d \odot \nabla d) \quad \text{in } \mathbb{R}^3.$$

By the Calderon-Zygmund  $L^p$ -theory we have

$$\begin{aligned} \int_{B_\tau} |\tilde{P}(t)|^{\frac{3}{2}} &\leq \int_{\mathbb{R}^3} |\tilde{P}(t)|^{\frac{3}{2}} \lesssim \int_{\mathbb{R}^3} \eta^3 |(u - u_1(t)) \otimes (u - u_1(t)) + \nabla d \odot \nabla d|^{\frac{3}{2}} \\ &\lesssim \int_{B_1} (|u - u_1(t)|^3 + |\nabla d|^3). \end{aligned}$$

Integrating this inequality over  $t \in (-\tau^2, 0)$  yields

$$\frac{1}{\tau^2} \int_{P_\tau} |\tilde{P}|^{\frac{3}{2}} \leq \frac{C}{\tau^2} \int_{P_1} (|u - u_1(t)|^3 + |\nabla d|^3). \quad (4.43)$$

Since the function  $Q := P - \tilde{P} \in L^{\frac{3}{2}}(P_1)$  satisfies

$$\Delta Q(t) = 0 \quad \text{in } B_{\frac{1}{2}}, \quad \forall t \in [-\frac{1}{4}, 0],$$

we have by the Harnack inequality that for any  $0 < \tau < \frac{1}{2}$ ,

$$\begin{aligned} \frac{1}{\tau^2} \int_{B_\tau} |Q|^{\frac{3}{2}} &\leq C\tau \int_{B_{\frac{1}{2}}} |Q|^{\frac{3}{2}} \leq C\tau \left[ \int_{B_1} |P|^{\frac{3}{2}} + \int_{B_1} |\tilde{P}|^{\frac{3}{2}} \right] \\ &\leq C\tau \left[ \int_{B_1} |P|^{\frac{3}{2}} + \int_{B_1} |u - u_1(t)|^3 + |\nabla d|^3 \right]. \end{aligned}$$

Integrating this inequality over  $t \in [-\tau^2, 0]$  implies

$$\frac{1}{\tau^2} \int_{P_\tau} |Q|^{\frac{3}{2}} \leq C\tau \left[ \int_{P_1} |P|^{\frac{3}{2}} + \int_{P_1} |u - u_1(t)|^3 + |\nabla d|^3 \right]. \quad (4.44)$$

It is now readily seen that (4.40) follows by adding the inequalities (4.43) and (4.44). Using interpolation and the Sobolev inequality, we have

$$\int_{B_1} |u - u_1|^3 \leq C \left( \int_{B_1} |u|^2 \right)^{\frac{3}{4}} \left( \int_{B_1} |\nabla u|^2 \right)^{\frac{3}{4}}. \quad (4.45)$$

Inserting (4.45) into (4.40) yields (4.41).  $\square$

Continuing to iterate the above process, we have

**Corollary 4.8.** *Under the same assumptions as Lemma 4.5, there exists  $\alpha \in (0, 1)$  such that for any  $z_1 \in P_{\frac{r_0}{2}}(z_0)$  and  $0 < \tau < r < \frac{r_0}{2}$ , it holds*

$$\begin{aligned} &\left( \frac{1}{\tau^2} \int_{P_\tau(z_1)} |u|^3 \right)^{\frac{1}{3}} + \left( \frac{1}{\tau^2} \int_{P_\tau(z_1)} |P|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \frac{1}{\tau^2} \int_{P_\tau(z_1)} |\nabla d|^3 \right)^{\frac{1}{3}} \\ &\leq \left( \frac{\tau}{r} \right)^\alpha \left[ \left( \frac{1}{r^2} \int_{P_r(z_1)} |u|^3 \right)^{\frac{1}{3}} + \left( \frac{1}{r^2} \int_{P_r(z_1)} |P|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \frac{1}{r^2} \int_{P_r(z_1)} |\nabla d|^3 \right)^{\frac{1}{3}} \right]. \end{aligned} \quad (4.46)$$

*Proof.* Set  $r_1 = \frac{r_0}{2}$  and  $\epsilon_1 = 2^{\frac{8}{3}}\epsilon_0$ . Then it follows from (4.9) and (4.10) that for any  $z_1 \in P_{\frac{r_0}{2}}(z_0)$ , both (4.9) and (4.10) also hold for  $(u, P, d)$  with  $z_0, r_0$  and  $\epsilon_0$  replaced by  $z_1, r_1$  and  $\epsilon_1$  respectively. For  $0 < \rho < r_1$ , define  $\Phi(\rho)$  by

$$\Phi(\rho) := \left( \frac{1}{\rho^2} \int_{P_\rho(z_1)} |u|^3 \right)^{\frac{1}{3}} + \left( \frac{1}{\rho^2} \int_{P_\rho(z_1)} |P|^{\frac{3}{2}} \right)^{\frac{2}{3}} + \left( \frac{1}{\rho^2} \int_{P_\rho(z_1)} |\nabla d|^3 \right)^{\frac{1}{3}}.$$

Then applying Lemma 4.5 for  $(u, P, d)$  on  $P_{r_1}(z_1)$ , there exists  $\theta_0 \in (0, \frac{1}{2})$  such that for any  $0 < r \leq r_1$ , it holds that

$$\Phi(\theta_0 r) \leq \frac{1}{2} \Phi(r) \leq \frac{1}{2} \epsilon_1.$$

Iterating (4)  $k$ -times,  $k \geq 1$ , yields

$$\Phi(\theta_0^k r) \leq 2^{-k} \Phi(r).$$

It is well known that this implies that there exists  $\alpha \in (0, 1)$  such that for any  $0 < \tau < r \leq r_1$ ,  $\Phi(\tau) \leq \left(\frac{\tau}{r}\right)^\alpha \Phi(r)$ . Therefore (4.46) holds.  $\square$

**Proof of Theorem 4.4.** We will now prove the smoothness of  $(u, d)$  in  $P_{\frac{r_0}{4}}(z_0)$  by the estimate (4.46). The idea is based on the Riesz potential estimates between Morrey spaces, that is analogous to those of Huang-Wang [10] and Lin-Wang [22].

First, let's recall the notion of Morrey spaces on  $\mathbb{R}^3 \times \mathbb{R}$ , equipped with the parabolic metric  $\delta$ :

$$\delta((x, t), (y, s)) = \max \left\{ |x - y|, \sqrt{|t - s|} \right\}, \quad \forall (x, t), (y, s) \in \mathbb{R}^3 \times \mathbb{R}.$$

For any open set  $U \subset \mathbb{R}^{3+1}$ ,  $1 \leq p < +\infty$ , and  $0 \leq \lambda \leq 5$ , define the Morrey Space  $M^{p, \lambda}(U)$  by

$$M^{p, \lambda}(U) := \left\{ v \in L^p_{\text{loc}}(U) : \|v\|_{M^{p, \lambda}(U)} \equiv \sup_{z \in U, r > 0} r^{\lambda-5} \int_{P_r(z) \cap U} |v|^p < \infty \right\}. \quad (4.47)$$

By Corollary 4.8 we have that for some  $\alpha \in (0, 1)$ ,

$$u, \nabla d \in M^{3, 3(1-\alpha)} \left( P_{\frac{r_0}{2}}(z_0) \right). \quad (4.48)$$

Write the equation (1.1)<sub>3</sub> as

$$\partial_t d - \Delta d = f, \quad \text{with } f := (|\nabla d|^2 d - u \cdot \nabla d). \quad (4.49)$$

By (4.48), we see that

$$f \in M^{\frac{3}{2}, 3(1-\alpha)} \left( P_{\frac{r_0}{2}}(z_0) \right).$$

As in [22] and [10], let  $\eta \in C_0^\infty(\mathbb{R}^{3+1})$  be a cut-off function of  $P_{\frac{r_0}{2}}(z_0)$ :  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  in  $P_{\frac{r_0}{2}}(z_0)$ , and  $|\partial_t \eta| + |\nabla^2 \eta| \leq C r_0^{-2}$ . Set  $w = \eta^2 d$ . Then we have

$$\partial_t w - \Delta w = F, \quad F := \eta^2 f + (\partial_t \eta^2 - \Delta \eta^2) d - 2 \nabla \eta^2 \cdot \nabla d. \quad (4.50)$$

It is easy to check that  $F \in M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^{3+1})$  and satisfies the estimate

$$\|F\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^{3+1})} \leq C \left[ 1 + \|f\|_{M^{\frac{3}{2}, 3(1-\alpha)}(P_{\frac{r_0}{2}}(z_0))} \right] \leq C(1 + \epsilon_0). \quad (4.51)$$

Let  $\Gamma(x, t)$  denote the fundamental solution of the heat equation on  $\mathbb{R}^3$ . Then by the Duhamel formula for (4.50) and the estimate (see [10] Lemma 3.1):

$$|\nabla \Gamma|(x, t) \lesssim \frac{1}{\delta^4((x, t), (0, 0))}, \quad \forall (x, t) \neq (0, 0),$$

we have

$$|\nabla w(x, t)| \leq \int_0^t \int_{\mathbb{R}^3} |\nabla \Gamma(x - y, t - s)| |F(y, s)| \leq C \int_{\mathbb{R}^4} \frac{|F(y, s)|}{\delta^4((x, t), (y, s))} := C \mathcal{I}_1(|F|)(x, t), \quad (4.52)$$

where  $\mathcal{I}_\beta$  is the Riesz potential of order  $\beta$  on  $\mathbb{R}^4$  ( $\beta \in [0, 5]$ ), defined by

$$\mathcal{I}_\beta(g) = \int_{\mathbb{R}^4} \frac{|g(y, s)|}{\delta((x, t), (y, s))^{5-\beta}}, \quad g \in L^p(\mathbb{R}^4). \quad (4.53)$$

Applying the Riesz potential estimates (see [10] Theorem 3.1), we conclude that  $\nabla w \in M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)$  and

$$\|\nabla w\|_{M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)} \lesssim \|F\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \lesssim \left[ 1 + \|f\|_{M^{\frac{3}{2}, 3(1-\alpha)}(P_{\frac{r_0}{2}}(z_0))} \right] \lesssim (1 + \epsilon_0). \quad (4.54)$$

Choosing  $\alpha \uparrow \frac{1}{2}$  and using  $\lim_{\alpha \uparrow \frac{1}{2}} \frac{3(1-\alpha)}{1-2\alpha} = +\infty$ , we can conclude that for any  $1 < q < \infty$ ,  $\nabla w \in L^q(P_{r_0}(z_0))$  and

$$\|\nabla w\|_{L^q(P_{r_0}(z_0))} \leq C(q, r_0, \epsilon_0). \quad (4.55)$$

Since  $(d - w)$  solves

$$\partial_t(d - w) - \Delta(d - w) = 0 \text{ in } P_{\frac{r_0}{2}}(z_0),$$

it follows from the standard estimate on the heat equation that for any  $1 < q < +\infty$ ,  $\nabla d \in L^q(P_{\frac{r_0}{4}}(z_0))$  and

$$\left\| \nabla d \right\|_{L^q(P_{\frac{r_0}{4}}(z_0))} \leq C(q, r_0, \epsilon_0). \quad (4.56)$$

Now we proceed with the estimation of  $u$ . Let  $v : \mathbb{R}^3 \times [0, +\infty) \rightarrow \mathbb{R}^3$  solve the Stokes equation:

$$\begin{cases} \partial_t v - \Delta v + \nabla Q = -\nabla \cdot [\eta^2(\nabla d \odot \nabla d + u \otimes u)] & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \nabla \cdot v = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ v(\cdot, 0) = 0 & \text{in } \mathbb{R}^3. \end{cases} \quad (4.57)$$

By using the Oseen kernel (see Leray [14]), an estimate for  $v$ , similar to (4.52), can be given by

$$|v(x, t)| \leq C \int_0^t \int_{\mathbb{R}^3} \frac{|X(y, s)|}{\delta((x, t), (y, s))^{3+1}} \leq C \mathcal{I}_1(|X|)(x, t), \quad (x, t) \in \mathbb{R}^3 \times (0, +\infty), \quad (4.58)$$

where  $X = \eta^2(\nabla d \odot \nabla d + u \otimes u)$ . As above, we can check that  $X \in M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)$  and

$$\left\| X \right\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \leq C \left[ \left\| \nabla d \right\|_{M^{3, 3(1-\alpha)}(P_{\frac{r_0}{2}}(z_0))}^2 + \left\| u \right\|_{M^{3, 3(1-\alpha)}(P_{\frac{r_0}{2}}(z_0))}^2 \right].$$

Hence, by [10] Theorem 3.1, we have that  $v \in M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)$ , and

$$\left\| v \right\|_{M^{\frac{3(1-\alpha)}{1-2\alpha}, 3(1-\alpha)}(\mathbb{R}^4)} \leq C \left\| X \right\|_{M^{\frac{3}{2}, 3(1-\alpha)}(\mathbb{R}^4)} \leq C \left[ \left\| \nabla d \right\|_{M^{3, 3(1-\alpha)}(P_{\frac{r_0}{2}}(z_0))}^2 + \left\| u \right\|_{M^{3, 3(1-\alpha)}(P_{\frac{r_0}{2}}(z_0))}^2 \right]. \quad (4.59)$$

By sending  $\alpha \uparrow \frac{1}{2}$ , (4.59) implies that for any  $1 < q < +\infty$ ,  $v \in L^q(P_{r_0}(z_0))$  and

$$\left\| v \right\|_{L^q(P_{r_0}(z_0))} \leq C(q, r_0, \epsilon_0). \quad (4.60)$$

Note that  $(u - v)$  satisfies the linear homogeneous Stokes equation in  $P_{\frac{r_0}{2}}(z_0)$ :

$$\partial_t(u - v) - \Delta(u - v) + \nabla(P - Q) = 0, \quad \nabla \cdot (u - v) = 0 \quad \text{in } P_{\frac{r_0}{2}}(z_0).$$

It is well-known that  $(u - v) \in L^\infty(P_{\frac{r_0}{4}}(z_0))$ . Therefore we conclude that for any  $1 < q < +\infty$ ,  $u \in L^q(P_{\frac{r_0}{4}}(z_0))$ , and

$$\left\| u \right\|_{L^q(P_{\frac{r_0}{4}}(z_0))} \leq C(q, r_0, \epsilon_0). \quad (4.61)$$

It is now standard that by (4.56) and (4.61), and estimates for the linear parabolic equation and the linear Stokes equation,  $(u, d) \in C^\infty(P_{\frac{r_0}{4}}(z_0), \mathbb{R}^3 \times S^2)$  and the estimate (4.11) holds.  $\square$

## 5. EXISTENCE OF $L^3_{\text{uloc}}$ -SOLUTIONS AND PROOFS OF THEOREM 1.2

In this section, we will prove our main result – Theorem 1.2.

**Proof of Theorem 1.2.** First, observe that by the scaling invariance of (1.1),  $(u, P, d) : \mathbb{R}^3 \times [0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  solves (1.1) under the initial condition  $(u_0, d_0)$  if and only if for any  $\lambda > 0$ ,  $(u^\lambda, P^\lambda, d^\lambda) : \mathbb{R}^3 \times [0, T^\lambda) \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  solves (1.1) under the initial condition  $(u_0^\lambda, d_0^\lambda)$ . Here

$$T^\lambda = \lambda^{-2}T, \quad (u_0^\lambda(x), d_0^\lambda(x)) = (\lambda u_0(\lambda x), d_0(\lambda x)) \quad \text{for } x \in \mathbb{R}^3;$$

and

$$(u^\lambda(x, t), P^\lambda(x, t), d^\lambda(x, t)) = (\lambda u(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t), d(\lambda x, \lambda^2 t)) \quad \text{for } (x, t) \in \mathbb{R}^3 \times [0, T^\lambda).$$

Therefore it suffices to prove Theorem 4.4 for  $R = 1$ . We divide the proof into six steps.

**Step 1.** *Approximation of  $(u_0, d_0)$  by smooth initial data.* We summarize this step into the following lemma.

**Lemma 5.1.** *For a sufficiently small  $\epsilon_0 > 0$ , let  $(u_0, d_0) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times S^2$ , with  $u_0 \in L^3_{\text{loc}}(\mathbb{R}^3)$  divergence free and  $(d_0 - e_0) \in L^3(\mathbb{R}^3)$  for some  $e_0 \in S^2$ , satisfy*

$$\| (u_0, \nabla d_0) \|_{L^3_1(\mathbb{R}^3)} \leq \epsilon_0. \quad (5.1)$$

Then there exist a large constant  $C_0 > 0$  and

$$\{(u_0^k, d_0^k)\} \subset C^\infty(\mathbb{R}^3, \mathbb{R}^3 \times S^2) \cap \bigcap_{p=2}^3 (L^p(\mathbb{R}^3, \mathbb{R}^3) \times \dot{W}^{1,p}(\mathbb{R}^3, S^2))$$

such that the following properties hold:

(i)  $\nabla \cdot u_0^k = 0$  in  $\mathbb{R}^3$  for all  $k \geq 1$ .

(ii) As  $k \rightarrow \infty$ ,

$$(u_0^k, d_0^k) \rightarrow (u_0, d_0) \text{ and } \nabla d_0^k \rightarrow \nabla d_0 \text{ in } L^p_{\text{loc}}(\mathbb{R}^3) \text{ for } p = 2, 3. \quad (5.2)$$

(iii) There exists  $k_0 > 1$  such that for any  $k \geq k_0$ ,

$$\| (u_0^k, \nabla d_0^k) \|_{L^3_1(\mathbb{R}^3)} \leq C_0 \epsilon_0. \quad (5.3)$$

We assume Lemma 5.1 for the moment and continue the proof of Theorem 1.2. By modifying the proof of the local existence Theorem 3.1 of Lin-Lin-Wang [21]<sup>2</sup>, we can conclude that there exist  $0 < T_k < +\infty$  and smooth solutions  $(u^k, P^k, d^k) : \mathbb{R}^3 \times [0, T_k] \rightarrow \mathbb{R}^3 \times \mathbb{R} \times S^2$  of (1.1), under the initial condition  $(u^k, d^k)|_{t=0} = (u_0^k, d_0^k)$ . Observe that by applying the proof of Lemma 4.2 with  $\phi \equiv 1$ , the following energy inequality holds:

$$\int_{\mathbb{R}^3} (|u^k(t)|^2 + |\nabla d^k(t)|^2) + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u^k|^2 + |\Delta d^k + |\nabla d^k|^2 d^k|^2 = \int_{\mathbb{R}^3} (|u_0^k|^2 + |\nabla d_0^k|^2), \quad 0 \leq t \leq T_k. \quad (5.4)$$

In particular, we have that  $(u_k, d_k) \in C([0, T_k], L^2(\mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3))$ .

**Step 2.** *Uniform lower bounds of  $T_k$ .* To see this, we first need to show

*Claim.* There exists  $\tau_0 > 0$  such that if  $T_k$  is the maximal time interval for the smooth solutions  $(u^k, d^k)$  obtained in step 1, then  $T_k \geq \tau_0$ , and

$$\sup_{0 \leq t \leq \tau_0} \| (u^k(t), \nabla d^k(t)) \|_{L^3_{\frac{1}{2}}(\mathbb{R}^3)}^3 \leq 2C_0^3 \epsilon_0^3. \quad (5.5)$$

To see (5.5), note that (5.3) implies that there exists a maximal time  $t_k^* \in (0, T_k]$  such that

$$\sup_{0 \leq t \leq t_k^*} \| (u^k(t), \nabla d^k(t)) \|_{L^3_{\frac{1}{2}}(\mathbb{R}^3)}^3 \leq 2C_0^3 \epsilon_0^3. \quad (5.6)$$

Hence

$$\| (u^k(t_k^*), \nabla d^k(t_k^*)) \|_{L^3_{\frac{1}{2}}(\mathbb{R}^3)}^3 = 2C_0^3 \epsilon_0^3. \quad (5.7)$$

By a simple covering argument, we see that (5.6) implies

$$\sup_{0 \leq t \leq t_k^*} \sup_{x \in \mathbb{R}^3} \int_{B_1(x)} (|u^k(t)|^3 + |\nabla d^k(t)|^3) \leq C \epsilon_0^3. \quad (5.8)$$

For any fixed  $x_0 \in \mathbb{R}^3$ , let  $\phi_0 \in C^\infty_0(\mathbb{R}^3)$  be a cut-off function of  $B_{\frac{1}{2}}(x_0)$ :

$$0 \leq \phi_0 \leq 1, \quad \phi_0 \equiv 1 \text{ on } B_{\frac{1}{2}}(x_0), \quad \phi_0 \equiv 0 \text{ outside } B_1(x_0), \text{ and } |\nabla \phi_0| \leq 4.$$

For convenience, we set for  $0 \leq t \leq t_k^*$ ,

$$\mathcal{E}_3^k(\phi_0; (x_0, t)) := \int_{\mathbb{R}^3} [|u^k(t)|^3 + |\nabla d^k(t)|^3] \phi_0^2. \quad (5.9)$$

<sup>2</sup>For  $K > 0$  and  $0 < \alpha < 1$ , first choose the solution space

$$X_T = \left\{ (u, d) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 : \nabla \cdot u = 0, \nabla^2 f, \partial_t f \in C_b(\mathbb{R}^3 \times [0, T]) \cap C^\alpha(\mathbb{R}^3 \times [0, T]), \right. \\ \left. (u, d)|_{t=0} = (u_0^k, d_0^k), \| (u - u_0^k, d - d_0^k) \|_{C^{\alpha,1}(\mathbb{R}^3 \times [0,1])} \leq K \right\},$$

then follow the fixed point argument as in [21] with slight modifications, one can obtain the local existence of smooth solutions.

Then by (3.6) and (5.8) we have that for any  $0 \leq t \leq t_k^*$ ,

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_3^k(\phi_0; (x_0, t)) + (1 - C\epsilon_0^2) \int_{\mathbb{R}^3} [|\nabla(|u^k(t)|^{\frac{3}{2}}\phi_0)|^2 + |\nabla(|\nabla d^k(t)|^{\frac{3}{2}}\phi_0)|^2] \\ & \leq C \int_{\mathbb{R}^3} (|u^k(t)|^3 + |\nabla d^k(t)|^3) |\nabla \phi_0|^2 + C \sup_{y \in \mathbb{R}^3} \left( \int_{B_1(y)} |u^k(t)|^3 + |\nabla d^k(t)|^3 \right)^{\text{sup}} \\ & \leq C\epsilon_0^3 + C\epsilon_0^5 \leq C\epsilon_0^3. \end{aligned} \quad (5.10)$$

Integrating (5.10) with respect to  $t \in [0, t_k^*]$  yields

$$\begin{aligned} & \mathcal{E}_3^k(\phi_0; (x_0, t_k^*)) + (1 - C\epsilon_0^2) \int_0^{t_k^*} \int_{\mathbb{R}^3} [|\nabla(|u^k|^{\frac{3}{2}}\phi_0)|^2 + |\nabla(|\nabla d^k|^{\frac{3}{2}}\phi_0)|^2] \\ & \leq C\epsilon_0^3 t_k^* + \mathcal{E}_3^k(\phi_0; (x_0, 0)) \leq C\epsilon_0^3 t_k^* + C_0^3 \epsilon_0^3, \end{aligned} \quad (5.11)$$

where we have used (5.3) in the last step. Therefore if  $\epsilon_0 > 0$  is chosen such that  $1 - C\epsilon_0^2 \geq 0$ , then (5.11) implies

$$\mathcal{E}_3^k(\phi_0; (x_0, t_k^*)) \leq C\epsilon_0^3 t_k^* + C_0^3 \epsilon_0^3.$$

Taking the supremum of  $\mathcal{E}_3^k(\phi_0; (x_0, t_k^*))$  over  $x_0 \in \mathbb{R}^3$ , we obtain

$$2C_0^3 \epsilon_0^3 = |||(u^k(t_k^*), \nabla d^k(t_k^*))|||_{L^{\frac{3}{2}}(\mathbb{R}^3)}^3 \leq \sup_{x_0 \in \mathbb{R}^3} E_3^k(\phi_0; (x_0, t_k^*)) \leq C\epsilon_0^3 t_k^* + C_0^3 \epsilon_0^3.$$

This clearly implies that there exists  $\tau_0 > 0$  such that  $T_k \geq t_k^* \geq \tau_0$ . By the definition of  $t_k^*$ , we also see that the estimate (5.5) holds.

**Step 3.** *Uniform estimation of  $(u^k, d^k)$ .* Note that  $P^k$  satisfies

$$\Delta P^k = -\text{div}^2(u^k \otimes u^k + \nabla d^k \odot \nabla d^k) \quad \text{in } \mathbb{R}^3.$$

It follows from (5.4), (5.5) and Lemma 3.2 that

$$\sup_{0 \leq t \leq \tau_0} \sup_{x \in \mathbb{R}^3} \|P^k(t) - c_x^k(t)\|_{L^3(B_1(x))} \leq C\epsilon_0, \quad (5.12)$$

where  $c_x^k(t) \in \mathbb{R}$  depends on both  $x \in \mathbb{R}^3$  and  $t \in [0, \tau_0]$ . By (5.5) and (5.12), we see that for any  $x_0 \in \mathbb{R}^3$ ,  $(u^k, P^k - c_{x_0}^k, d^k)$  satisfies the conditions of Theorem 4.4 in  $P_{\sqrt{\tau_0}}(x_0, \tau_0) := B_{\sqrt{\tau_0}}(x_0) \times [0, \tau_0]$ . Hence by Theorem 4.4 we obtain that  $(u^k, d^k) \in C^\infty(\mathbb{R}^3 \times (0, \tau_0), \mathbb{R}^3 \times S^2)$ , and

$$\sup_k \|(u^k, \nabla d^k)\|_{C^m(\mathbb{R}^3 \times [\delta, \tau_0])} \leq C(m, \delta, \epsilon_0) \quad (5.13)$$

holds for any  $0 < \delta < \frac{\tau_0}{2}$  and  $m \geq 0$ .

**Step 4.** *Passage to the limit.* Based on the estimates of  $(u^k, d^k)$ , we may assume, after taking subsequences, that  $(u, d) \in \bigcap_{0 < \delta < \tau_0} C_b^\infty(\mathbb{R}^3 \times [\delta, \tau_0], \mathbb{R}^3 \times S^2)$ , with  $(u, \nabla d) \in L^\infty([0, \tau_0], L^3_{\text{uloc}}(\mathbb{R}^3))$ , such that

$(u^k, \nabla d^k) \rightarrow (u, \nabla d)$  weakly in  $L^3(\mathbb{R}^3 \times [0, \tau_0])$ ,  $(u^k, d^k) \rightarrow (u, d)$  in  $C^m(B_R \times [\delta, \tau_0])$ ,  $\forall m \geq 0, R > 0, \delta < \tau_0$ .

Sending  $k \rightarrow \infty$  in (5.8) yields

$$\sup_{0 \leq t \leq \tau_0} \|(u, \nabla d)\|_{L^3_1(\mathbb{R}^3)} \leq C\epsilon_0.$$

We can check from (1.1) and (5.8) that for any  $R > 0$ ,

$$\|(\partial_t u^k, \partial_t d^k)\|_{L^{\frac{3}{2}}([0, \tau_0], W^{-1, \frac{3}{2}}(B_R))} \leq C(R) < +\infty.$$

This implies that

$$(u(t), \nabla d(t)) \rightarrow (u_0, \nabla d_0) \text{ strongly in } L^3_{\text{loc}}(\mathbb{R}^3) \text{ as } t \downarrow 0. \quad (5.14)$$

In particular, we have that  $(u_0, \nabla d_0) \in C_*^0([0, \tau_0], L^3_{\text{uloc}}(\mathbb{R}^3))$ .

**Step 5.** *Characterization of the maximal time interval  $T_0$ .* Let  $T_0 > \tau_0$  be the maximal time interval in which the solution  $(u, d)$  constructed in step 4 exists. Suppose that  $T_0 < +\infty$  and (1.8) were false. Then there exists  $r_0 > 0$  so that

$$\limsup_{t \uparrow T_0} |||(u(t), \nabla d(t))|||_{L^3_{r_0}(\mathbb{R}^3)} \leq \epsilon_0.$$

In particular, there exists  $r_1 \in (0, r_0]$  such that

$$\sup_{T_0 - r_1^2 \leq t \leq T_0} \|(u(t), \nabla d(t))\|_{L_{r_1}^3(\mathbb{R}^3)} \leq \epsilon_0.$$

Hence by Theorem 4.4, we conclude that  $(u, d) \in C_b^\infty(\mathbb{R}^3 \times [0, T_0]) \cap L^\infty([0, T_0], L_{\text{uloc}}^3(\mathbb{R}^3))$ . This contradicts the maximality of  $T_0$ . Hence (1.8) holds.

**Step 6. Uniqueness.** Let  $(u_1, d_1), (u_2, d_2) : \mathbb{R}^3 \times [0, T_0] \rightarrow \mathbb{R}^3 \times S^2$  be two solutions of (1.1), under the same initial condition  $(u_0, d_0)$ , that satisfy the properties of Theorem 1.2. We first show  $(u_1, d_1) \equiv (u_2, d_2)$  in  $\mathbb{R}^3 \times [0, \tau_0]$ . This can be done by the argument of [27] page 15-16. For convenience, we sketch it here.

Set  $u = u_1 - u_2, d = d_1 - d_2$ . Then  $(u, d)$  satisfies

$$\begin{cases} \partial_t u - \Delta u = -\mathbb{P}\nabla \cdot [u_1 \otimes u_1 - u_2 \otimes u_2 + \nabla d_1 \odot \nabla d_1 - \nabla d_2 \odot \nabla d_2] \\ \partial_t d - \Delta d = -(u_1 \cdot \nabla d_1 - u_2 \cdot \nabla d_2) + |\nabla d_1|^2 d_1 - |\nabla d_2|^2 d_2 \\ (u, d)|_{t=0} = (0, 0). \end{cases}$$

By the Duhamel formula, we have

$$\begin{cases} u(t) = -\mathbb{V}[u_1 \otimes u_1 - u_2 \otimes u_2 + \nabla d_1 \odot \nabla d_1 - \nabla d_2 \odot \nabla d_2] \\ d(t) = -\mathbb{S}[(u_1 \cdot \nabla d_1 - u_2 \cdot \nabla d_2) - (|\nabla d_1|^2 d_1 - |\nabla d_2|^2 d_2)], \end{cases}$$

where

$$\mathbb{S}f(t) = \int_0^t e^{-(t-s)\Delta} f(s) ds, \quad \mathbb{V}f(t) = \int_0^t e^{-(t-s)\Delta} \mathbb{P}\nabla \cdot f(s) ds, \quad \forall f : \mathbb{R}^3 \times [0, +\infty) \rightarrow \mathbb{R}^3.$$

Recall the three function spaces used in [27]. Let  $\mathbf{X}_{\tau_0}$  denote the space of functions  $f$  on  $\mathbb{R}^3 \times [0, \tau_0]$  such that

$$\|f\|_{\mathbf{X}_{\tau_0}} := \sup_{0 < t \leq \tau_0} \|f(t)\|_{L^\infty(\mathbb{R}^3)} + \|f\|_{X_{\tau_0}} < +\infty,$$

where

$$\|f\|_{\mathbf{X}_{\tau_0}} := \sup_{0 < t \leq \tau_0} \sqrt{t} \|\nabla f(t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x \in \mathbb{R}^3, 0 < r \leq \sqrt{\tau_0}} (r^{-3} \int_{P_r(x, r^2)} |\nabla f|^2)^{\frac{1}{2}},$$

$\mathbf{Y}_{\tau_0}$  denote the space of functions  $g$  on  $\mathbb{R}^3 \times [0, \tau_0]$  such that

$$\|g\|_{\mathbf{Y}_{\tau_0}} := \sup_{0 < t \leq \tau_0} t \|g(t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x \in \mathbb{R}^3, 0 < r \leq \sqrt{\tau_0}} r^{-3} \int_{P_r(x, r^2)} |g| < +\infty,$$

and  $\mathbf{Z}_{\tau_0}$  the space of functions  $h$  on  $\mathbb{R}^3 \times [0, \tau_0]$  such that

$$\|h\|_{\mathbf{Z}_{\tau_0}} := \sup_{0 < t \leq \tau_0} \sqrt{t} \|h(t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x \in \mathbb{R}^3, 0 < r \leq \sqrt{\tau_0}} (r^{-3} \int_{P_r(x, r^2)} |h|^2)^{\frac{1}{2}} < +\infty.$$

Since  $(u_i, d_i) \in L^\infty([0, \tau_0], L^2(\mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3))$  satisfies (1.7) for  $i = 1, 2$ , Theorem 4.4 and the Hölder inequality imply that  $u_i \in \mathbf{Z}_{\tau_0}, d_i \in \mathbf{X}_{\tau_0}$  for  $i = 1, 2$ , and

$$\sum_{i=1}^2 (\|u_i\|_{\mathbf{Z}_{\tau_0}} + \|d_i\|_{\mathbf{X}_{\tau_0}}) \leq C\epsilon_0.$$

It follows from Lemma 3.1 and Lemma 4.1 of [27] that

$$\begin{aligned} \|u\|_{\mathbf{Z}_{\tau_0}} + \|d\|_{\mathbf{X}_{\tau_0}} &\lesssim \left\| (|u_1| + |u_2|)|u| + (|\nabla d_1| + |\nabla d_2|)|\nabla d| \right\|_{\mathbf{Y}_{\tau_0}} \\ &\quad + \left\| |u|\nabla d_2 + |u_1|\nabla d + (|\nabla d_1| + |\nabla d_2|)|\nabla d| + |\nabla d_2|^2 |d| \right\|_{\mathbf{Y}_{\tau_0}} \\ &\lesssim \left[ \sum_{i=1}^2 (\|d_i\|_{\mathbf{X}_{\tau_0}} + \|u_i\|_{\mathbf{Z}_{\tau_0}}) \|u\|_{\mathbf{Z}_{\tau_0}} + \left[ \sum_{i=1}^2 (\|u_i\|_{\mathbf{Z}_{\tau_0}} + \|d_i\|_{\mathbf{X}_{\tau_0}}) \right] \|d\|_{\mathbf{X}_{\tau_0}} \right] \\ &\lesssim \epsilon_0 [\|u\|_{\mathbf{Z}_{\tau_0}} + \|d\|_{\mathbf{X}_{\tau_0}}]. \end{aligned}$$

This clearly implies that  $(u_1, d_1) \equiv (u_2, d_2)$  in  $\mathbb{R}^3 \times [0, \tau_0]$ . Since  $(u_1, d_1)$  and  $(u_2, d_2)$  are classical solutions of (1.1) in  $\mathbb{R}^3 \times [\tau_0, T_0)$ , and  $(u_1, d_1) = (u_2, d_2)$  at  $t = \tau_0$ , it is well-known that  $(u_1, d_1) \equiv (u_2, d_2)$  in  $\mathbb{R}^3 \times [\tau_0, T_0)$ .

The proof is complete.  $\square$

Finally, we provide the proof of Lemma 5.1.

**Proof of Lemma 5.1:** Let  $\theta \in C^\infty([0, +\infty))$  be such that

$$\theta(r) = 1 \text{ for } 0 \leq r \leq 1; \quad 0 \leq \theta(r) \leq 1 \text{ for } 1 \leq r \leq 2; \quad \theta(r) = 0 \text{ for } r \geq 2.$$

Let  $\eta \in C_0^\infty(\mathbb{R}^3)$  be a standard mollifier, and define for  $k \geq 1$

$$\eta_{\frac{1}{k}}(x) = k^3 \eta(kx) \text{ and } \theta_k(x) = \theta\left(\frac{|x|}{k}\right) \text{ for } x \in \mathbb{R}^3.$$

*Step 1. Approximation of  $d_0$ .* This will be done by two rounds of approximation. It follows from  $(d_0 - e_0) \in L^3(\mathbb{R}^3)$  that there exists  $k_0 > 1$  such that for any  $k \geq k_0$ , it holds

$$\int_{\mathbb{R}^3 \setminus B_{k-1}} |d_0 - e_0|^3 \leq \epsilon_0^3. \quad (5.15)$$

By the Fubini theorem, we may assume that for  $k \geq k_0$ , it also holds

$$\begin{cases} \int_{\partial B_k} |d_0 - e_0|^3 dH^2 \leq 2 \int_{\mathbb{R}^3 \setminus B_{k-1}} |d_0 - e_0|^3 \leq 2\epsilon_0^3, \\ \sup_{x \in \partial B_k} \int_{\partial B_k \cap B_2(x)} |\nabla d_0|^3 dH^2 \leq 4 \|\nabla d_0\|_{L^3_2(\mathbb{R}^3)}^3 \leq C\epsilon_0^3, \\ \int_{\partial B_k} |\nabla d_0|^3 dH^2 \leq 2 \int_{B_{k+1}} |\nabla d_0|^3 \lesssim k^3 \|\nabla d_0\|_{L^3_1(\mathbb{R}^3)}^3 \leq k^3 \epsilon_0^3. \end{cases} \quad (5.16)$$

Define the approximate sequence  $\widetilde{d}_0^k : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$\widetilde{d}_0^k(x) = \begin{cases} d_0(x) & \text{if } |x| \leq k \\ (|x| - k)e_0 + (k + 1 - |x|)d_0\left(k \frac{x}{|x|}\right) & \text{if } k \leq |x| \leq k + 1 \\ e_0 & \text{if } |x| \geq k + 1. \end{cases}$$

Then by direct calculations we have that

$$\begin{aligned} \|\nabla \widetilde{d}_0^k\|_{L^p(\mathbb{R}^3)}^p &= \int_{B_k} |\nabla d_0|^p + \int_{B_{k+1} \setminus B_k} |\nabla \widetilde{d}_0^k|^p \\ &\lesssim \int_{B_k} |\nabla d_0|^p + \int_{\partial B_k} |\nabla d_0|^p dH^2 + \int_{\partial B_k} |d_0 - e_0|^p dH^2 \\ &\lesssim k^p \epsilon_0^p < +\infty, \text{ for } p = 2, 3, \\ \left\| \nabla \widetilde{d}_0^k \right\|_{L^3_1(\mathbb{R}^3)}^3 &\lesssim \|\nabla d_0\|_{L^3_1(\mathbb{R}^3)}^3 + \sup_{x \in \partial B_k} \int_{\partial B_k \cap B_1(x)} |d_0 - e_0|^3 dH^2 \\ &\quad + \sup_{x \in \partial B_k} \int_{\partial B_k \cap B_1(x)} |\nabla d_0|^3 dH^2 \\ &\leq C\epsilon_0^3, \end{aligned}$$

and for any  $x_0 \in B_{k+1} \setminus B_k$ ,

$$\begin{aligned} \text{dist}(\widetilde{d}_0^k(x_0), S^2) &\leq \frac{1}{|B_1|} \int_{B_1(x_0)} \left| \widetilde{d}_0^k(x_0) - d_0(y) \right| \\ &\lesssim \int_{B_1(x_0)} \left| (|x_0| - k)e_0 + (k + 1 - |x_0|)d_0\left(k \frac{x_0}{|x_0|}\right) - d_0(y) \right| \\ &\lesssim \int_{B_1(x_0)} \left| d_0(y) - e_0 \right| + \left| d_0(y) - d_0\left(k \frac{x_0}{|x_0|}\right) \right| \\ &\lesssim \left( \int_{\mathbb{R}^3 \setminus B_k} |d_0 - e_0|^3 \right)^{\frac{1}{3}} + \|\nabla d_0\|_{L^3_1(\mathbb{R}^3)} \leq 2\epsilon_0. \end{aligned}$$

This implies

$$\sup_{x_0 \in \mathbb{R}^3} \text{dist}(\widetilde{d}_0^k(x_0), S^2) = \sup_{x_0 \in B_{k+1} \setminus B_k} \text{dist}(\widetilde{d}_0^k(x_0), S^2) \leq 2\epsilon_0$$

so that if  $\epsilon_0 > 0$  is chosen sufficiently small then  $\widetilde{d}_0^k(x)$  remains close to  $S^2$  uniformly for  $x \in \mathbb{R}^3$ . Therefore we can project  $\widetilde{d}_0^k$  onto  $S^2$  to get  $\widehat{d}_0^k(x) = \frac{\widetilde{d}_0^k(x)}{|\widetilde{d}_0^k(x)|}$  for  $x \in \mathbb{R}^3$ . It is easy to see that  $\widehat{d}_0^k : \mathbb{R}^3 \rightarrow S^2$  satisfies:

$$\widehat{d}_0^k = d_0 \text{ in } B_k, \widehat{d}_0^k = e_0 \text{ in } \mathbb{R}^3 \setminus B_{k+1}, \left\| \nabla \widehat{d}_0^k \right\|_{L^3_1(\mathbb{R}^3)} \leq C\epsilon_0, \text{ and } \int_{\mathbb{R}^3} \left| \nabla \widehat{d}_0^k \right|^p \leq Ck^p \epsilon_0^p < +\infty \text{ (} p = 2, 3 \text{)}. \quad (5.17)$$

For any  $l, k \geq 1$ , define  $d_0^{k,l}(x) = \left( \eta_{\frac{1}{l}} * \widehat{d}_0^k \right)(x)$  for  $x \in \mathbb{R}^3$ . Then  $d_0^{k,l} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$  satisfies

$$\left\| \nabla d_0^{k,l} \right\|_{L^3_1(\mathbb{R}^3)} \leq C\epsilon_0, \text{ and } \int_{\mathbb{R}^3} \left| \nabla d_0^{k,l} \right|^p \leq Ck^p \epsilon_0^3 < +\infty, \forall l \geq 1, \text{ (} p = 2, 3 \text{)}, \quad (5.18)$$

and by the modified Poincaré inequality it holds that

$$\sup_{x \in \mathbb{R}^3} \text{dist}(d_0^{k,l}(x), S^2) \lesssim \left\| \nabla d_0^{k,l} \right\|_{L^3_1(\mathbb{R}^3)} \leq C\epsilon_0, \forall l \geq 1, \quad (5.19)$$

and for any  $k \geq 1$ ,

$$\lim_{l \rightarrow \infty} \left( \left\| d_0^{k,l} - d_0 \right\|_{L^p(B_{k-1})} + \left\| \nabla(d_0^{k,l} - d_0) \right\|_{L^p(B_{k-1})} \right) = 0, \text{ for } p = 2, 3.$$

Therefore, by the Cauchy diagonal process we may conclude that, after taking possible subsequences, there exist  $l(k) \rightarrow \infty$  as  $k \rightarrow \infty$  such that

$$d_0^k(x) = \frac{d_0^{k,l(k)}}{\left| d_0^{k,l(k)} \right|}(x), \forall x \in \mathbb{R}^3,$$

satisfies the desired properties of approximation:  $d_0^k \in C^\infty(\mathbb{R}^3, S^2) \cap \dot{W}^{1,p}(\mathbb{R}^3, S^2)$  ( $p = 2, 3$ ), and

$$\left\| \nabla d_0^k \right\|_{L^3_1(\mathbb{R}^3)} \leq C_0 \epsilon_0, \quad (5.20)$$

and for any  $0 < R < +\infty$ ,

$$\lim_{k \rightarrow \infty} \left[ \left\| d_0^k - d_0 \right\|_{L^p(B_R)} + \left\| \nabla(d_0^k - d_0) \right\|_{L^p(B_R)} \right] = 0, \text{ for } p = 2, 3. \quad (5.21)$$

Next we would like to obtain the desired approximation of  $u_0$ , whose proof is similar to [2] Theorem 1.4. For the completeness, we outline the detail below.

*Step 2. Approximation of  $u_0$ .* Let  $\mathbb{P} : L^2(\mathbb{R}^3) \rightarrow \mathbb{P}L^2(\mathbb{R}^3)$  denote the Leray projection operator. For  $k \geq 1$ , define

$$\widetilde{u}_0^k(x) = \mathbb{P}[\theta_k u_0](x), \quad x \in \mathbb{R}^3.$$

Since  $\theta_k u_0 \in L^p(\mathbb{R}^3, \mathbb{R}^3)$  and  $\mathbb{P} : L^p(\mathbb{R}^3) \rightarrow \mathbb{P}L^p(\mathbb{R}^3)$  is bounded, it follows that  $\nabla \cdot \widetilde{u}_0^k = 0$  in  $\mathbb{R}^3$  and  $\widetilde{u}_0^k \in L^p(\mathbb{R}^3)$  for  $p = 2, 3$ . Now we want to show

$$\left\| \widetilde{u}_0^k \right\|_{L^3_1(\mathbb{R}^3)} \lesssim \left\| u_0 \right\|_{L^3_1(\mathbb{R}^3)}, \quad (5.22)$$

and

$$\widetilde{u}_0^k \rightarrow u_0 \text{ strongly in } L^p_{\text{loc}}(\mathbb{R}^3) \text{ for } p = 2, 3. \quad (5.23)$$

Since

$$\widetilde{u}_0^k(x) = (\theta_k u_0)(x) - \nabla \Delta^{-1} \nabla \cdot [\theta_k u_0](x),$$

and  $\|\theta_k u_0\|_{L^3_1(\mathbb{R}^3)} \leq \|u_0\|_{L^3_1(\mathbb{R}^3)}$ , it suffices to show

$$\left\| \nabla \Delta^{-1} \nabla \cdot [\theta_k u_0] \right\|_{L^3_1(\mathbb{R}^3)} \lesssim \left\| u_0 \right\|_{L^3_1(\mathbb{R}^3)}.$$

Set  $\Phi = \nabla \Delta^{-1} \nabla \cdot [\theta_k u_0]$ . Then we have

$$\widetilde{u}_0^k(x) = \theta\left(\frac{x}{k}\right)u_0(x) - \Phi(x), \quad x \in \mathbb{R}^3.$$

It follows from  $\nabla \cdot u_0 = 0$  that we have

$$\begin{aligned}\Phi(x) &= \nabla \Delta^{-1} \nabla \cdot [\theta_k u_0](x) = \nabla \Delta^{-1} [(\nabla \theta_k) \cdot u_0](x) \\ &= \frac{1}{k} \int_{\mathbb{R}^3} K(x-y) \nabla \theta\left(\frac{y}{k}\right) \cdot u_0(y) \\ &= \frac{1}{k} \int_{B_k(x)} K(x-y) \nabla \theta\left(\frac{y}{k}\right) \cdot u_0(y) + \frac{1}{k} \int_{\mathbb{R}^3 \setminus B_k(x)} K(x-y) \nabla \theta\left(\frac{y}{k}\right) \cdot u_0(y) \\ &= I(x) + II(x),\end{aligned}$$

where  $K(x) = c_3 \frac{x}{|x|^3}$ ,  $c_3 = \frac{1}{3|B_1|}$ , is the kernel of the operator  $\nabla \Delta^{-1}$ . We estimate  $I$  and  $II$  separately as follows. It is easy to see that

$$\|I\|_{L^3_1(\mathbb{R}^3)} \leq \frac{1}{k} \|K\|_{L^1(B_k)} \left\| \nabla \theta\left(\frac{\cdot}{k}\right) \cdot u_0 \right\|_{L^3_1(\mathbb{R}^3)} \leq C \|u_0\|_{L^3_1(\mathbb{R}^3)},$$

while

$$|II(x)| \leq \frac{C}{k^3} \int_{B_{2k}} |u_0(y)| \leq C \|u_0\|_{L^3_1(\mathbb{R}^3)},$$

so that

$$\|II\|_{L^3_1(\mathbb{R}^3)} \leq C \|u_0\|_{L^3_1(\mathbb{R}^3)}.$$

Combining these two estimates implies (5.22).

For any fixed compact set  $E \subset \mathbb{R}^3$  and  $x \in E$ , we write

$$\begin{aligned}\Phi(x) &= \frac{c_3}{k} \int_{\mathbb{R}^3} \left( \frac{x-y}{|x-y|^3} + \frac{y}{|y|^3} \right) \nabla \theta\left(\frac{y}{k}\right) \cdot u_0(y) - \frac{c_3}{k} \int_{\mathbb{R}^3} \frac{y}{|y|^3} \nabla \theta\left(\frac{y}{k}\right) \cdot u_0(y) \\ &= III_k(x) + IV_k(u_0).\end{aligned}$$

Since  $\nabla \theta\left(\frac{y}{k}\right)$  has its support in  $B_{2k} \setminus B_k$ , for  $k$  sufficiently large we have that

$$\left| \frac{x-y}{|x-y|^3} + \frac{y}{|y|^3} \right| \leq \frac{C_E}{k^3}, \text{ for } x \in E \text{ and } y \in B_{2k} \setminus B_k,$$

and hence it holds

$$|III_k(x)| \leq \frac{C_E}{k^4} \int_{B_{2k} \setminus B_k} |u_0(y)| \leq \frac{C_E}{k} \|u_0\|_{L^3_1(\mathbb{R}^3)} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

while it is easy to bound  $IV_k(u_0)$  by

$$|IV_k(u_0)| \lesssim \frac{1}{k^3} \int_{B_{2k}} |u_0(y)| \lesssim \|u_0\|_{L^3_1(\mathbb{R}^3)}.$$

Hence we may assume that there exists a constant vector  $c \in \mathbb{R}^3$ , with  $|c| \leq C \|u_0\|_{L^3_1(\mathbb{R}^3)}$ , such that

$$\lim_{k \rightarrow \infty} IV_k(u_0) = c.$$

Now we define

$$\widehat{u}_0^k(x) = \widetilde{u}_0^k(x) + \frac{3}{2} \mathbb{P}\left[\theta\left(\frac{x}{k}\right)c\right], \quad x \in \mathbb{R}^3.$$

Then we have that  $\widehat{u}_0^k \in L^p(\mathbb{R}^3)$  for  $p = 2, 3$ , and

$$\left\| \widehat{u}_0^k \right\|_{L^3_1(\mathbb{R}^3)} \leq C \|u_0\|_{L^3_1(\mathbb{R}^3)}.$$

It is easy to check that for any  $x \in E$ , if  $k \rightarrow \infty$  then

$$\mathbb{P}\left[\theta\left(\frac{x}{k}\right)c\right] = \theta\left(\frac{x}{k}\right)c - \nabla \Delta^{-1} \nabla \cdot \left[\theta\left(\frac{x}{k}\right)c\right] = \theta\left(\frac{x}{k}\right)c + o(1) + \frac{c_3}{k} \int_{\mathbb{R}^3} \frac{y}{|y|^3} \nabla \theta\left(\frac{y}{k}\right) \cdot c \rightarrow \frac{2}{3}c.$$

Therefore, for any  $x \in E$ , if  $k \rightarrow \infty$  then

$$\begin{aligned}\widehat{u}_0^k(x) - u_0(x) &= \left(\theta\left(\frac{x}{k}\right) - 1\right)u_0(x) - \Phi(x) + \frac{3}{2} \mathbb{P}\left[\theta\left(\frac{x}{k}\right)c\right] \\ &= \left(\theta\left(\frac{x}{k}\right) - 1\right)u_0(x) - III_k(x) - IV_k(u_0) + \frac{3}{2} \mathbb{P}\left[\theta\left(\frac{x}{k}\right)c\right] \rightarrow 0.\end{aligned}$$

This clearly implies (5.23). The proof of Lemma 5.1 is not complete yet, since  $\widehat{u_0^k} \notin C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ . To overcome this, we mollify  $\widehat{u_0^k}$  to get

$$u_0^{k,l}(x) = \left( \eta_{\frac{1}{l}} * u_k \right) (x), \quad x \in \mathbb{R}^3, \forall l \geq 1.$$

Then it is straightforward to check that  $u_0^{k,l} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$  for  $p = 2, 3$ ,  $\nabla \cdot u_0^{k,l} = 0$ ,

$$\left\| u_0^{k,l} \right\|_{L_1^3(\mathbb{R}^3)} \leq \left\| \widehat{u_0^k} \right\|_{L_1^3(\mathbb{R}^3)} \leq C \left\| u_0 \right\|_{L_1^3(\mathbb{R}^3)},$$

and for any  $k \geq 1$ ,

$$u_0^{k,l} \rightarrow \widehat{u_0^k} \text{ strongly in } L_{\text{loc}}^p(\mathbb{R}^3) \text{ for } p = 2, 3, \text{ as } l \rightarrow \infty.$$

Thus, by the Cauchy diagonal process we may assume that there exist  $l(k) \rightarrow \infty$  as  $k \rightarrow \infty$  such that

$$u_0^k(x) = u_0^{k,l(k)}(x), \quad x \in \mathbb{R}^3$$

satisfies the required properties of approximation of  $u_0$ :  $u_0^k \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$  for  $p = 2, 3$ ,  $\nabla \cdot u_0^k = 0$ ,

$$\left\| u_0^k \right\|_{L_1^3(\mathbb{R}^3)} \leq C \left\| u_0 \right\|_{L_1^3(\mathbb{R}^3)},$$

and

$$u_0^k \rightarrow u_0 \text{ strongly in } L_{\text{loc}}^p(\mathbb{R}^3) \text{ for } p = 2, 3, \text{ as } k \rightarrow \infty.$$

This completes the proof of Lemma 5.1. □

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