An Introduction of Infinity Harmonic Functions

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Abstract

This note serves as a basic introduction on the analysis of infinity harmonic functions, a subject that has received considerable interests very recently. The author discusses its connection with absolute minimal Lipschitz extension, present several equivalent characterizations of infinity harmonic functions. He presents the celebrated theorem by R. Jensen [17] on the uniqueness of infinity harmonic functions, the linear approximation property of infinity harmonic functions by Crandall and Evans [9] and the asymptotic behavior near an isolated singularity of infinity harmonic functions by Savin, Wang, and Yu [25].

Contents

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1 Absolute Minimal Lipschitz Extension

For $n \geq 1$ and any subset $E \subset \mathbf{R}^n$, we define the space of Lipschitz continuous functions on E by

$$\mathrm{Lip}(E) := \left\{ f : E \to \mathbf{R} : \ \mathrm{Lip}_E(f) \equiv \sup_{x,y \in E, x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < +\infty \right\}.$$

1.1 Minimal Lipschitz Extension or MLE

The problem of minimal Lipschitz extension (MLE) states that for any domain $\Omega \subset \mathbf{R}^n$ and a given Lipschitz continuous function $g : \partial\Omega \to \mathbf{R}$, find a Lipschitz continuous extension $G : \overline{\Omega} \to \mathbf{R}$ such that $G|_{\partial\Omega} = g$, and

$$\mathrm{Lip}_\Omega(G) = \min \left\{ \mathrm{Lip}_\Omega(H) : \ H \in \mathrm{Lip}(\Omega), \ H\big|_{\partial\Omega} = g \right\}. \tag{1}$$

The existence of MLEs is well-known. In fact, Mcshane [23] and Whitney [27] constructed two such extensions:

$$g^+(x) = \inf_{y \in \partial\Omega} \left\{ g(y) + \mathrm{Lip}_{\partial\Omega}(g)\,|x - y| \right\}, \ x \in \overline{\Omega}, \tag{2}$$

$$g^-(x) = \sup_{y \in \partial\Omega} \left\{ g(y) - \mathrm{Lip}_{\partial\Omega}(g)\,|x - y| \right\}, \ x \in \overline{\Omega}. \tag{3}$$

By the definition, we know that any Lipschitz extension $G \in \mathrm{Lip}(\Omega)$ of $g \in \mathrm{Lip}(\partial\Omega)$ automatically satisfies

$$\mathrm{Lip}_\Omega(G) \geq \mathrm{Lip}_{\partial\Omega}(g). \tag{4}$$

For g^+ and g^-, we can check that

(i) $g^+\big|_{\partial\Omega} = g^-\big|_{\partial\Omega} = g$,

(ii) $\mathrm{Lip}_\Omega(g^+) = \mathrm{Lip}_\Omega(g^-) = \mathrm{Lip}_{\partial\Omega}(g)$.

In fact, for any $x \in \partial\Omega$, it follows from the definition that

$$g^+(x) \leq g(x) + \mathrm{Lip}_{\partial\Omega}(g)|x - x| = g(x),$$

and since

$$g(y) + \mathrm{Lip}_{\partial\Omega}(g)|x - y| \geq g(x), \ \forall y \in \partial\Omega,$$

we have

$$g^+(x) = \inf_{y \in \partial\Omega} \{g(y) + \mathrm{Lip}_{\partial\Omega}(g)|x - y|\} \geq g(x).$$

Hence $g^+ = g$ on $\partial\Omega$. Similarly, one can check $g^- = g$ on $\partial\Omega$.

To verify (ii), let $x_1, x_2 \in \Omega$ be any pair of points. By the definition, for any $\epsilon > 0$, there exists $y_1 \in \partial\Omega$ such that

$$g^+(x_1) \geq g(y_1) + \mathrm{Lip}_{\partial\Omega}(g)|x_1 - y_1| - \epsilon.$$

Since

$$g^+(x_2) \leq g(y_1) + \mathrm{Lip}_{\partial\Omega}(g)|x_2 - y_1|,$$

we then obtain

$$\begin{aligned} g^+(x_2) - g^+(x_1) &\leq \mathrm{Lip}_{\partial\Omega}(g)\left(|x_2 - y_1| - |x_1 - y_1|\right) + \epsilon \\ &\leq \mathrm{Lip}_{\partial\Omega}(g)|x_2 - x_1| + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, this implies

$$g^+(x_2) - g^+(x_1) \leq \mathrm{Lip}_{\partial\Omega}(g)|x_2 - x_1|. \tag{5}$$

By interchanging x_1 and x_2 in (5), we conclude that $\mathrm{Lip}_\Omega(g^+) \leq \mathrm{Lip}_{\partial\Omega}(g)$. We can verify (ii) for g^- similarly. Thus (ii) holds.

It follows from (4) and (i)-(ii) that

$$\mathrm{Lip}_\Omega(G) = \mathrm{Lip}_{\partial\Omega}(g), \quad \text{whenever } G \in \mathrm{Lip}(\Omega) \text{ is a MLE of } g \in \mathrm{Lip}(\partial\Omega). \tag{6}$$

Furthermore, it is easy to check that any MLE $G \in \mathrm{Lip}(\Omega)$ of $g \in \mathrm{Lip}(\partial\Omega)$ satisfies:

$$g^-(x) \leq G(x) \leq g^+(x)\ \forall x \in \Omega. \tag{7}$$

Thus g^+ (g^-, respectively) is called the *maximal* (*minimal*, respectively) MLE of $g \in \mathrm{Lip}(\partial\Omega)$.

In general, one can construct a domain Ω and $g \in \mathrm{Lip}(\partial\Omega)$ such that $g^+ \neq g^-$. Hence there is *no uniqueness* of MLE. Here is an example, due to Jensen [17].

Example 1.1 Let $\Omega = B(0,1) \subseteq \mathbf{R}^2$, the unit disk centered at 0. Set $g(x,y) = 2xy$ on $\partial\Omega$. For any $\alpha \in (0, \frac{1}{2})$, define

$$u^\alpha = \begin{cases} 0 & \text{if } x^2 + y^2 \leq \alpha^2 \\ \frac{2xy(\sqrt{x^2+y^2} - \alpha)}{(1-\alpha)(x^2+y^2)} & \text{if } \alpha^2 \leq x^2 + y^2 \leq 1. \end{cases}$$

Then u^α is a MLE of g.

Another issue on MLE is that it is *unstable* under small compact perturbations. More precisely, let $G \in \mathrm{Lip}(\Omega)$ be a MLE of $g \in \mathrm{Lip}(\partial\Omega)$, and $\phi \in \mathrm{Lip}(\Omega)$ with compact support, $\mathrm{supp}(\phi) \subset\subset \Omega$. It is *not true* that

$$\mathrm{Lip}_{\mathrm{supp}(\phi)}(G + t\phi) \geq \mathrm{Lip}_{\mathrm{supp}(\phi)}(G), \quad \text{for small } |t|. \tag{8}$$

Because of (8), there is no valid *first order variation* available for MLEs. (8) motivates the introduction of absolute minimal Lipschitz extensions (AMLE), a notion first introduced by Aronsson [1, 2, 3, 4] in 1960's.

1.2 Absolute Minimal Lipschitz Extension or AMLE

We first recall the definition of AMLE [3].

Definition 1.2 For any domain $\Omega \subset \mathbf{R}^n$ and $g \in \mathrm{Lip}(\partial\Omega)$, a MLE $G \in \mathrm{Lip}(\Omega)$ of g is called to be an *absolute minimal Lipschitz extension or AMLE* of g, if

$$\mathrm{Lip}_U(G) \le \mathrm{Lip}_U(H),\ \forall U \subset\subset \Omega \ \text{ and } H \in \mathrm{Lip}(U) \text{ with } H\big|_{\partial U} = G\big|_{\partial U}, \tag{9}$$

or equivalently,

$$\mathrm{Lip}_U(G) = \mathrm{Lip}_{\partial U}(G),\ \forall U \subset\subset \Omega. \tag{10}$$

Remark 1.3 Recall that $u : \Omega \to \mathbf{R}$ is a locally Lipschitz continuous function on Ω, denoted as $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, if $u \in \mathrm{Lip}(K)$ for any compact $K \subset\subset \Omega$. We can also define AMLE for $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ by the criterion:

$$\mathrm{Lip}_U(u) = \mathrm{Lip}_{\partial U}(u),\ \forall U \subset\subset \Omega. \tag{11}$$

Remark 1.4 The existence of AMLE was first obtained by Aronsson [3] by Perron's method. See also Juutinen [22].

Two basic questions on AMLEs are their *uniqueness* and *regularity*. To study these questions, we introduce an alternative notion of AMLE as follows.

1.3 AMLE by L^p-approximation

For $1 < p < +\infty$, let $W^{1,p}(\Omega)$ denote the Sobolev space that consists of functions $u \in L^p(\Omega)$, whose distributional derivative $\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) \in L^p(\Omega)$. For $g \in \mathrm{Lip}(\partial\Omega)$, let $W^{1,p}_g(\Omega)$ be the set that consists of all $u \in W^{1,p}(\Omega)$ such that $u|_{\partial\Omega} = g$ in the sense of trace.

Recall that $u_p \in W^{1,p}_g(\Omega)$ is called a p-harmonic function, if it minimizes the Dirichlet p-energy:

$$\int_\Omega |\nabla u_p|^p \le \int_\Omega |\nabla v|^p,\ \forall v \in W^{1,p}_g(\Omega). \tag{12}$$

It is well-known that any p-harmonic function $u_p \in W^{1,p}_g(\Omega)$ satisfies the p-Laplace equation:

$$\Delta_p u_p := \mathrm{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = 0, \quad \text{in } \Omega, \tag{13}$$

$$u_p = g, \quad \text{on } \partial\Omega. \tag{14}$$

Moreover, by the additivity of L^p-integral, we have the following local minimality of p-harmonic functions:

$$\int_U |\nabla u_p|^p \le \int_U |\nabla v|^p,\ \forall U \subset\subset \Omega,\ \forall v \in W^{1,p}(U) \text{ with } v\big|_{\partial U} = u_p\big|_{\partial U}. \tag{15}$$

By the standard energy estimate, there is a subsequence $p_i \uparrow +\infty$ and $u \in \mathrm{Lip}(\Omega)$ such that $u = \lim_{p_i\uparrow+\infty} u_{p_i}$ weakly in $\cap_{q>1} W^{1,q}(\Omega)$. It is not hard to show that (15) can yield:

$$\|\nabla u\|_{L^\infty(U)} \le \|\nabla v\|_{L^\infty(U)},\ \forall U \subset\subset \Omega,\ v \in \mathrm{Lip}(U) \text{ with } v\big|_{\partial U} = u\big|_{\partial U}. \quad (16)$$

Definition 1.5 For $g \in \mathrm{Lip}(\partial\Omega)$, $u \in \mathrm{Lip}(\Omega)$ is an AMLE of g if (16) holds.

Remark 1.6 Note that for any open subset $U \subset \mathbf{R}^n$, it holds

$$\|\nabla u\|_{L^\infty(U)} \le \mathrm{Lip}_U(u). \quad (17)$$

Moreover, if U is convex, then $\|\nabla u\|_{L^\infty(U)} = \mathrm{Lip}_U(u)$. For a non-convex U, the inequality in (17) may be a strict inequality. Hence it is a nontrivial fact that the notion of AMLE given by Definition 1.2 and 1.5 is equivalent, whose proof will be given Chapter 2 below.

1.4 Euler-Lagrange equation of AMLE

Here we provide two formal derivations of the Euler-Lagrange equation of an AMLE, which is assumed to be in $C^2(\Omega)$. This fact was first derived by Aronsson [3].

Proposition 1.7 *If $u \in C^2(\Omega)$ is an AMLE, then u solves the Infinity Laplace Equation:*

$$\Delta_\infty u := \sum_{1\le i,j\le n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad in\ \Omega. \quad (18)$$

Proof. Here we outline a proof given by Jensen [17]. For simplicity, assume $0 \in \Omega$, $u(0) = 0$, and $\Omega \supset B_\epsilon$, the ϵ-ball center at 0, for small $\epsilon > 0$. We want to verify (18) for u at 0. For small $\alpha \in \mathbf{R}$, construct

$$w(x) = u(x) + \frac{\alpha}{2}(\epsilon^2 - |x|^2),\ x \in B_\epsilon.$$

Since $w = u$ on ∂B_ϵ and u is an AMLE, we have

$$\|\nabla u\|_{L^\infty(B_\epsilon)} \le \|\nabla w\|_{L^\infty(B_\epsilon)}.$$

Since $u, w \in C^2(B_\epsilon)$, by the Taylor expansion we have

$$u(x) = \sum_i u_i(0)x_i + \frac{1}{2}\sum_{i,j} u_{ij}(0)x_i x_j + o(|x|^2),$$

$$w(x) = \frac{\alpha}{2}\epsilon^2 + \sum_i u_i(0)x_i + \frac{1}{2}\sum_{i,j}(u_{ij}(0) - \alpha\delta_{ij})x_i x_j + o(|x|^2),$$

where

$$u_i = \frac{\partial u}{\partial x_i},\ u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

If $\sum_j u_j(0)u_{ij}(0) = 0$ for all $i \in \{1, \cdots, n\}$, then (18) trivially holds at 0. We may assume

$$\sum_j u_j(0)u_{ij}(0) \neq 0 \text{ for some } i \in \{1, \cdots, n\}.$$

Hence

$$\sum_{k,j} u_k(0)u_{ik}(0)u_j(0)u_{ij}(0) > 0,$$

and

$$\sum_{k,j} u_k(0)u_j(0)\left(u_{ik}(0) - \alpha\delta_{ik}\right)\left(u_{ij}(0) - \alpha\delta_{ij}\right) > 0,$$

provide α is chosen to be sufficiently small. Observe that

$$|\nabla u|^2(x) = |\nabla u|^2(0) + \sum_{i,j} u_i(0)u_{ij}(0)x_j + o(\epsilon),$$

and

$$|\nabla w|^2(x) = |\nabla u|^2(0) + \sum_{i,j} u_i(0)(u_{ij}(0) - \alpha\delta_{ij})x_j + o(\epsilon),$$

for $x \in B_\epsilon$. Hence we have

$$\|\nabla u\|^2_{L^\infty(B_\epsilon)} = |\nabla u|^2(0) + \epsilon\sqrt{\sum_{i,j,k} u_j(0)u_k(0)u_{ij}(0)u_{ik}(0)} + o(\epsilon),$$

and

$$\|\nabla w\|^2_{L^\infty(B_\epsilon)} = |\nabla u|^2(0) + \epsilon\sqrt{\sum_{i,j,k} u_j(0)u_k(0)(u_{ij}(0)_\alpha\delta_{ij})(u_{ik}(0) - \alpha\delta_{ik})} + o(\epsilon).$$

Therefore, we have

$$\begin{aligned} F(\alpha) &= \sqrt{\sum_{i,j,k} u_j(0)u_k(0)(u_{ij}(0) - \alpha\delta_{ij})(u_{ik}(0) - \alpha\delta_{ik})} \\ &\quad - \sqrt{\sum_{i,j,k} u_j(0)u_k(0)u_{ij}(0)u_{ik}(0)} \\ &\geq 0. \end{aligned}$$

This implies

$$0 = \frac{d}{d\alpha}|_{\alpha=0}F(\alpha) = \frac{\sum_{i,k} u_i(0)u_k(0)u_{ik}(0)}{\sum_{i,j,k} u_j(0)u_k(0)u_{ij}(0)u_{ik}(0)}.$$

This yields (18). □

Remark 1.8 One can also derive the infinity Laplace equation for AMLE by the L^p-approximation scheme. In fact, suppose that u_p is a harmonic function and $u_p \to u$ in $C^2_{\text{loc}}(\Omega)$. Then u solves (18) [3, 17].

Proof. For any $x_0 \in \Omega$, we will verify that $\Delta_\infty u(x_0) = 0$. Without loss of generality, assume $\nabla u(x_0) \neq 0$. Since u_p solves (13), we have

$$\Delta_p u_p = (p-2)|\nabla u_p|^{p-4}\left(\frac{|\nabla u|^2}{p-2}\Delta_2 u_p + \Delta_\infty u_p\right) = 0.$$

In particular,

$$\frac{|\nabla u|^2}{p-2}\Delta_2 u_p + \Delta_\infty u_p = 0.$$

Sending p to ∞, this implies $\Delta_\infty u = 0$ in Ω. □

It is important to point out that the informal derivations given by both the proposition and remark above can be formally justified by employing the notion of viscosity solutions, see Aronsson-Crandall-Juutinen [6] and Chapter 2 below.

1.5 Degeneracy of ellipticity of the infinity Laplace equation

We can write (18) as

$$\Delta_\infty u = \sum_{i,j} a_{ij}(x) u_{ij},$$

where

$$a_{ij}(x) = u_i u_j, \ 1 \le i, j \le n.$$

It is easy to see that

$$0 \le \sum_{i,j} a_{ij}(x)\xi_i\xi_j = (\xi \cdot \nabla u)^2, \ \forall \xi = (\xi_1, \cdots, \xi_n).$$

Therefore, $\Delta_\infty u(x)$ is elliptic that is degenerate along the $(n-1)$-dimensional hyperplane that perpendicular to $\nabla u(x)$, $\{\xi \in \mathbf{R}^n : \xi \perp \nabla u(x)\}$.

Remark 1.9 It is this degeneracy of ellipticity that yields great difficulties to attack both the uniqueness and regularity of the infinity Laplace equation (18). Although the uniqueness of (18) has been settled by Jensen [17], the regularity of (18) has remained to be open for $n \ge 3$. It was proved by Savin [24], Evans-Savin [16] that an infinity harmonic function in dimension two is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$.

1.6 Characteristics of the infinity Laplace equation

Suppose that $u \in C^2(\Omega)$ solves $\Delta_\infty u = 0$ in Ω. Then for any bounded subdomain $U \subset\subset \Omega$ and $x \in U$, consider the *gradient flow*:

$$\begin{cases} X'(t) = \nabla u(X(t)), \quad t > 0, \\ X(0) = x \in U. \end{cases}$$

Since $\nabla u \in C^1(\Omega)$, the standard theory of ODE implies that there exists a maximal time interval $-\infty < T_1 < 0 < T_2 < \infty$ such that the above ODE has a unique solution $X \in C^2((T_1, T_2), U)$, which can be viewed as the characteristic curve of (18). In fact, we have

(i) $|\nabla u|(X(t)) = |\nabla u(x)|, \ T_1 < t < T_2.$

(ii) $u(X(t)) = u(x) + t|\nabla u(x)|^2, \ T_1 < t < T_2.$

(iii) $X(T_1), \ X(T_2) \in \partial U.$

To see this, calculate

$$\begin{aligned} \frac{d}{dt}|\nabla u|^2(X(t)) &= 2\langle \nabla u(X(t)), \nabla^2 u(X(t)) \cdot X'(t)\rangle \\ &= 2\langle \nabla u(X(t)), \nabla^2 u(X(t)) \cdot \nabla u(X(t))\rangle \\ &= 2\Delta_\infty u(X(t)) = 0. \end{aligned}$$

Hence $|\nabla u|(X(t)) =$ constant and (i) follows. To see (ii), note that

$$\frac{d}{dt}(u(X(t))) = \nabla u(X(t)) \cdot X'(t) = |\nabla u|^2(X(t)) = |\nabla u(x)|^2.$$

By integration, this implies (ii). Since $X(t) \in U$ and u is bounded on U, (ii) implies that both T_1 and T_2 are finite. Moreover, $X(T_1), \ X(T_2) \in \partial U$, for otherwise, neither T_1 nor T_2 is a maximal time interval.

Corollary 1.10 *Suppose that $u \in C^2(\Omega)$ solves (18). Then u is an AMLE on Ω.*

Proof. Suppose the conclusion were false. Then there exist a subdomain $U_0 \subset\subset \Omega$ and a function $v_0 \in \text{Lip}(U_0)$ such that $v_0 = u$ on ∂U_0, but

$$\|\nabla u\|_{L^\infty(U_0)} > c_0 \equiv \|\nabla v_0\|_{L^\infty(U_0)}. \tag{19}$$

Hence there exists $x_0 \in U_0$ such that $|\nabla u|(x_0) > c_0$. Now let $X(t) : (T_1, T_2) \to U_0$ be the gradient flow of u with $X(0) = x_0$. Then (i)-(iii) implies

$$\begin{aligned} v_0(X(T_2)) - v_0(X(T_1)) &= u(X(T_2)) - u(X(T_1)) \\ &= |\nabla u|^2(x_0)(T_2 - T_1) \\ &> |\nabla u|(x_0)c_0(T_2 - T_1) \end{aligned} \tag{20}$$

On the other hand, since $v_0(X(t)) \in \mathrm{Lip}((T_1, T_2))$, we have

$$\begin{aligned} v_0(X(T_2) - v_0(X(T_1)) &= \int_{T_1}^{T_2} \frac{d}{dt} v_0(X(t))\, dt \\ &= \int_{T_1}^{T_2} \nabla v_0(X(t)) \cdot X'(t)\, dt \\ &= \int_{T_1}^{T_2} \nabla v_0(X(t)) \cdot \nabla u(X(t))\, dt \\ &\leq \int_{T_1}^{T_2} |\nabla v_0|(X(t))|\nabla u|(X(t))\, dt \\ &= |\nabla u|(x_0) \int_{T_1}^{T_2} |\nabla v_0|(X(t))\, dt \\ &\leq c_0(T_2 - T_1)|\nabla u|(x_0). \end{aligned} \tag{21}$$

It is clear that (21) contradicts (20). □

Remark 1.11 Strictly speaking, we need to assume $v_0 \in C^1(U_0)$ in the above proof. However, we can employ the standard mollification method to easily extend the argument to cover the case that $v_0 \in \mathrm{Lip}(U_0)$. We leave it to reader as an exercise.

1.7 Cone solutions of the infinity Laplace equation

Suppose that $u \in C^1(\Omega)$ solves the Eikonal equation:

$$|\nabla \psi| = k \ \text{ in } \Omega \tag{22}$$

for some $k \geq 0$. Then it is easy to see that ψ solves (18). In fact, we have $\nabla |\nabla \psi|^2 \equiv 0$ in Ω. On the other hand, observe that

$$\Delta_\infty \psi = \frac{1}{2} \nabla \psi \cdot \nabla |\nabla \psi|^2.$$

Hence $\Delta_\infty \psi = 0$ in Ω.

The above observation enables us to build many cone type solutions of the infinity Laplace equation (18): for any $x_0 \in \mathbf{R}^n$ and $a,\ b \in \mathbf{R}$, define

$$C(x) = b + a|x - x_0|,\ x \in \mathbf{R}^n \setminus \{x_0\}.$$

Then since

$$|\nabla C(x)| = |b|, \ \forall x \neq x_0,$$

we have that $C \in C^1(\mathbf{R}^n \setminus \{x_0\})$ solves (18). Through the important work by Crandall, Evans, Gariepy [10], it becomes evident that such a cone type solution plays a crucial role in the study of the infinity Laplace equation (18). We will discuss the detail in subsequent chapters.

1.8 AMLE may not have C^2-regularity

In this section, we present the example by Aronsson [5], where he constructed an AMLE that only has $C^{1,\frac{1}{3}}$-regularity.

Denote by $B, \subset \mathbf{R}^2$, the unit ball in $\mathbf{R}^2$. Let $\psi(x,y) = x^{\frac{4}{3}} - y^{\frac{4}{3}} : B \to \mathbf{R}$. It is obvious that $\psi \in C^\infty(B \setminus \{(x,y) \in \mathbf{R}^2 : \ x = 0 \text{ or } y = 0\}) \bigcap \in C^{1,\frac{1}{3}}(B)$. We claim that ψ is an AMLE on B. To see this, we first show that ψ solves (18) on $B \setminus \{(x,y) \in \mathbf{R}^2 : \ x = 0 \text{ or } y = 0\}$. In fact, if $x \neq 0$ and $y \neq 0$, then

$$\nabla\psi(x,y) = \frac{4}{3}\left(x^{\frac{1}{3}}, -y^{\frac{1}{3}}\right), \ \nabla^2\psi(x,y) = \frac{4}{9}\begin{pmatrix} x^{-\frac{2}{3}} & 0 \\ 0 & -y^{-\frac{2}{3}} \end{pmatrix}.$$

Hence

$$\Delta_\infty\psi(x,y) = \frac{4^3}{3^4}\left(x^{\frac{1}{3}}, -y^{\frac{1}{3}}\right)\begin{pmatrix} x^{-\frac{2}{3}} & 0 \\ 0 & -y^{-\frac{2}{3}} \end{pmatrix}\begin{pmatrix} x^{\frac{1}{3}} \\ -y^{\frac{1}{3}} \end{pmatrix} = 0.$$

Therefore, Corollary 1.10 implies that ψ is an AMLE on $B \setminus \{(x,y) \in \mathbf{R}^2 : x = 0 \text{ or } y = 0\}$.

Now we want to verify that if $U \subset\subset B$ that contains either $(x_0, 0)$ or $(0, y_0)$ for some $x_0, y_0 \in \mathbf{R}$, then

$$\|\nabla\psi\|_{L^\infty(U)} \leq \|\nabla v\|_{L^\infty(U)} \ \text{ whenever } v \in \text{Lip}(U) \ \text{ with } v\big|_{\partial U} = \psi\big|_{\partial U}. \tag{23}$$

Suppose that (23) were false. Then there exist $(x_1, y_1) \in U$ and $v_1 \in \text{Lip}(U)$ with $v_1 = \psi$ on ∂U such that

$$|\nabla\psi|(x_1, y_1) > c_1 \equiv \|\nabla v_1\|_{L^\infty(U)}.$$

Now let

$$\gamma = \left\{(x,y) \in \mathbf{R}^2 : \ x^{\frac{2}{3}} + y^{\frac{2}{3}} = x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}\right\}.$$

By direct calculations, we see that γ is the orbit of the gradient flow of ψ that passes through (x_1, y_1), and

$$|\nabla\psi|^2\Big|_\gamma \equiv |\nabla\psi|^2(x_1, y_1) = \frac{4^2}{3^2}\left(x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}\right).$$

Moreover, one can check that either
(i) there is a unique pair of distinct points $(x_2, y_2), (x_3, y_3) \in \gamma \cap \partial U$ so that

$$\psi(x_3, y_3) - \psi(x_2, y_2) = |\nabla\psi|^2(x_1, y_1)l(\gamma) > c_1^2 l(\gamma), \tag{24}$$

where $l(\gamma)$ is the length of the curve γ; or
(ii) there is a $(x_4, 0) = \gamma \cap \{x - \text{axis}\} \in U$ so that we can extend γ by joining the line segment from $(x_4, 0)$ to $(x_5, 0) = \{x - \text{axis}\} \cap \partial U$, denoted as $\tilde{\gamma} = \gamma \cup [(x_4, 0), (x_5, 0)]$. In this case, since $|\nabla\psi|\big|_{[(x_4,0),(x_5,0)]} \geq |\nabla\psi|(x_1, y_1)$, we also have

$$\psi(x_5, 0) - \psi(x_2, y_2) \geq c_1^2 l(\gamma);$$

or
(iii) there is a $(0, y_4) = \gamma \cap \{y - \text{axis}\} \in U$ so that we can extend γ by joining the line segment from $(0, y_4)$ to $(0, y_5) = \{y - \text{axis}\} \cap \partial U$, denoted as $\hat{\gamma} = \gamma \cup [(0, y_4), (0, y_5)]$. In this case, since $|\nabla\psi|\big|_{[(0,y_4),(0,y_5)]} \geq |\nabla\psi|(x_1, y_1)$, we also have

$$\psi(0, y_5) - \psi(x_2, y_2) \geq c_1^2 l(\gamma);$$

or
(iv) both (ii) and (iii) happen. In this case, we can extend γ by adding both the segments to obtain $\overline{\gamma}$ so that the inequality (24) holds.

On the other hand, we can show that

$$\begin{aligned} \psi(x_3, y_3) - \psi(x_2, y_2) &= v_1(x_3, y_3) - v_1(x_2, y_2) \\ &\leq \|\nabla v_1\|^2_{L^\infty(U)} l(\gamma) = c_1^2 l(\gamma). \end{aligned}$$

This yields the desired contradiction. □

Remark 1.12 In chapter 2 below, we will introduce the notion of viscosity solution to (18) and show that ψ is a weak (or viscosity) solution of the infinity Laplace equation (18). We will also present the theorem, first due to Jensen [17], that any AMLE is equivalent to a viscosity solution of the infinity Laplace equation (18).

2 Infinity harmonic function

From the previous Chapter, we have that an AMLE may not enjoy regularity better than $C^{1,\frac{1}{3}}$. It is also clear that the infinity Laplace equation (18) may not have classical solutions, due to the high degeneracy of its ellipticity. This motives us to introduce the notion of weak solutions to (18). Thanks to the theory of viscosity solutions to second order elliptic equations developed by Crandall and Lions, Evans,, Jensen, and many others, we can introduce the notion of viscosity solutions to the infinity Laplace equation. The reader can consult Crandall-Ishii-Lions [12] for the background material and many reference therein. But we would like to point out that the notion is based on the following weak maximum principle.

2.1 Notion of infinity harmonic function

Proposition 2.1 *Suppose that $u \in C^2(\Omega)$ satisfies $\Delta_\infty u \geq (\leq) 0$ in Ω. Then for any $(x_0, \phi) \in \Omega \times C^2(\Omega)$, if*

$$0 = (\phi - u)(x_0) \leq (\geq)(\phi - u)(x) \ \forall x \in \Omega,$$

then

$$\Delta_\infty \phi(x_0) \geq (\leq) 0.$$

Proof. Assume

$$0 = (\phi - u)(x_0) \leq (\phi - u)(x) \ \forall x \in \Omega.$$

Then by the Taylor expansion we have

$$\nabla \phi(x_0) = \nabla u(x_0), \ \nabla^2 \phi(x_0) \geq \nabla^2 u(x_0).$$

Hence

$$\begin{aligned} \Delta_\infty \phi(x_0) &= \langle \nabla \phi(x_0), \nabla^2 \phi(x_0) \cdot \nabla \phi(x_0) \rangle \\ &= \langle \nabla u(x_0), \nabla^2 \phi(x_0) \cdot \nabla u(x_0) \rangle \\ &\geq \langle \nabla u(x_0), \nabla^2 u(x_0) \cdot \nabla u(x_0) \rangle \\ &= \Delta_\infty u(x_0) \geq 0. \end{aligned}$$

The second part of the conclusion can be proven exactly in the same way. □

Definition 2.2 [12] A function $u \in C(\Omega)$ is called a viscosity subsolution of the infinity Laplace equation (18) if for any pair $(x_0, \phi) \in \Omega \times C^2(\Omega)$,

$$0 = (\phi - u)(x_0) \leq (\phi - u)(x), \ \forall x \in \Omega \tag{25}$$

implies

$$\Delta_\infty \phi(x_0) \geq 0. \tag{26}$$

A function $u \in C(\Omega)$ is called a viscosity supersolution of the infinity Laplace equation (18) if $-u$ is a viscosity subsolution of (18). A function $u \in C(\Omega)$ is called a viscosity solution of (18) if u is both a viscosity subsolution and a viscosity supersolution of (18).

Definition 2.3 A function $u \in C(\Omega)$ is called an infinity harmonic function if u is a viscosity solution of the infinity Laplace equation (18).

It is easy to see from Proposition 2.1 that any C^2-solution of (18) is a viscosity solution of (18). The next proposition shows that the converse *may* not be true.

Proposition 2.4 *$\psi(x,y) = x^{\frac{4}{3}} - y^{\frac{4}{3}} : \mathbf{R}^2 \to \mathbf{R}$ is an infinity harmonic function.*

Proof. See also [6]. It suffices to show that for any $x_0, y_0 \in \mathbf{R}$, if $\phi \in C^2(\mathbf{R}^2)$ touches $p_0 = (x_0, 0)$, or $(0, y_0)$ from above (below), then

$$\Delta_\infty \psi(p_0) \geq (\leq) 0.$$

For simplicity, consider
(i) $(x_0, 0) = (1, 0)$. Then we have

$$\nabla\phi(1,0) = \nabla\psi(1,0) = (\frac{4}{3}, 0)$$

so that

$$\Delta_\infty \phi(1,0) = \phi_x^2(1,0)\phi_{xx}(1,0) = \frac{4^2}{3^2}\phi_{xx}(1,0).$$

On the other hand, we have

$$(\psi - \phi)_{xx}(1,0) \leq 0$$

so that

$$\phi_{xx}(1,0) \geq \psi_{xx}(1,0) = \frac{4}{9}.$$

Therefore we have

$$\Delta_\infty \phi(1,0) \geq \frac{4^3}{3^4} > 0.$$

(ii) $(0, y_0) = (0, 1)$. Then we have

$$\psi(x,1) - \phi(x,1) \leq \psi(0,1) - \phi(0,1),$$

i.e.

$$x^{\frac{4}{3}} \leq \phi(x,1) - \phi(0,1) = \phi_x(0,1)x + \frac{1}{2}\phi_{xx}(0,1)x^2 + o(|x|^2).$$

Since

$$\phi_x(0,1) = \psi_x(0,1) = 0,$$

this implies

$$x^{\frac{4}{3}} \le \frac{1}{2}\phi_{xx}(0,1)x^2 + o(|x|^2).$$

This is clearly impossible for small x.
This verifies that ψ is a viscosity subsolution of (18). Similarly, one can verify ψ is a viscosity supersolution of (18). □

Remark 2.5 It is not hard to verify that for any $x_0 \in \mathbf{R}^n$ and $a, b \in \mathbf{R}$, $C(x) = b + a|x - x_0| \in \text{Lip}(\mathbf{R}^n)$ is a viscosity subsolution (or supersolution) but not a supersolution (subsolution) of (18) when $a > 0$ (or $a < 0$).

2.2 Cone comparison of infinity harmonic functions

In this section we present a geometric characterization of infinity harmonic functions, namely the cone comparison principle, which was established by Crandall, Evans, Gariepy [10].

Definition 2.6 A function $u \in C(\Omega)$ enjoys the cone comparison from above (or below), or $u \in CCA(\Omega)$ or $CCB(\Omega)$, if for any subdomain $U \subset\subset \Omega$ and $x_0 \in \Omega \setminus U$ and $a, b \in \mathbf{R}$

$$u(x) \le (\ge) C(x) = b + a|x - x_0| \quad \text{on } \partial U \tag{27}$$

implies

$$u(x) \le (\ge) C(x) = b + a|x - x_0| \quad \text{in } U. \tag{28}$$

A u enjoys the cone comparison, or $u \in CC(\Omega)$, if $u \in CCA(\Omega) \bigcap CCB(\Omega)$.

Theorem 2.7 *Suppose that $u \in C(\Omega)$ is an infinity harmonic function. Then $u \in CC(\Omega)$.*

Proof. Suppose that $u \notin CCA(\Omega)$. Then there are $U \subset\subset \Omega$, $x_0 \in \Omega \setminus U$, $a, b \in \mathbf{R}$, and $x_1 \in U$ such that

$$\alpha_1 \equiv u(x_1) - C(x_1) = \max\{u(x) - C(x) : \ x \in \overline{U}\} > 0 \ge \max_{x \in \partial U} (u(x) - C(x)), \tag{29}$$

where $C(x) = b + a|x - x_0|$. For $\epsilon > 0$ small, set

$$C_1(x) = C(x) - \epsilon|x - x_1|^2 = b + a|x - x_0| - \epsilon|x - x_1|^2.$$

Then we have

$$(u - C_1)(x_1) = \alpha_1 = \max\{(u - C_1)(x) : \ x \in \overline{U}\}.$$

This implies that $C_1 - \alpha_1 \in C^2(U)$ touches u from above at x_1. Hence

$$\Delta_\infty C_1(x_1) \ge 0.$$

On the other hand, direct calculations imply

$$\nabla C_1(x_1) = a\frac{x_1 - x_0}{|x_1 - x_0|},$$

$$\nabla^2 C_1(x_1) = a\left(\frac{I_n}{|x_1 - x_0|} - \frac{(x_1 - x_0)\otimes(x_1 - x_0)}{|x_1 - x_0|^3}\right) - 2\epsilon I_n$$

so that

$$\begin{aligned}\Delta_\infty C_1(x_1) &= a^3\langle\frac{x_1 - x_0}{|x_1 - x_0|},\ \left(\frac{I_n}{|x_1 - x_0|} - \frac{(x_1 - x_0)\otimes(x_1 - x_0)}{|x_1 - x_0|^3}\right)\cdot\frac{x_1 - x_0}{|x_1 - x_0|}\rangle \\ &\quad -2\epsilon a^2\langle\frac{x_1 - x_0}{|x_1 - x_0|},\ I_n\cdot\frac{x_1 - x_0}{|x_1 - x_0|}\rangle \\ &= -2a^2\epsilon < 0.\end{aligned}$$

We get the desired contradiction. Similarly, we can prove that $u \in CCB(\Omega)$. This completes the proof. □

Remark 2.8 It follows from the proof of theorem 2.7 that if $u \in C(\Omega)$ is a viscosity subsolution (or supersolution) of (18), then $u \in CCA(\Omega)$ or $CCB(\Omega)$.

An immediate consequence of theorem 2.7 is the following monotone slope property.

Corollary 2.9 *If $u \in C(\Omega) \in CCA(\Omega)$, then for any $x_0 \in \Omega$ and $0 < r < dist(x_0, \partial\Omega)$,*

$$\begin{aligned} S_r^+(u, x_0) &= \max_{x\in\partial B_r(x_0)} \frac{u(x) - u(x_0)}{r} \\ &= \inf\{k \geq 0\ |u(x) \leq u(x_0) + k|x - x_0|\ \forall x \in \partial B_r(x_0)\}\end{aligned} \quad (30)$$

is monotonically nondecreasing with respect to r. Hence

$$S^+(u, x_0) = \lim_{r\downarrow 0} S_r^+(u, x_0) \quad (31)$$

exists for any $x \in \Omega$ and is upper semicontinuous on Ω. Furthermore, u is locally Lipschitz continuous on Ω.

Proof. By the definition of $S_r^+(u, x_0)$, we have that

$$u(x) \leq u(x_0) + S_r^+(u, x_0)|x - x_0| \text{ on } \partial(B_r(x_0) \setminus \{x_0\}).$$

Since $u \in CCA(\Omega)$, this implies

$$u(x) \leq u(x_0) + S_r^+(u, x_0)\ \forall x \in B_r(x_0).$$

Therefore, for any $0 < s \le r$ and $x \in \partial B_s(x_0)$, we have

$$\frac{u(x) - u(x_0)}{s} \le S_r^+(u, x_0).$$

This yields that $S_s^+(u, x_0) \le S_r^+(u, x_0)$ and $S_r^+(u, x_0)$ is monotonically nondecreasing with respect to r. The existence and the upper semicontinuity of $S^+(u, x_0)$ then follow directly. To see the local Lipschitzity of u, let $K \subset\subset \Omega$ be a compact set and $x, y \in K$. We may assume that $u(y) \ge u(x)$ so that

$$0 \le u(y) - u(x) \le S_{|y-x|}^+(u, x)|y - x|.$$

Since $u \in L^\infty(K)$ and $S_r^+(u, x)$ is monotonically nondecreasing, we see that $S_r^+(u, x)$ is bounded for $x \in \Omega$ and $0 < r \le \mathrm{diam}(K)$. Thus $u \in \mathrm{Lip}(K)$. The proof is complete. □

Similarly we have

Corollary 2.10 *If $u \in CCB(\Omega)$, then for any $x_0 \in \Omega$ and $0 < r < dist(x_0, \partial\Omega)$,*

$$S_r^-(u, x_0) = \min_{\partial B_r(x_0)} \frac{u(x_0) - u(x)}{r} \tag{32}$$

is monotonically nondecreasing with respect to r. In particular,

$$S^-(u, x_0) = \lim_{r \downarrow 0} S_r^-(u, x_0) \tag{33}$$

exists and upper semicontinuous for $x \in \Omega$. Furthermore, $u \in Lip_{loc}(\Omega)$.

Proof. Note that $u \in CCB(\Omega)$ is equivalent to $-u \in CCA(\Omega)$. Hence the conclusion follows from Corollary 2.9. □

By combining Corollary 2.9 with 2.10, we have

Theorem 2.11 *Suppose that $u \in CC(\Omega)$. Then $u \in Lip_{loc}(\Omega)$, and for any $x \in \Omega$, both $S_r^+(u, x)$ and $S_r^-(u, x)$ are monotonically nondecreasing with respect to $0 < r < dist(x, \partial\Omega)$. Moreover,*

$$S^+(u, x) = S^-(u, x), \ \forall x \in \Omega. \tag{34}$$

If ∇u exists at x, then

$$S^+(u, x) = S^-(u, x) = |\nabla u|(x).$$

Proof. In order to prove (34), we introduce the point-wise Lipschitz norm of u:

$$T_u(y) = \lim_{r \downarrow 0} \mathrm{Lip}_{B_r(y)}(u), \ y \in \Omega, \tag{35}$$

whose existence is clear, as $\mathrm{Lip}_{B_r(y)}(u)$ is bounded and monotonically nondecreasing with respect to $r > 0$. Now we claim

$$T_u(y) = S^+(u, y), \ \forall y \in \Omega. \tag{36}$$

In fact, for any $r > 0$ and $[x_1, x_2] \subset B_r(y)$. By the Lipschitz continuity of u,

$$g(t) := u(x_1 + t(x_2 - x_1))$$

is Lipschitz continuous in $t \in [0, 1]$ and hence is differentiable for a.e. $t \in (0, 1)$. Fix $t \in (0, 1)$, for small $h > 0$, by the definition of S^+ we have

$$\begin{aligned}\frac{g(t+h) - g(t)}{h} &= \frac{u\left(x_1 + (t+h)(x_2 - x_1)\right) - u\left(x_1 + t(x_2 - x_1)\right)}{h} \\ &\leq S^+_{h|x_2 - x_1|}(u, x_1 + t(x_2 - x_1))|x_2 - x_1|.\end{aligned}$$

Sending $h \downarrow 0$ we have

$$g'(t) \leq S^+_{h|x_2-x_1|}(u, x_1 + t(x_2 - x_1))|x_2 - x_1| \leq \left(\sup_{z\in[x_1,x_2]} S^+(u, z)\right) |x_2 - x_1|$$

at any point of differentiability of g. Therefore

$$u(x_2) - u(x_1) = g(1) - g(0) = \int_0^1 g'(t)\, dt \leq \left(\sup_{z\in[x_1,x_2]} S^+(u, z)\right) |x_2 - x_1|. \tag{37}$$

Therefore, we obtain

$$\mathrm{Lip}_{B_r(y)}(u) \leq \sup_{z\in B_r(y)} S^+(u, z).$$

Sending $r \downarrow 0$ and recalling the upper semicontinuity of $S^+(u, \cdot)$, we have

$$T_u(y) \leq S^+(u, y).$$

On the other hand, it is easy to see

$$S^+(u, y) \leq S^+_r(u, y) = \max_{z\in\partial B_r(y)} \frac{u(z) - u(y)}{r} \leq \mathrm{Lip}_{B_r(y)}(u)$$

so that $S^+(u, y) \leq T_u(y)$. The gives (36). Since $T_{-u}(y) = T_u(y)$, we can show similarly that

$$S^-(u, y) = T_{-u}(y) = T_u(y).$$

Thus (34) follows.

At the end of this section, we prove the converse of theorem 2.11. The proof we give here is from Crandall, Wang, Yu [13].

Theorem 2.12 *(i) If $u \in CCA(\Omega)$, then u is a viscosity subsolution of (18).*
(ii) If $u \in CCB(\Omega)$, then u is a viscosity supersolution of (18).
(iii) If $u \in CC(\Omega)$, then u is an infinity harmonic function.

Proof. For simplicity, we only prove (i). Assume $x_0 = 0 \in \Omega$, $\phi \in C^2(\Omega)$ touches u from above at 0. Note that if $\nabla\phi(0) = 0$, then $\Delta_\infty \phi(0) = 0$. Hence we can assume $\nabla\phi(0) \neq 0$. To proceed, we first need
Claim 1: $|\nabla\phi(0)| \leq N^+(u, 0)$. Assume this claim for the moment, we continue as follows. For small $r > 0$, let $x_r \in \partial B_r$ be such that

$$u(x_r) = u(0) + rS_r^+(u, 0)(\geq rS^+(u, 0)).$$

Then we have

$$\phi(x_r) - \phi(0) \geq u(x_r) - u(0) \geq rS^+(u, 0) \geq r|\nabla\phi(0)|.$$

Since $|\frac{x_r}{r}| = 1$, we may assume after taking subsequences that there is $q \in \mathbf{R}^n$ with $|q| = 1$ such that

$$\lim_{r\downarrow 0} \frac{x_r}{r} = q.$$

Taking r to 0, we have

$$\nabla\phi(0) \cdot q = \lim_{r\downarrow 0} \frac{\phi(x_r) - \phi(0)}{r} \geq |\nabla\phi(0)|.$$

This implies

$$q = \frac{\nabla\phi(0)}{|\nabla\phi(0)|}.$$

Since

$$\phi(x_r) - \phi(0) = \int_0^1 \nabla\phi(tx_r) \cdot x_r \, dt \leq r \int_0^1 |\nabla\phi(tx_r)| \, dt,$$

there exists $t_0 \in (0, 1)$ such that

$$|\nabla\phi(t_0 x_r)| \geq |\nabla\phi(0)|.$$

This implies

$$\frac{|\nabla\phi(t_0 x_r)|^2 - |\nabla\phi(0)|^2}{t_0 r} \geq 0.$$

Sending r to 0, we obtain

$$\begin{aligned} 0 &\leq \nabla|\nabla\phi|^2(0) \cdot \lim_{r\downarrow 0} \frac{x_r}{r} \\ &= \nabla|\nabla\phi|^2(0) \cdot q \\ &= \frac{1}{|\nabla\phi(0)|} \Delta_\infty \phi(0). \end{aligned}$$

Therefore $\Delta_\infty \phi(0) \geq 0$ and (i) holds.

Now we return to the proof of Claim 1. By modifying ϕ to $\phi + \epsilon|x|^2$, we may assume

$$0 = (\phi - u)(0) < (\phi - u)(x), \ \forall x \neq 0. \tag{38}$$

Let u_ϵ be the standard ϵ-mollification of u. Then for small $r > 0$ there exists $x_\epsilon \in B_r$ such that

$$(\phi - u_\epsilon)(x_\epsilon) = \min_{x \in \overline{B_r}} (\phi - u_\epsilon)(x).$$

Since $u_\epsilon \to u$ uniformly on B_r, it is easy to see from (38) that

$$\lim_{\epsilon \downarrow 0} x_\epsilon = 0.$$

Since x_ϵ is the minimum point of $\phi - u_\epsilon$, we have

$$|\nabla \phi(x_\epsilon)| = |\nabla u_\epsilon(x_\epsilon)|.$$

Sending ϵ to 0, we then have

$$\begin{aligned} |\nabla \phi(0)| &= \lim_{\epsilon \downarrow 0} |\nabla \phi(x_\epsilon)| = \lim_{\epsilon \downarrow 0} |\nabla u_\epsilon(x_\epsilon)| \\ &\leq \lim_{r \downarrow 0} \lim_{\epsilon \downarrow 0} \|\nabla u\|_{L^\infty(B_{r+\epsilon})} \\ &= \lim_{r \downarrow 0} \sup_{x \in B_{2r}} S^+(u,x) = S^+(u,0), \end{aligned}$$

here we have used the upper semicontinuity of S^+ in the last inequality,

$$|\nabla u_\epsilon(x_\epsilon)| \leq \int_{\mathbf{R}^n} \eta_\epsilon(x_\epsilon - y)|\nabla u(y)|\, dy \leq \|\nabla u\|_{L^\infty(B_{r+\epsilon})},$$

and $\nabla u(x) = S^+(u,x)$ for a.e. $x \in B_{2r}$. Hence Claim 1 holds. The proof of (i) is complete. □

2.3 Equivalence between AMLE and CC

In this section, we will establish the equivalence between absolute minimal Lipschitz extension property and cone comparison principle. We start with an end point estimate.

Lemma 2.13 *Suppose that $u \in CCA(\Omega)$. Then for any $x_0 \in \Omega$ and $0 < r <$ dist$(x_0, \partial\Omega)$, if $x_r \in \partial B_r(x_0)$ satisfies*

$$u(x_r) = u(x_0) + rS_r^+(u,x_0), \tag{39}$$

then

$$S_R^+(u,x_r) \geq S_r^+(u,x_0), \ \forall R > 0. \tag{40}$$

In particular,

$$S^+(u,x_r) \geq S^+(u,x_0). \tag{41}$$

Proof. For any $\theta \in (0,1)$, we have

$$u(x_0 + \theta(x_r - x_0)) \le u(x_0) + \theta r S_r^+(u, x_0),$$

and

$$u(x_r) = u(x_0) + r S_r^+(u, x_0).$$

Hence

$$\frac{u(x_r) - u(x_0 + \theta(x_r - x_0))}{(1-\theta)r} \ge S_r^+(u, x_0).$$

This means

$$S_{(1-\theta)r}^+(u, x_0 + \theta(x_r - x_0)) \ge S_r^+(u, x_0).$$

For any $R > 0$, choose $\theta \in (0,1)$ such that $R \ge (1-\theta)r$. Then we obtain

$$S_R^+(u, x_0 + \theta(x_r - x_0)) \ge S_r^+(u, x_0).$$

Sending $\theta \uparrow 1$ yields

$$S_R^+(u, x_r) \ge S_r^+(u, x_0).$$

Sending $R \downarrow 0$ and applying the upper semicontinuity of S^+, we obtain (41). □As an immediate consequence of lemma 2.13, we have

Corollary 2.14 *Suppose that $u \in C(\mathbf{R}^n)$ is a viscosity subsolution of (18) and is bounded from above. Then u is constant.*

Proof. Suppose that u were not constant. Then there exists $x_0 \in \mathbf{R}^n$ such that $S^+(u, x_0) > 0$. Let $x_1 \in \partial B_1(x_0)$ be such that

$$u(x_1) = u(x_0) + S_1^+(u, x_0) \ge u(x_0) + S^+(u, x_0).$$

Then lemma 2.13 implies

$$S^+(x_1) \ge S^+(x_0).$$

Now let $x_2 \in \partial B_1(x_1)$ be such that

$$u(x_2) = u(x_1) + S_1^+(u, x_1).$$

Then we have

$$S^+(u, x_2) \ge S^+(u, x_1).$$

By induction, we obtain a sequence of $\{x_i\} \subset \mathbf{R}^n$ such that for all $i \ge 1$
(i) $|x_i - x_{i-1}| = 1$.
(ii) $u(x_i) = u(x_{i-1}) + S_1^+(u, x_{i-1}) \ge u(x_{i-1}) + S^+(u, x_{i-1})$.
(iii) $S^+(u, x_i) \ge S^+(u, x_{i-1})$.
Therefore we have

$$u(x_i) \ge u(x_0) + i S^+(u, x_0), \ \forall i \ge 1.$$

This implies that u is not bounded from above, which is impossible. □

Now we are ready to prove

Proposition 2.15 *If $u \in C(\Omega)$ is an AMLE, then $u \in CC(\Omega)$.*

Proof. Suppose that $u \notin CCA(\Omega)$. Then there exist a subdomain $W \subset\subset \Omega$, $x_0 \in \Omega \setminus W$, and $a, b \in \mathbf{R}$ such that

$$u(x) = b + a|x - x_0| \quad \text{on } \partial W, \tag{42}$$

but

$$u(x) > b + a|x - x_0| \quad \text{in } W. \tag{43}$$

Hence by the absolute minimality we have

$$\|\nabla u\|_{L^\infty(W)} \leq \|\nabla(b + a|x - x_0|)\|_{L^\infty(W)} = |a|.$$

In particular, if $[x_1, x_2] \subset W$, then

$$|u(x_1) - u(x_2)| \leq |a||x_2 - x_1|.$$

For simplicity, assume $x_0 = 0$ and $a \geq 0$. Let $\hat{x} \in \partial W$ be such that $(1+\delta)\hat{x} \in W$ for small $\delta > 0$. Then

$$|u((1+\delta)\hat{x}) - u(\hat{x})| \leq a\delta|\hat{x}|.$$

Hence

$$u((1+\delta)\hat{x}) \leq u(\hat{x}) + a\delta|\hat{x}| = b + a|\hat{x}| + a\delta|\hat{x}| = b + a|(1+\delta)\hat{x}|.$$

This contradicts (45). Similarly, we can prove $u \in CCB(\Omega)$. □

Proposition 2.16 *If $u \in CC(\Omega)$, then u is an AMLE.*

Proof. Suppose that u were not an AMLE. Then there exist $W \subset\subset \Omega$ and $v \in \mathrm{Lip}(W)$ with $v = u$ on ∂W such that for some $\delta > 0$

$$\|\nabla u\|_{L^\infty(W)} \geq \|\nabla v\|_{L^\infty(W)} + 2\delta. \tag{44}$$

Let $x_0 \in W$ be such that $\nabla u(x_0)$ exists and

$$S^+(u, x_0) = |\nabla u(x_0)| \geq \|\nabla v\|_{L^\infty(W)} + \delta. \tag{45}$$

Let $x_1 \in \Omega$, with $|x_1 - x_0| = \mathrm{dist}(x_0, \partial W)$, be such that

$$u(x_1) = u(x_0) + S^+_{\mathrm{dist}(x_0, \partial W)}(u, x_0).$$

Then lemma 2.13 implies

$$S^+(u, x_1) \geq S^+(u, x_0).$$

Repeating this procedure, we obtain a maximal set of points $\{x_i\}_{i=0}^{J_1} \subset \Omega$ such that for $1 \le i \le J_1$,
(i) $|x_i - x_{i-1}| = \delta_i := \text{dist}(x_{i-1}, \partial W)$.
(ii) $u(x_i) - u(x_{i-1}) = \delta_i S^+_{\delta_i}(u, x_{i-1})$.
(iii) $S^+(u, x_i) \ge S^+(u, x_{i-1})$.
Applying the same procedure to $-u$, we obtain another maximal sequence of points $\{x_j\}_{j=-J_2}^{0}$ such that for $-J_2 \le j \le -1$,
(i) $|x_{i+1} - x_i| = \delta_{i+1} := \text{dist}(x_i, \partial W)$.
(ii) $u(x_{i+1}) - u(x_i) = \delta_{i+1} S^-_{\delta_{i+1}}(u, x_i)$.
(iii) $S^-(u, x_{i+1}) \ge S^-(u, x_i)$.
By combining these two sequences of points and using the monotonicity of S^+, S^-, we obtain a sequence of points $\{x_i\}_{i=-J_2}^{J_1} \subset \Omega$ such that for $-J_2 \le i \le J_1 - 1$,
(i) $|x_{i+1} - x_i| = \delta_{i+1}$.
(ii) $u(x_{i+1}) - u(x_i) \ge \delta_{i+1} S^+(u, x_0)$.
Here we have used the fact that $S^+(u, x) = S^-(u, x)$ and the monotonicity:

$$S^+(u, x_i) = S^-(u, x_i) \ge S^+(u, x_0) = S^-(u, x_0), \ \forall i.$$

Note that if J_1 (or J_2) is finite, then $x_{J_1} \in \partial W$ (or $x_{J_2} \in \partial W$). Otherwise, $J_2 = J_1 = +\infty$, we must have

$$x_{-\infty} = \lim_{j \to -\infty} x_j \in \partial W, \ x_{+\infty} = \lim_{j \to +\infty} x_j \in \partial W.$$

This follows from (i). Now we have

$$u(x_{+\infty}) - u(x_{-\infty}) \ge S^+(u, x_0) \sum_{j=-\infty}^{+\infty} |x_{j+1} - x_j|. \tag{46}$$

On the other hand, we have

$$v(x_{+\infty} - v(x_{-\infty}) \le \|\nabla v\|_{L^\infty(W)} \sum_{j=-\infty}^{+\infty} |x_{j+1} - x_j|. \tag{47}$$

Since $u = v$ on ∂W, we have $u(x_{+\infty}) - u(x_{-\infty}) = v(x_{+\infty}) - v(x_{-\infty})$. Hence (46) contradicts (47). □

As a byproduct of the above argument, we can easily observe the following maximum principle for $S^+(u, x)$. The reader can see Bhattacharaya [7] for a different proof.

Corollary 2.17 *Suppose $u \in C(\Omega)$ is an AMLE. Then for any subdomain $W \subset\subset \Omega$, we have*

$$\max\{S^+(u, x) : x \in \overline{W}\} = \max\{S^+(u, x) : x \in \partial W\}. \tag{48}$$

2.4 Convexity property of infinity harmonic functions

In this section, we give another characterization of infinity harmonic functions in term of the convexity of $\max_{\partial B_r(x)} u$ as a function of r. First, we have a weak version

Proposition 2.18 *Suppose that $u \in C(\Omega)$ is an infinity harmonic function. For $x \in \Omega$ and $0 < r < dist(x, \partial\Omega)$, denote $\rho(r) = \max_{\partial B_r(x)} u$ and $\eta(r) = \min_{\partial B_r(x)} u$. Then $\rho(r)$ (or $\eta(r)$) is a viscosity subsolution (or supersolution) of*

$$\rho'(r)^2 \rho''(r) = 0. \tag{49}$$

Proof. For $0 < r_0 < \text{dist}(x, \partial\Omega)$, suppose that $\phi \in C^2(0, \text{dist}(x, \partial\Omega)$ touches ρ at $r = r_0$ from above. Let $x_{r_0} \in \partial B_{r_0}(x)$ be such that

$$u(x_{r_0}) = \rho(r_0).$$

Then it is easy to see that $\phi(|x|) \in C^2(0, \text{dist}(x, \partial\Omega)$ touches u from above at $x = x_{r_0}$. Hence

$$\Delta_\infty \left(\phi(|x|)\right) \Big|_{x=x_{r_0}} \geq 0.$$

Direct calculations imply

$$\nabla(\phi(|x|)) = \phi'(|x|)\frac{x}{|x|}, \;\; \nabla^2(\phi(|x|)) = \phi''(|x|)\frac{x \otimes x}{|x|^2} + \phi'(|x|)\left(\frac{I_n}{|x|} - \frac{x \otimes x}{|x|^3}\right).$$

Hence we have, at $x = x_{r_0}$

$$\begin{aligned} 0 &\leq \Delta_\infty(\phi(|x|)) \\ &= \phi'(|x|)^2\phi''(|x|) + \phi'(|x|)^2\left(\frac{I_n : x \otimes x}{|x|^3} - \frac{x \otimes x : x \otimes x}{|x|^5}\right) \\ &= \phi'(|x|)^2\phi''(|x|). \end{aligned}$$

This proves that $\rho(r)$ is a viscosity subsolution of (49). The conclusion of $\eta(r)$ can be proven similarly. □

Now we present a stronger version of the convexity of ρ.

Theorem 2.19 *A function $u \in C(\Omega)$ is an infinity harmonic function if and only if for any $x \in \Omega$,*

(i) $g^+(r) = \max_{y \in \partial B_r(x)} u(y)$ is convex for $0 < r < dist(x, \partial\Omega)$.

(ii) $g^-(r) = \min_{y \in \partial B_r(x)} u(y)$ is concave for $0 < r < dist(x, \partial\Omega)$.

Proof. "⇒": For simplicity, we only verify (i) since (ii) can be proven similarly. Without loss of generality, assume $x = 0 \in \Omega$. For $0 < r_1 < r_2 < \mathrm{dist}(0, \partial\Omega)$, define a cone function:

$$C(x) = \rho^+(r_2)\frac{|x| - r_1}{r_2 - r_1} + \rho^+(r_1)\frac{r_2 - |x|}{r_2 - r_1}, \ r_1 \le |x| \le r_2.$$

Then it is easy to see that

$$u(x) \le C(x) \ \text{ on } \partial(B_{r_2} \setminus B_{r_1}).$$

Since $u \in CCA(\Omega)$, it follows

$$u(x) \le C(x) \ \text{ in } B_{r_2} \setminus B_{r_1}.$$

Hence for any $\theta \in (0,1)$ and $|x| = \theta r_1 + (1-\theta)r_2$, we have

$$\begin{aligned} \rho^+(\theta r_1 + (1-\theta)r_2) &= \max\{u(x) : |x| = \theta r_1 + (1-\theta)r_2\} \\ &\le C(\theta r_1 + (1-\theta)r_2) \\ &= (1-\theta)\rho^+(r_2) + \theta\rho^+(r_1). \end{aligned}$$

This implies the convexity of $\rho^+(r)$.

"⇐": Suppose that (i) holds. Then for $x = 0 \in \Omega$ and $0 < r < R < \mathrm{dist}(0, \partial\Omega)$, we have

$$\rho^+(\theta r + (1-\theta)R) \le \theta\rho^+(r) + (1-\theta)\rho^+(R), \ \forall 0 < \theta < 1.$$

Sending $r \downarrow 0$, since $\rho(r) \to u(0)$, we obtain

$$\rho^+((1-\theta)\theta R) \le \theta u(0) + (1-\theta)\rho^+(R), \ \forall \theta \in (0,1).$$

This clearly implies that

$$u(x) \le u(0) + \max_{y \in \partial B_R} \frac{u(y) - u(0)}{R}|x|, \ \forall x \in B_R.$$

Thus $u \in CCA(\Omega)$ and $\Delta_\infty u \ge 0$ in the viscosity sense. □

2.5 Two alternative notions of AMLE are equivalent

In this section, we show that the two alternative notions of AMLE given by definition 1.2.1 and definition 1.3.1 are equivalent. More precisely, we have

Theorem 2.20 *Suppose $u \in Lip_{loc}(\Omega)$. Then the following are equivalent:*

(i) $Lip_U(u) = Lip_{\partial U}(u)$ for any $U \subset\subset \Omega$.

(ii) $\|\nabla u\|_{L^\infty(U)} \le \|\nabla v\|_{L^\infty(U)} \ \forall\, U \subset\subset \Omega$ and $v \in Lip(U)$ with $v\big|_{\partial U} = u\big|_{\partial U}$.

Proof. See also [6]. It suffices to show that (i) is equivalent to that $u \in CC(\Omega)$, which was proved to be equivalent to (ii) through previous discussions.

Suppose that u satisfies (i). We want to show that $u \in CCA(\Omega)$. Assume that it were false. Then there are $V \subset\subset \Omega$, $x_0 \in \Omega \setminus V$, and $a, b \in \mathbf{R}$ such that

$$u(x) = b + a|x - x_0| \quad \text{on } \partial V, \tag{50}$$

but

$$u(x) > b + a|x - x_0| \quad \text{in } V. \tag{51}$$

For simplicity, assume $x_0 = 0$ and $a \geq 0$. By (i), we have

$$\mathrm{Lip}_V(u) = \mathrm{Lip}_{\partial V}(b + a|x - x_0|) = a.$$

Therefore, if $\hat{x} \in \partial W$ is such that $(1+\delta)\hat{x} \in W$ for small $\delta > 0$, then we have

$$u((1+\delta)\hat{x}) \leq u(\hat{x}) + \mathrm{Lip}_V(u)\delta|\hat{x}| \leq u(\hat{x}) + a\delta|\hat{x}| = b + a|\hat{x}| + a\delta|\hat{x}| = b + a|(1+\delta)\hat{x}|.$$

This contradicts (51). Applying the same argument to $-u$ yields $u \in CCB(\Omega)$.

Suppose now that $u \in CC(\Omega)$. We want to show that (i) holds. For any $V \subset\subset \Omega$ and $x \in V$, we claim that

$$\mathrm{Lip}_{\partial(V \setminus \{x\})}(u) = \mathrm{Lip}_{\partial V}(u). \tag{52}$$

This amounts to verify that for any fixed $y \in \partial V$,

$$u(y) - \mathrm{Lip}_{\partial V}|y - z| \leq u(z) \leq u(y) + \mathrm{Lip}_{\partial V}|y - z| \tag{53}$$

holds for $z \in V$. Since (53) holds for any $z \in \partial V$, $y \notin V$, and $u \in CC(\Omega)$, we conclude that (53) holds for all $z \in V$. Therefore (52) holds. For any $x, y \in V$, applying (52) twice, we obtain

$$\mathrm{Lip}_{\partial(V \setminus \{x,y\})}(u) = \mathrm{Lip}_{\partial(V \setminus \{x\})}(u) = \mathrm{Lip}_{\partial V}(u).$$

Therefore, we have $|u(x) - u(y)| \leq \mathrm{Lip}_{\partial V}(u)$ and (i) holds. □

2.6 Harnack inequality of infinity harmonic functions

A fundamental property for solutions to 2nd order uniformly elliptic equations are the Harnack inequality. Although the infinity Laplace equation is degenerate elliptic, its solutions enjoy the Harnack inequality as well.

Theorem 2.21 *Suppose $u \in CCA(\Omega)$ and u is nonpositive in Ω. Then for any ball $B_R \subset \Omega$ such that $B_{4R} \subset \Omega$, we have*

$$\sup_{B_R} u \leq \frac{1}{3} \inf_{B_R} u. \tag{54}$$

Proof. See also [6]. For any $x, y \in B_R$, we have $d(y) = \text{dist}(y, \partial\Omega) \geq 3R$. Since $B_d(y) \subset \Omega$ and $u \in CCA(\Omega)$, we have

$$u(x) \leq u(y) + \max_{z \in \partial B_d(y)} \frac{u(z) - u(y)}{d(y)} |x - y|. \tag{55}$$

Since $u \leq 0$, (55) yields

$$u(x) - u(y) \leq -u(y) \left(\frac{|x - y|}{d(y)} \right).$$

Since $|x - y| \leq 2R$ and $\frac{|x-y|}{d(y)} \leq \frac{2}{3}$, we obtain

$$u(x) \leq \frac{1}{3} u(y).$$

This clearly implies (54). □

For $u \in CCA(\Omega)$ that changes sign, we have the following

Corollary 2.22 *Suppose $u \in CCA(\Omega)$. Then for any ball $B_R \subset \Omega$ such that $B_{4R} \subset \Omega$, we have*

$$|u(x) - u(y)| \leq \left(\sup_{B_{4R}} u - \sup_{B_R} u \right) \frac{|x - y|}{R}, \quad \forall x, y \in B_R. \tag{56}$$

Proof. Observe that if $u \in CCA(\Omega)$, then $v := u - \sup_{B_{4R}} u \in CCA(\Omega)$. Applying (55) with u replaced by v, we have

$$\begin{aligned} u(x) - u(y) &\leq -\left(u(y) - \sup_{B_{4R}} u \right) \frac{|x - y|}{d(y)} \\ &\leq -\inf_{B_R} \left(u - \sup_{B_{4R}} u \right) \frac{|x - y|}{3R} \\ &\leq -\sup_{B_R} \left(u - \sup_{B_{4R}} u \right) \frac{|x - y|}{R} \\ &= \left(\sup_{B_{4R}} u - \sup_{B_R} u \right) \frac{|x - y}{R}, \end{aligned} \tag{57}$$

where we have used (54) in the second of last step. By interchanging x with y, we obtain (56). □

Remark 2.23 It is clear that (56) also implies that if $u \in CCA(\mathbf{R}^n)$ is bounded from above, then u is constant.

3 Linear approximation of infinity harmonic functions

In this chapter, we will establish some preliminary regularity property for infinity harmonic function, namely the so-called linear approximation property due to Crandall, Evans [9].

3.1 D. Preiss' example

In this section, we provide an example, due to D. Preiss, that a function of linear approximation may not be differentiable.

Example 3.1 The linear approximation property in theorem (3.2) itself implies neither differentiability nor the infinity harmonicity. Here is an example

$$\begin{cases} u(x) = x\sin(\log|\log|x||), & 0 \neq x \in (-1,1), \\ u(0) = 0. \end{cases}$$

Then for any $t \in [-1,1]$, there exists $r_i \downarrow 0$ such that

$$\frac{u(r_i h)}{h} \to th$$

locally uniformly in $\mathbf{R}$.

Proof. First, it is easy to check that u is not differentiable at 0. Second, since the infinity Laplace equation in 1-dimension is:

$$(u'(t))^2 u''(t) = 0,$$

one can check that u is not an infinity harmonic function. It is also easy to check that u is Lipschitz near 0 and $T_u(0) = 1$. Furthermore, one sees that

$$\frac{u(r_i h)}{r_i} = h\sin(\log|\log r_i + \log|h||),\ h \in \mathbf{R}.$$

Furthermore, by choosing properly the sequence $r_i \downarrow 0$ as $i \to \infty$, one can readily achieve

$$\sup\left\{\left|\frac{u(t_i h)}{t_i} - th\right| :\ h \in K\right\} \to 0$$

as $i \to \infty$ for every compact $K \subset \mathbf{R}$. □

3.2 Crandall-Evans' theorem

The main theorem of this section is the following

Theorem 3.2 *Suppose that $u \in C(\Omega)$ is an infinity harmonic function. Then for any $x_0 \in \Omega$ and $r_i \downarrow 0$, there exists a vector $e \in \mathbf{R}^n$ that may depend on both x_0 and r_i's, with $|e| = S^+(u, x_0)$, such that after passing to subsequences, for any $0 < R < +\infty$ it holds*

$$\lim_{r_i \to 0} \left\| \frac{u(x_0 + r_i x) - u(x_0)}{r_i} - e \cdot x \right\|_{C^0(B_R)} = 0 \tag{58}$$

Remark 3.3 It is easy to see that theorem 3.2 implies that any cone function $C(x) = b + a|x - x_0|$ for $x_0 \in \mathbf{R}^n$, $b \in \mathbf{R}$, and $0 \neq a \in \mathbf{R}$, is not an infinity harmonic function.

Proof. Note that for any $r > 0$,

$$\frac{C(x_0 + rx) - C(x_0)}{r} = a|x|,$$

so that there exists no $e \in \mathbf{R}^n$ such that

$$\lim_{r \downarrow 0} \frac{C(x_0 + rx) - C(x_0)}{r} = e \cdot x$$

locally uniformly on $\mathbf{R}^n$. □

The proof of theorem 3.2 relies on several lemmas. We start with the tightness lemma.

Lemma 3.4 *Suppose that $u : \mathbf{R}^n \to \mathbf{R}$ satisfies:*

(i) $Lip_{\mathbf{R}^n}(u) = 1$.

(ii) There is an $e \in \mathbf{R}^n$ with $|e| = 1$ such that

$$u(te) = t, \ \forall t \in \mathbf{R}.$$

Then $u(x) = e \cdot x$ for all $x \in \mathbf{R}^n$.

Proof. Assume $e = (0', 1) \in \mathbf{R}^n$. Then we have that $u(0, \cdots, 0, t) = t$ for all $t \in \mathbf{R}$. For any $x = (x_1, \cdots, x_{n-1}, x_n) \in \mathbf{R}^n$, we then have

$$|v(x_1, \cdots, x_{n-1}, x_n) - v(0, \cdots, 0, t)|^2 \leq x_1^2 + \cdots x_{n-1}^2 + (x_n - t)^2.$$

This implies

$$v^2 - 2tv + t^2 \leq x_1^2 + \cdots x_{n-1}^2 + x_n^2 - 2tx_n + t^2.$$

For $t > 0$, this gives

$$\frac{v^2 - 2vt}{t} \leq \frac{|x|^2 - 2tx_n}{t}.$$

Sending $t \uparrow +\infty$ yields

$$v(x_1, \cdots, x_{n-1}, x_n) \geq x_n.$$

On the other hand, for $t < 0$, we have

$$\frac{v^2 - 2vt}{t} \geq \frac{|x|^2 - 2tx_n}{t},$$

this implies $v(x_1, \cdots, x_{n-1}, x_n) \leq x_n$ after taking t to $-\infty$. This completes the proof. □

We also need the following lemma on differentiability.

Lemma 3.5 *Suppose $u : B_1 \to \mathbf{R}$ has $Lip_{B_1}(u) = 1$, and there exists $e \in \partial B_1$ such that*

$$u(te) - u(0) = t \ \forall -1 \leq t \leq 1.$$

Then u is differentiable at te for any $-1 < t < 1$ and $\nabla u(te) = e$.

Proof. Fix $t_0 \in (-1, 1)$. For $\lambda_i \downarrow 0$, consider the blow-up sequence

$$v_i(x) = \frac{u(t_0 e + \lambda_i x) - u(t_0 e)}{\lambda_i}, \ x \in B_{\frac{1-|t_0|}{\lambda_i}}.$$

Then we have
(i) $\mathrm{Lip}_{B_{\frac{1-|t_0|}{\lambda_i}}}(v_i) = 1$,
(ii) $v_i(te) = t, \ \forall t \in (\frac{(-1-t_0)e}{\lambda_i}, \frac{(1-t_0)e}{\lambda_i})$.
Therefore, after passing to subsequences, we may assume that there exists $w \in \mathrm{Lip}(\mathbf{R}^n)$ such that $v_i \to w$ locally uniformly on $\mathbf{R}^n$. Moreover, by the lower semicontinuity of the Lipschitz norm, we have
(iii) $\mathrm{Lip}_{\mathbf{R}^n}(w) \leq 1$.
(iv) $w(te) = t$ for all $t \in \mathbf{R}$.
It is easy to see that (iv) and (iii) imply $\mathrm{Lip}_{\mathbf{R}^n}(w) = 1$. Hence, by lemma 3.4, we conclude that

$$w(x) = e \cdot x.$$

Hence for any $x \in \mathbf{R}^n$ we have

$$\lim_{i \to \infty} \frac{u(t_0 e + \lambda_i x) - u(t_0 e)}{\lambda_i} = e \cdot x,$$

Since e is independent of λ_i, we have

$$\lim_{\lambda \to 0} \frac{u(t_0 e + \lambda x) - u(t_0 e)}{\lambda} = e \cdot x.$$

Hence u is differentiable at $t_0 e$ and $\nabla u(t_0 e) = e$. □

Now we start to prove theorem 3.2.

Proof of Theorem 3.2. Without loss of generality, assume $x_0 = 0$ and $u(0) = 0$. Note that if $S^+(u, 0) = 0$, then u is differentiable at 0 and $\nabla u(0) = 0$ so that the conclusion holds. Hence we can assume $S^+(u, 0) > 0$. For any $\lambda_i \downarrow 0$, define

$$v_i(x) = \frac{u(\lambda_i x)}{\lambda_i},\ x \in B_{\lambda_i^{-1}}.$$

Then it is easy to check that $v_i(0) = 0$ and $\mathrm{Lip}_{B_{\lambda_i^{-1}}}(v_i) = \mathrm{Lip}(u, B_1)$. Hence v_i is uniformly bounded on B_R for any $R > 0$. We can assume that, after passing to subsequences, there exists $v \in \mathrm{Lip}(\mathbf{R}^n)$, with $v(0) = 0$, such that for any $R > 0$
(i) $v_i \to v$ uniformly on B_R.
(ii) $\nabla v_i \to \nabla v$ weak* in $L^\infty(B_R)$. By the compactness of viscosity solutions (see [12], we have that v is an infinity harmonic function on $\mathbf{R}^n$. By the lower semicontinuity, we have

$$\begin{aligned} \|\nabla v\|_{L^\infty(B_R)} &\le \lim_{i\to\infty} \|\nabla v_i\|_{L^\infty(B_R)} = \lim_{i\to\infty} \|\nabla u\|_{L^\infty(B_{\lambda_i R})} \\ &= \lim_{i\to\infty} \sup\{S^+(u, x) :\ x \in B_{\lambda_i R}\} = S^+(u, 0), \end{aligned}$$

where we have used the upper semicontinuity of S^+ in the last step. Sending $R \to \infty$, we have

$$\|\nabla v\|_{L^\infty(\mathbf{R}^n)} \le S^+(u, 0). \tag{59}$$

On the other hand, note that for any $r > 0$ and $x \in \mathbf{R}^n$,

$$\begin{aligned} S_r^+(v, x) &= \max_{y \in \partial B_r(x)} \frac{v(y) - v(x)}{r} \\ &= \lim_{i\to\infty} \max_{y \in \partial B_{\lambda_i r}(\lambda_i x)} \frac{u(y) - u(\lambda_i x)}{\lambda_i r} \\ &= \lim_{i\to\infty} S_{\lambda_i r}^+(u, \lambda_i x) \\ &\le S^+(u, 0), \end{aligned}$$

where we have used in the last step the following upper semicontinuity of $S^+(u, \cdot)$: for any $z_i \in \mathbf{R}^n$ with $z_i \to z_0$ and $R_i \to 0$, it holds

$$\lim_{i\to\infty} S_{R_i}^+(u, z_i) \le S^+(u, z_0).$$

Therefore, for v we have
(iii) $\|\nabla v\|_{L^\infty(\mathbf{R}^n)} = S^+(v, 0) (= S^+(u, 0))$.

(iv) $S_R^+(v,x) \le S^+(v,0)$ for any $x \in \mathbf{R}^n$ and any $R > 0$.
(v) $S_R^+(v,0) = S^+(v,0)$ for any $R > 0$.
Now we claim that

$$v(x) = e \cdot x, \ \forall x \in \mathbf{R}^n \tag{60}$$

for some $e \in \mathbf{R}^n$ with $|e| = S^+(u,0)$.

Note by the end point estimate, there exists $e \in \partial B_1$ such that

$$v(e) = S_1^+(v,0) = S^+(v,0). \tag{61}$$

This and (iii) imply that

$$v(te) = tS^+(v,0), \ \forall 0 \le t \le 1. \tag{62}$$

(For, otherwise, there exists $t_0 \in (0,1)$ such that

$$v(t_0 e) < t_0 S^+(v,0).$$

Since

$$v(e) - v(t_0 e) \le (1-t_0) \|\nabla v\|_{L^\infty(\mathbf{R}^n)} = (1-t_0)S^+(v,0),$$

we then have

$$v(e) = v(e) - v(t_0 e) + v(t_0 e) < t_0 S^+(v,0) + (1-t_0)S^+(v,0) = S^+(v,0).$$

This contradicts (61)).

Now we define

$$T_M = \max\{t \ge 0 \mid v(se) = sS^+(v,0) \ \forall 0 \le s \le t\}.$$

We now claim

$$T_M = +\infty.$$

For, otherwise, $1 \le T_M < +\infty$. Set $x_1 = T_M e$. Then the upper semicontinuity of S^+ implies

$$S^+(v,x_1) = S^+(v,0).$$

Let $x_2 \in \partial B_1(x_1)$ be such that

$$v(x_2) = v(x_1) + S_1^+(v,x_1) \ge v(x_1) + S^+(v,x_1) = v(x_1) + S^+(v,0).$$

On the other hand, since $S_1^+(v,x_1) \le S^+(v,0)$, we have

$$v(x_2) \le v(x_1) + S^+(v,0).$$

Therefore we have

$$v(x_2) = v(x_1) + S^(v,0).$$

By the end point estimate, we have

$$S^+(v, x_2) \geq S^+(v, x_1).$$

Hence we have

$$S^+(v, x_2) = S^+(v, 0).$$

Now we have

$$v(x_2) = T_M S^+(v, 0) + S^+(v, 0) = (T_M + 1)S^+(v, 0).$$

Since

$$v(x_2) \leq |x_2| S^+_{|x_2|}(v, 0) \leq |x_2| S^+(v, 0),$$

we obtain

$$T_M + 1 = |x_1| + |x_2 - x_1| \leq |x_2|.$$

This forces

$$x_2 = (T_M + 1)e.$$

Hence

$$v((T_M + 1)e) = (T_M + 1)S^+(v, 0).$$

This contradicts the definition of T_M and hence the claim holds. This implies

$$v(te) = tS^+(v, 0), \forall t \geq 0. \tag{63}$$

Similarly, by applying the above argument to $-v$, we conclude that there exists $f \in \partial B_1$ such that

$$v(tf) = tS^-(v, 0) = tS^+(v, 0),\ \forall t \leq 0. \tag{64}$$

Combining (63) with (64), we obtain

$$v(e) - v(-f) = 2S^+(v, 0) \leq \|\nabla v\|_{L^\infty(\mathbf{R}^n)} |e + f| \leq S^+(v, 0)|e + f|,$$

this implies

$$2 \leq |e + f|.$$

Hence $e = f$. Finally, we have that

$$v(te) = tS^+(v, 0),\ \forall t \in \mathbf{R}. \tag{65}$$

Since $\mathrm{Lip}_{\mathbf{R}^n}(v) = S^+(v, 0)$, we can apply lemma 3.4 to conclude that

$$v(x) = (S^+(v, 0)e) \cdot x.$$

This implies (60). The proof is complete. □

To end this section, we present a corollary on differentiability at the maximum point of $S^+(u, \cdot)$. This observation is from Depauw-Wang [14].

Corollary 3.6 *Suppose that $u \in C(\Omega)$ is an infinity harmonic function, and $x_0 \in \Omega$ is a local maximum point of $S^+(u,x)$. Then u is differentiable at x_0.*

Proof. Let $r_0 > 0$ be such that

$$S^+(u,x_0) = \max\left\{S^+(u,x):\ x \in \overline{B_{r_0}(x_0)}\right\}. \tag{66}$$

For $0 < r_1 < r_0$, let $x_1 \in \partial B_{r_1}(x_0)$ be such that

$$u(x_1) = u(x_0) + r_1 S^+_{r_1}(u,x_0) \geq u(x_0) + r_1 S^+(u,x_0).$$

On the other hand, it follows from both the end point estimate and (66) that

$$S^+_{r_1}(u,x_0) \leq S^+(u,x_1) \leq S^+(u,x_0).$$

Hence we have

$$u(x_1) = u(x_0) + r_1 S^+(u,x_0).$$

As in the proof of theorem 3.2 that

$$u(x_0 + \theta(x_1 - x_0)) = u(x_0) + \theta r_1 S^+(u,x_0),\ \forall \theta \in [0,1]. \tag{67}$$

By theorem 3.2, we know that there exists $e \in \mathbf{R}^n$ with $|e| = S^+(u,x_0)$ such that for some $r_i \downarrow 0$,

$$\lim_{i\to 0}\left\|\frac{u(x_0 + r_i x) - u(x_0)}{r_i} - e\cdot x\right\|_{C^0(B_2)} = 0.$$

Hence by (67) we have

$$S^+(u,x_0) = e\cdot\frac{x_1 - x_0}{r_1}.$$

This implies

$$e = S^+(u,0)\frac{x_1 - x_0}{r_1}.$$

In other word, e is uniquely determined by x_1 and $S^+(u,x_0)$, and is independent of $r_i \downarrow 0$. This means that u is differentiable at x_0. □

4 Asymptotic behavior near isolated singularity

In this chapter, we will outline the theorem by Savin, Wang, Yu [25] on the asymptotic behavior of an infinity harmonic function near its isolated singular point.

Recall that for $1 < p \leq \infty$, if a function $u \in W^{1,p}(\Omega) \cap C(\Omega)$ satisfies the p-Laplace equation $\Delta_p u = 0$ in $\Omega \setminus \{x_0\}$, then we call x_0 as an *isolated singular point* of u. Furthermore, x_0 is called a *removable singularity* if u can be extended to be a solution of the p-Laplace equation on Ω. Otherwise, we call x_0 a *non-removable isolated singularity.* It is a classical theorem by J. Serrin [26] that for $1 < p < n$ a nonnegative p-harmonic function is comparable to the fundamental solution to the p-Laplace equation near its non-removable isolated singular points.

4.1 Preliminary analysis on asymptotic behaviors

The main theorem of this chapter is

Theorem 4.1 *Suppose that $n \geq 2$ and $u \in C(\Omega \setminus \{x_0\})$ is a non-negative infinity harmonic function in $\Omega \setminus \{x_0\}$. Then, $u \in Lip_{loc}(\Omega)$ and the following holds:*
either (i) x_0 is a removable singularity,
or (ii) there exists a fixed constant $c \neq 0$ such that

$$u(x) = u(x_0) + c|x - x_0| + o(|x - x_0|), \tag{68}$$

i.e.

$$\lim_{x \to x_0} \frac{|u(x) - u(x_0) - c|x - x_0||}{|x - x_0|} = 0.$$

In particular, in case (ii), u has either a local maximum or a local minimum at x_0 and

$$|c| = T_u(x_0) := \lim_{r \to 0} \|\nabla u\|_{L^\infty(B_r(x_0))}.$$

Remark 4.2 (i) Theorem 4.1 indicates that up to first order, an infinity harmonic function in dimensions $n \geq 2$ behaves exactly like $C(x) = u(x_0) + c|x - x_0|$ near any non-removable isolated singularity x_0.
(ii) When $n = 1$, theorem 4.1 fails. In fact, for any $a, b \in \mathbf{R}$ with $a \neq b$,

$$u(x) = \begin{cases} ax, & x \geq 0, \\ bx, & x \leq 0 \end{cases}$$

is an infinity harmonic function with 0 being a non-removable isolated singularity.

A preliminary analysis of an infinity harmonic function near its non-removable, isolated singularity gives

Lemma 4.3 *For $n \geq 1$, suppose that $u \in C(B_1)$ and is a viscosity solution of $\Delta_\infty u = 0$ in $B_1(0) \setminus \{0\}$. Then u is either a viscosity supersolution or a viscosity subsolution of $\Delta_\infty u = 0$ in B_1. Moreover, if u is not a viscosity supersolution of $\Delta_\infty u = 0$ in B_1, then there exist $\epsilon > 0$ and $0 \neq p \in \mathbf{R}^n$ such that*

$$u(x) \leq u(0) + p \cdot x + \epsilon|x|, \quad x \in B_\epsilon(0). \tag{69}$$

If u is not a viscosity subsolution of $\Delta_\infty u = 0$ in B_1, then there exist $\epsilon > 0$ and $0 \neq p \in \mathbf{R}^n$ such that

$$u(x) \geq u(0) + p \cdot x - \epsilon|x|, \; x \in B_\epsilon(0). \tag{70}$$

Proof. Without loss of generality, we assume that $u(0) = 0$. Suppose that u is not a viscosity supersolution. Then there exists $\phi \in C^2(B_1(0))$ such that

$$(\phi - u)(x) < (\phi - u)(0) = 0 \text{ for } x \in B_1(0) \setminus \{0\}, \tag{71}$$

but

$$\Delta_\infty \phi(0) > 0. \tag{72}$$

Let $p = \nabla\phi(0)$. Then $p \neq 0$. We claim that (69) holds for this choice of p. For otherwise, then for any $m \in \mathbf{N}$ there exists $x_m \in B_{\frac{1}{m}}(0)$ such that

$$u(x_m) < \phi(x_m) + \frac{1}{m}|x_m|. \tag{73}$$

It is clear that $x_m \neq 0$. Denote

$$\phi_m(x) = \phi_(x) + \frac{x_m}{m|x_m|}x.$$

Let $y_m \in B_1(0)$ be such that

$$u(y_m) - \phi_m(y_m) = \min_{B_1(0)}(u - \phi_m).$$

Then we have that $y_m \neq 0$ and $\lim_{m\to\infty} y_m = 0$. Hence,

$$\Delta_\infty \phi_m(y_m) \leq 0.$$

Since

$$\nabla\phi_m = \nabla\phi + \frac{x_m}{m|x_m|}, \; \nabla^2\phi_m = \nabla^2\phi,$$

sending $m \to \infty$ implies

$$\Delta_\infty \phi(0) \leq 0.$$

This is a contradiction. Hence (69) holds. Similarly, we can show that if u is not a viscosity subsolution $\Delta_\infty u = 0$ at 0, then (70) holds. □

A careful examination of the above proof yields

Corollary 4.4 *Suppose that $u \in C(B_1)$ is a viscosity solution of $\Delta_\infty u = 0$ in $B_1 \setminus \{0\}$. If u is differentiable at 0. Then 0 is removable.*

Proof. Suppose that u is not a viscosity supersolution, then there are $\epsilon > 0$ and $0 \neq p \in \mathbf{R}^n$ such that

$$u(x) \geq u(0) + p \cdot x + \epsilon |x|,$$

this implies clearly that u is not differentiable at 0. Similarly, if u is not a viscosity subsolution, then (70) implies u is not differentiable at 0. Both are impossible by the assumption. □

4.2 Liouville property of infinity harmonic functions on $\mathbf{R}^n \setminus \{0\}$.

The next lemma is the key in the proof of theorem 4.1. It is a Liouville type result.

Lemma 4.5 *Suppose that $u : \mathbf{R}^n \to \mathbf{R}$ satisfies:*

(i) $\|\nabla u\|_{L^\infty(\mathbf{R}^n)} \leq 1$.

(ii) $u(0) = 0$ *and for some* $\epsilon > 0$, $u(x) \leq (1-\epsilon)|x|$ *for all* $x \in \mathbf{R}^n$.

(iii) u *is a viscosity subsolution of (18) in* $\mathbf{R}^n \setminus \{0\}$.

(iv) There exists $e \in \mathbf{R}^n$, *with* $|e| = 1$, *such that*

$$u(-te) = -t \text{ for all } t \geq 0. \tag{74}$$

Then

$$u(x) = -|x|. \tag{75}$$

Proof. Note that (i) and (iv) imply that $\mathrm{Lip}_{\mathbf{R}^n}(u) = 1$. For simplicity, assume $e = (0', 1)$. For $\epsilon > 0$, denote

$$\mathcal{S}_\epsilon = \{v : \mathbf{R}^n \to \mathbf{R} : \ (i) - (iv) \text{ are all satisfied}\}.$$

Define

$$w(x) = \sup_{v \in \mathcal{S}_\epsilon} v(x), \ x \in \mathbf{R}^n.$$

Then it is not hard to see that $w \in \mathcal{S}_\epsilon$. Moreover, for any $\lambda > 0$, since

$$w_\lambda(x) = \frac{w(\lambda x)}{\lambda} \in \mathcal{S}_\epsilon,$$

we conclude that

$$w_\lambda(x) \le w(x), \ \forall x \in \mathbf{R}^n.$$

Since $\lambda > 0$, this implies that $w_\lambda(x) = w(x)$ in $\mathbf{R}^n$ for any $\lambda > 0$. Hence w is positively homogeneous of degree one. It follows from (v) and the proof of lemma 3.4 that

$$w(x_1, \cdots, x_n) \le x_n, \ \forall (x_1, \cdots, x_n) \in \mathbf{R}^n.$$

We now claim

$$\max_{x' \in \partial B_1^{n-1}} w(x', 0) < 0. \tag{76}$$

For, otherwise, there is $x_0' \in \partial B_1^{n-1}$ such that $w(x_0', 0) = 0$. By (i), this implies

$$w(x_0', -t) = -t, \ \forall t \ge 0.$$

Hence

$$S^+(w, (x_0', 0)) = S^-(w, (x_0', 0)) = 1 = \|\nabla w\|_{L^\infty(B_1(x_0', 0))}.$$

There exists $y_0 \in \partial B_1((x_0', 0))$ such that

$$w(y_0) = w(x_0', 0) + S_1^+(x_0', 0) = 1,$$

since

$$S_1^+(x_0', 0) = 1.$$

We claim that $y_0 = (x_0', 1)$. For, otherwise,

$$\begin{aligned} w(y_0) - w(x_0', -1) &= 1 + 1 \\ &\le |y_0 - (x_0' - 1)| \\ &< |y_0 - (x_0', 0)| + |(x_0', 0) - (x_0', -1)| \\ &= 1 + 1. \end{aligned}$$

Therefore we have

$$w(x_0', t) = t, \ \forall - \infty \le t \le 1.$$

Continuing the same argument at $(x_0', 1)$ yields that

$$w(x_0', t) = t, \ \forall t \in \mathbf{R}.$$

This contradicts (ii).

For $\delta > 0$, we now define

$$\Gamma_\delta = \{x = (x', x_n) \in \mathbf{R}^n \ |x_n \le 0 \ \text{ or } x_n \le \delta |x'|\}.$$

Then $w \le 0$ in Γ_δ. We also define

$$C_\delta = \left\{x = (x', x_n) \in \mathbf{R}^n \ |x_n \le 0 \text{ and } x_n^2 \ge \delta^2 |x'|^2\right\}.$$

Observe that for any $x \in C_\delta$, $B_{|x|}(x) \subset \Gamma_\delta$. Since $w \in CC(\mathbf{R}^n \setminus \{0\})$, we have

$$S^+(w, x) \leq \max_{y \in \partial B_{|x|}(x)} \frac{w(y) - w(x)}{|x|} \leq -\frac{w(x)}{|x|}.$$

The 1-homogeneity of w implies

$$x \cdot \nabla w(x) = w(x).$$

Hence we obtain

$$\nabla w(x) = \frac{w(x)}{|x|^2} x,$$

and

$$|\nabla w|(x) = -\frac{w(x)}{|x|}.$$

Taking derivatives gives

$$\nabla(|\nabla w|) = -\nabla(\frac{w(x)}{|x|}) = -\frac{\nabla w}{|x|} + \frac{w}{|x|^2}\frac{x}{|x|} = 0.$$

Therefore

$$|\nabla w| \equiv 1, \text{ and } w(x) = -|x| \text{ in } C_\delta.$$

Since $S^{n-1} = \{x \in \mathbf{R}^n : \ |x| = 1\}$ is connected for $n \geq 2$, the continuity argument then implies

$$w(x) = -|x|, \ \forall x \in \mathbf{R}^n.$$

Since

$$-|x| \leq u(x) \leq w(x), \ x \in \mathbf{R}^n,$$

we conclude that $u(x) = -|x|$ in $\mathbf{R}^n$. □

With this lemma at hand, we can proceed with the proof of theorem 4.1.

Proof of Theorem 4.1. Assume $\Omega = B_1$, $x_0 = 0$, and $u(0) = 0$. We may assume that u is not a viscosity subsolution. Hence by lemma 4.3, we have that there are $0 \neq p \in \mathbf{R}^n$ and $\epsilon > 0$ such that

$$u(x) \leq p \cdot x - \epsilon|x|, \ \forall x \in B_\epsilon. \tag{77}$$

There exists a neighborhood V of 0 such that

$$u \geq p \cdot x - \delta \text{ in } V,$$

and

$$u = p \cdot x - \delta \text{ on } V.$$

Set

$$\bar{t} = \sup\{t \geq 0 \mid [0, -tp] \subset V\}.$$

Then

$$\begin{aligned} c &= \|\nabla u\|_{L^\infty(V)} \\ &\geq \frac{u(0) - u(-\bar{t}p)}{\bar{t}|p|} \\ &= \frac{\bar{t}p \cdot p + \delta}{\bar{t}|p|} > |p|. \end{aligned} \tag{78}$$

On the other hand, since u is an AMLE on $\mathbf{R}^n \setminus \{0\}$, we have

$$\begin{aligned} K &= \sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : \ x, y \in \partial(V \setminus \{0\}), x \neq y\right\} \\ &\geq \|\nabla u\|_{L^\infty(V)} = c > |p|. \end{aligned} \tag{79}$$

Note that

$$\sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : \ x, y \in \partial V, x \neq y\right\} = |p|.$$

Therefore, we have

$$K = \sup_{x \in \partial V} \frac{|u(x)|}{|x|} = -\frac{u(\bar{x})}{|\bar{x}|} \tag{80}$$

for some $\bar{x} \in \partial V$. Hence

$$u(\bar{x}) = -K|\bar{x}|.$$

Since $K > |p|$, we can show that $[0, \bar{x}] \subset V$ and hence $K \leq c$. Thus $K = c$ and

$$u(t\bar{x}) = -Kt|\bar{x}|, \ \forall 0 \leq t \leq 1. \tag{81}$$

Now we define for $\lambda \to 0$ the rescalled functions

$$u_\lambda(x) = \frac{u(\lambda x)}{K\lambda}, \ x \in \lambda^{-1}V.$$

Then we can check that
(1) $\mathrm{Lip}_{\lambda^{-1}V}(u_\lambda) = 1$.
(2)u_λ is a viscosity subsolution of (18) on $\lambda^{-1}V$.
(3)$u_\lambda(0) = 0$ and

$$u_\lambda(x) \leq (1 - \epsilon)|x|, \ \forall x \in B_{\lambda^{-1}\epsilon}.$$

(4) $u_\lambda(t\bar{x}) = -t \ \forall 0 \leq t \leq \lambda^{-1}$.
Hence, by taking subsequences, we can assume that there exists $w : \mathbf{R}^n \to \mathbf{R}$ that satisfies all the conditions (i)-(iv) (with $e = \frac{\bar{x}}{|\bar{x}|}$ in (iv)) of lemma 4.5 such that $u_\lambda \to w$ locally uniformly in $\mathbf{R}^n$. Hence lemma 4.5 implies $w(x) = \frac{\bar{x}}{|\bar{x}|} \cdot x$. The proof is complete. □

By extending the argument of the proof of lemma 4.5, we can prove a more general Liouville type theorem.

Theorem 4.6 *If $u : \mathbf{R}^n \to \mathbf{R}$ satisfies the following:*

(i) $\|Du\|_{L^\infty(\mathbf{R}^n)} = 1$.

(ii) for some $M \in \mathbf{R}$ *and* $\epsilon > 0$, $u(x) = M + (1-\epsilon)|x|$ *for all* $x \in \mathbf{R}^n$.

(iii) u *is an infinity harmonic function in* $\mathbf{R}^n \setminus \{0\}$. *Then*

$$u(x) = u(0) - |x|. \tag{82}$$

Proof. (ii) implies that for $R > 0$ sufficiently large, we have

$$u(x) \le u(0) + (1 - \frac{\epsilon}{2})R, \ \forall x \in \partial B_R.$$

Hence by the cone comparison property, we have

$$u(x) \le u(0) + (1 - \frac{\epsilon}{2})|x|, \ \forall x \in B_R.$$

Sending $R \to \infty$, we have

$$u(x) \le u(0) + (1 - \frac{\epsilon}{2})|x|, \ \forall x \in \mathbf{R}^n. \tag{83}$$

We first claim that

$$T_u(0) = 1. \tag{84}$$

For, otherwise, there exists $0 < \delta < \epsilon$ such that

$$\|\nabla u\|_{L^\infty(B_\delta)} \le 1 - \delta.$$

It follows from (i) that there exists $x_0 \in \mathbf{R}^n$ such that

$$S^+(u, x_0) \ge 1 - \frac{\delta}{2}.$$

Then $x_0 \notin B_\delta$. Let $x_1 \in \partial B_{\frac{\delta}{2}}(x_0)$ be such that

$$u(x_1) = u(x_0) + \frac{\delta}{2} S^+_{\frac{\delta}{2}}(u, x_0) \ge u(x_0) + \frac{\delta}{2} S^+(u, x_0).$$

Then since

$$S^+(u, x_1) \ge S^+(u, x_0) \ge 1 - \frac{\delta}{2},$$

we conclude that $x_1 \notin B_{\frac{\delta}{2}}$. Repeating this procedure, we obtain $\{x_m\} \subset \mathbf{R}^n \setminus B_{\frac{\delta}{2}}$ such that for all $m \ge 1$,
(i) $|x_m - x_{m-1}| = \frac{\delta}{2}$.

(ii) $u(x_m) - u(x_{m-1}) \geq (1 - \frac{\delta}{2})|x_m - x_{m-1}|$.
Hence we have

$$u(x_m) \geq u(x_0) + (1 - \frac{\delta}{2})|x_m - x_0|, \ \forall m \geq 1. \tag{85}$$

It is easy to see from this inequality that

$$\lim_{m\to\infty} |x_m| = +\infty.$$

Now we see that (85) contradicts (83). Hence (84) holds.

Assume that u is a viscosity subsolution of (18) on $\mathbf{R}^n$. Then (84) implies

$$S^+(u, 0) = 1.$$

Hence by the same argument as in the proof of lemma 4.5 that there exists $e \in \partial B_1$ such that

$$u(te) = t, \ \forall t \geq 0.$$

This contradiction with (83). Hence u must be a viscosity supersolution of (18) on $\mathbf{R}^n$. Then we have

$$S^-(u, 0) = 1,$$

and there exists a $e \in \partial B_1$ such that

$$u(-te) = -t, \ \forall t \geq 0.$$

Now we see that all these conditions (i)-(iv) of lemma 4.5 hold. Hence lemma 4.5 implies that $u(x) = u(0) - |x|$ in $\mathbf{R}^n$. The proof is complete. □

5 Uniqueness of infinity harmonic functions

The uniqueness for viscosity solutions to the infinity Laplace equation under the Dirichlet boundary condition remained to be a major open problem after Aronsson [1, 2, 3, 4] introduced it in 1960's. It was until 1993 that Jensen [17] finally established it. New proofs were found by Barles, Busca [8] in 2001, and by Crandall, Gunnarsson, Wang [11] in 2006. The reader can consult [6] for an account of [8]. The ideas by [11] was manifested by Jensen-Wang-Yu [20] in the content of uniqueness for Aronsson's equation, which will be presented in Part II below. The presentation of this Chapter follows the original proof by [17] closely.

There are several important ideas in [17]. The first is the sup (inf)-convolution, which can be used to approach viscosity subsolutions (supersolutions) to the infinity Laplace equation by semiconvex(seminconcave) ones. The second is the construction of viscosity subsolutions (supersoltuions) of the infinity Laplace equation with gradients away from zero. The third is the deformation of viscosity subsolutions (supersolutions) to strict subsolutions (supersolutions). The last is the well-known maximum principle for semiconvex functions by Jensen [18].

5.1 Sup/Inf-convolution of elliptic equations

Denote by $\mathcal{S}^{n\times n}$ the set of symmetric $n \times n$ matrices of real numbers. We first recall the definition of 2nd order (degenerate) elliptic operators.

Definition 5.1 Given a continuous function $F : \mathcal{S}^{n\times n} \times \mathbf{R}^n \times \Omega \to \mathbf{R}$. We call F is (degenerate) elliptic if for any $(p, x) \in \mathbf{R}^n \times \Omega$,

$$F(M, p, x) \geq F(N, p, x) \ \text{ whenever } M, N \in \mathcal{S}^{n\times n} \text{ satisfies } M \geq N. \tag{86}$$

It is easy to check that the infinity Laplace operator:

$$\Delta_\infty(M, p) = \langle p, Mp\rangle, \ (M, p) \in \mathcal{S}^{n\times n} \times \mathbf{R}^n$$

is (degenerate) elliptic.

Definition 5.2 Let $u \in C(\Omega)$ and $\epsilon > 0$. Then, for any $x \in \Omega$,

$$u^\epsilon(x) = \sup_{y\in\Omega} \left(u(y) - \frac{|x-y|^2}{2\epsilon} \right) \tag{87}$$

and

$$u_\epsilon(x) = \inf_{y\in\Omega} \left(u(y) + \frac{|x-y|^2}{2\epsilon} \right) \tag{88}$$

are called the sup-convolution and inf-convolution of u respectively.

Definition 5.3 For $u \in C(\Omega)$ and $K \in \mathbf{R}$, we say that u is semiconvex with constant K if $u(x) + \frac{K}{2}|x|^2, \ x \in \Omega$, is convex. Similarly, u is semiconcave with constant K if $u(x) - \frac{K}{2}|x|^2, \ x \in \Omega$, is concave. Note that a semiconvex (semiconcave) function with constant $K = 0$ is convex (concave).

The first observation on sup/inf-convolution is

Proposition 5.4 *For $u \in C(\Omega)$ and $\epsilon > 0$, the sup-convolution u^ϵ is semiconvex with constant $\frac{1}{\epsilon}$ and the inf-convolution u_ϵ is semiconcave with constant $\frac{1}{\epsilon}$.*

Proof. For any fixed $y \in \Omega$, note that

$$x \to u(y) - \frac{|x-y|^2}{2\epsilon}$$

is semiconvex with constant $\frac{1}{\epsilon}$. Moreover, since the supremum of semiconvex functions with the same constant $\frac{1}{\epsilon}$ is again semiconvex with constant $\frac{1}{\epsilon}$, we conclude that u^ϵ is semiconvex with constant $\frac{1}{\epsilon}$. Similarly, we can show that u_ϵ is semiconcave with constant $\frac{1}{\epsilon}$. □

Remark 5.5 By the Alexandrov's theorem on convex functions (Evans-Gariepy [15]), we have that for any $u \in C(\Omega)$, the sup-convolution u^ϵ and inf-convolution u_ϵ are locally Lipschitz continuous and twice differentiable almost everywhere in Ω. Moreover,

$$\nabla^2 u^\epsilon \geq -K I_n, \ \nabla^2 u_\epsilon \leq K I_n \tag{89}$$

almost everywhere in Ω.

For $\delta > 0$, denote

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}.$$

Now we prove the following fact on sup-convolution and inf-convolution, due to Jensen-Lions-Songanidis [19].

Proposition 5.6 *If $u \in C(\overline{\Omega})$, then*

$$Lip_\Omega(u^\epsilon)\,(Lip_\Omega(u_\epsilon)) \leq \frac{\sup_{\overline{\Omega}} |y|}{\epsilon}, \tag{90}$$

and for any $\delta > 0$,

$$\lim_{\epsilon \downarrow 0} u^\epsilon(x)\,(u_\epsilon(x)) = u(x) \ \textit{uniformly for} \ x \in \Omega_\delta. \tag{91}$$

Suppose that $F \in C(\mathcal{S}^{n\times n} \times \mathbf{R}^n)$ is elliptic and u is a viscosity solution to

$$F(\nabla^2 u, \nabla u) = 0 \ \ \textit{in} \ \Omega. \tag{92}$$

Then $u^\epsilon \in C(\Omega)$ is also a viscosity subsolution of (92). Similarly, if u is a viscosity supersolution of (92), then $u_\epsilon \in C(\Omega)$ is also a viscosity supersolution of (92).

Proof. For simplicity, we only consider the sup-convolution u^ϵ. For any $y \in \Omega$ fixed, since

$$u(y) - \frac{|x-y|^2}{2\epsilon}$$

is Lipschitz in Ω with Lipschitz constant at most

$$\frac{\sup_{z\in\overline{\Omega}} |z|}{\epsilon},$$

we can easily see that the Lipschitz constant of u^ϵ is no more than the same constant. This gives (90).

It is also obvious that u^ϵ is monotonically nondecreasing in $\epsilon > 0$. By definition, we have

$$u^\epsilon(x) = \sup_{y\in\Omega}\left\{u(y) - \frac{|x-y|^2}{2\epsilon}\right\} \geq u(x) - \frac{|x-x|^2}{2\epsilon} = u(x), \ \forall x \in \Omega.$$

For simplicity, let $y \in \Omega$ be such that

$$u(x) \leq u^\epsilon(x) = u(y) - \frac{|x-y|^2}{2\epsilon}$$

so that

$$|x-y|^2 \leq 2\epsilon(u(y) - u(x)) \leq 2\epsilon \sup_{\overline{\Omega}} |u|$$

and hence

$$u^\epsilon(x) = \sup_{\left\{y\in\Omega: |y-x| \leq \sqrt{2\epsilon \sup_{\overline{\Omega}} |u|}\right\}} \left(u(y) - \frac{|x-y|^2}{2\epsilon}\right).$$

Therefore, if $\delta > 0$ is such that $\delta > \sqrt{2\epsilon \sup_{\overline{\Omega}} |u|}$, then for any $x \in \Omega_\delta$ there is $x_\epsilon \in \Omega$ such that

$$u^\epsilon(x) = u(x_\epsilon) - \frac{|x_\epsilon - x|^2}{2\epsilon} \geq u(x)$$

so that

$$\begin{aligned}
\frac{|x_\epsilon - x|^2}{2\epsilon} &\leq u(x_\epsilon) - u(x) \\
&\leq \sup\{|u(x+y) - u(x)| : \ |y| \leq \sqrt{2\epsilon \sup_{\overline{\Omega}} |u|}, x \in \Omega_\delta\} \\
&\to 0
\end{aligned}$$

as $\epsilon \downarrow 0$ by the uniform continuity of u on $\overline{\Omega}$. Thus $u^\epsilon \to u$ uniformly on Ω_δ.

Now for $\delta > 0$, let $x_0 \in \Omega_\delta$ and $\phi \in \Omega \times C^2(\Omega)$ be such that

$$0 = (u^\epsilon - \phi)(x_0) \geq (u^\epsilon - \phi)(x) \ \forall x \in \Omega, \tag{93}$$

and let $y_0 \in \Omega$ be such that

$$u^\epsilon(x_0) = u(y_0) - \frac{|x_0 - y_0|^2}{2\epsilon}. \tag{94}$$

Then we have

$$0 = u(y_0) - \frac{|x_0 - y_0|^2}{2\epsilon} - \phi(x_0) \geq u(y) - \frac{|y - x|^2}{2\epsilon} - \phi(x), \ \forall x, y \in \Omega. \tag{95}$$

For $y \in \Omega_\delta$, choosing $x = y + x_0 - y_0 \in \Omega$ yields

$$u(y_0) - \phi(x_0) \geq u(y) - \phi(x_0 - y_0 + y). \tag{96}$$

This means that $\hat{\phi}(y) = \phi(x_0 - y_0 + y) + (u(y_0) - \phi(x_0)) \in C^2(\Omega_\delta)$ is a upper test function of u at y_0. Hence

$$F(\nabla^2 \hat{\phi}(y_0), \nabla \hat{\phi}(y_0)) \geq 0.$$

Since

$$F(\nabla^2 \hat{\phi}(y_0), \nabla \hat{\phi}(y_0)) = F(\nabla^2 \phi(x_0), \nabla \phi(x_0)),$$

we conclude that u^ϵ is a viscosity subsolution of (92). □

5.2 Viscosity subsolutions with non-zero gradients

In this section, we establish two auxillary equations for the infinity Laplace equation (18), whose solutions are subsolutions and supersolutions of the infinity Laplace equation with gradients bounded away from zero. The construction is based on the L^p-approximation scheme.

Theorem 5.7 *For $g \in Lip(\Omega)$ and $\epsilon > 0$, there exists*

(i) a viscosity solution $u_\epsilon \in Lip(\Omega)$ of

$$\max\{\epsilon - |\nabla u|, -\Delta_\infty u\} = 0 \quad in\ \Omega \tag{97}$$
$$u = g \quad on\ \partial\Omega. \tag{98}$$

(ii) a viscosity solution $v_\epsilon \in Lip(\Omega)$ of

$$\min\{|\nabla u| - \epsilon, -\Delta_\infty u\} = 0 \quad in\ \Omega \tag{99}$$
$$u = g \quad on\ \partial\Omega. \tag{100}$$

(iii) there exists a continuous, nondecreasing $\beta : [0, +\infty) \to [0, +\infty)$, with $\beta(0) = 0$, such that

$$\|u_\epsilon - v_\epsilon\|_{L^\infty(\Omega)} = \beta(\epsilon). \tag{101}$$

Proof. Since (i) can be done by the same way as (ii), we only outline the proof of (ii) and (iii) as follows. For $1 < p < +\infty$, let $u_p \in W^{1,p}_g(\Omega)$ be the unique minimizer of

$$F_p(v) = \int_\Omega \left(\frac{1}{p}|\nabla v|^p - \epsilon^{p-1} v\right),\ v \in W^{1,p}_g(\Omega).$$

Direct calculations imply that u_p solves

$$-\Delta_p u_p = \epsilon^{p-1} \text{ in } \Omega \tag{102}$$

in the sense of distributions. By the minimality of u_p, one has

$$\int_\Omega |\nabla u_p|^p \le \int_\Omega |\nabla g|^p + \epsilon^{p-1}\int_\Omega (u_p - g).$$

This, combined with the Poincaré inequality, implies

$$\|\nabla u_p\|_{L^p(\Omega)} \le C\mathrm{Lip}_\Omega(g)$$

for some $C > 0$ independent of $p > 1$. Hence by the Sobolev's embedding theorem, we may assume that there exists a $u_\epsilon \in \mathrm{Lip}(\Omega)$ with $u_\epsilon|_{\partial\Omega} = g$ such that after taking possible subsequences,

$$u_p \to u_\epsilon \text{ in } C(\overline{\Omega}) \bigcap \left(\bigcap_{q>1} W^{1,q}(\Omega)\right). \tag{103}$$

Now we claim that *for $p > n$, u_p is a viscosity solution of (102).*

For simplicity we only indicate u_p is a subsolution of (102) (see also Juutinen-Linqvist-Manfredi [21]). For, otherwise, there exists $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that

$$0 = (u - \phi)(x_0) > (u - \phi)(x)\ \forall x \in \Omega \setminus \{x_0\},$$

and

$$-\Delta_p \phi(x_0) - \epsilon^{p-1} = 2\alpha_0 > 0. \tag{104}$$

Hence, for $\delta_0 > 0$ sufficiently small,

$$-\Delta_p \phi - \epsilon^{p-1} \ge \alpha_0 \text{ in } B_{\delta_0}(x_0). \tag{105}$$

Since there exists a neighborhood $x_0 \in V \subset B_{\delta_0}(x_0)$ such that for $\phi_0 = \phi - \frac{\delta_0}{2}$,

$$u > \phi_0 \text{ in } V,\ u = \phi_0 \text{ on } \partial V,$$

we have

$$\int_V \langle |\nabla u_p|^{p-2}\nabla u_p, \nabla u_p - \nabla\phi_0\rangle = \epsilon^{p-1}\int_V (u_p - \phi_0).$$

On the other hand, since $u_p - \phi_0 \geq 0$ in V, multiplying (104) by $u_p - \phi_0$ and integrating over V, we have

$$\int_V \langle |\nabla \phi_0|^{p-2} \nabla \phi_0, \nabla u_p - \nabla \phi_0 \rangle > \epsilon^{p-1} \int_V (u_p - \phi_0).$$

Subtracting these two equations gives

$$\int_V \langle |\nabla u_p|^{p-2} \nabla u_p - |\nabla \phi_0|^{p-2} \nabla \phi_0, \nabla (u_p - \phi_0) \rangle < 0.$$

This is clearly impossible. Thus the claim holds.

Now we want to show u_ϵ is a viscosity solution of (99). For simplicity, we only show u_ϵ is a viscosity subsolution. Let $(x_0, \phi) \in \Omega \times C^2(\Omega)$ be such that

$$0 = (u - \phi)(x_0) > (u - \phi)(x), \ \forall x \in \Omega \setminus \{x_0\}. \tag{106}$$

We need to show

$$\min \{|\nabla \phi(x_0)| - \epsilon, -\Delta_\infty \phi(x_0)\} \leq 0. \tag{107}$$

If $|\nabla \phi(x_0)| \leq \epsilon$, then (107) holds. Hence we assume that for small $\delta > 0$,

$$|\nabla \phi(x_0)| \geq (1 + 2\delta)\epsilon,$$

Note that there exists $x_p \to x_0$ such that $(u_p - \phi)$ achieves its maximum at x_p. For p sufficiently large, we can assume

$$|\nabla \phi(x_p)| \geq (1 + \delta)\epsilon.$$

Moreover, we have

$$-\Delta_p \phi(x_p) \leq \epsilon^{p-1}.$$

Dividng both side by $(p-2)|\nabla \phi(x_p)|^{p-4}$, we obtain

$$\begin{aligned} -\Delta_\infty \phi(x_p) \quad &\leq \quad \left(\frac{\epsilon^3}{p-2}\right) \left(\frac{\epsilon}{|\nabla \phi(x_p)|}\right)^{p-4} \\ &+ \quad \left(\frac{|\nabla \phi|^2 \Delta \phi}{p-2}\right)(x_p). \end{aligned}$$

Sending $p \to \infty$, this implies

$$-\Delta_\infty \phi(x_0) \leq 0.$$

Thus u_ϵ is a viscosity subsolution of (99).

Now we want to establish (101). Note that v_ϵ is obtained as the limit of minimizers v_p to the functional

$$G_p(v) = \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \epsilon^{p-1} v\right), \quad v \in W_g^{1,p}(\Omega).$$

It is easy to see that v_p satisfies

$$-\Delta_p v_p = -\epsilon^{p-1} \quad \text{in } \Omega. \tag{108}$$

We have

$$\begin{aligned}
\int_\Omega \langle |\nabla u_p|^{p-2}\nabla u_p - |\nabla v_p|^{p-2}\nabla v_p, \nabla(u_p - v_p)\rangle &= \epsilon^{p-1}\int_\Omega (u_p - v_p) \\
&\leq \epsilon^{p-1}\|u_p - v_p\|_{L^1(\Omega)}.
\end{aligned}$$

Hence, by Poincaré inequality and Hölder inequality, we obtain

$$\|\nabla(u_p - v_p)\|_{L^1(\Omega)} \leq C(\Omega)^{\frac{1}{p}}|\Omega|\epsilon.$$

In particular, we conclude that there exists a continuous and nondecreasing $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(0) = 0$ such that

$$\|u_p - v_p\|_{L^1(\Omega)} \leq \beta(\epsilon).$$

Sending p to ∞, we obtain that $\|u_\epsilon - v_\epsilon\|_{L^1(\Omega)} \leq \beta(\epsilon)$. This, combined with the boundedness of $\mathrm{Lip}_\Omega(u_\epsilon)$ and $\mathrm{Lip}_\Omega(v_\epsilon)$, implies (101). The proof is now complete. □

5.3 Deformation of subsolutions to strict subsolutions

In this section, we indicate how to deform a viscosity subsolution/supersolution to the infinity Laplace equation (18) with gradient bounded away from zero to a strict subsolution/supersolution of (18). More precisely, we have

Theorem 5.8 *For $\epsilon > 0$, suppose that $u \in C(\overline{\Omega})$ is a viscosity subsolution of the equation (97). Then for any $\lambda > 0$, there exist $\mu = \mu(\epsilon, \lambda) > 0$ and $u_\lambda \in C(\overline{\Omega})$, with $\|u_\lambda - u\|_{L^\infty(\Omega)} \leq \lambda$, such that u_λ is a viscosity subsolution of*

$$\max\{\epsilon - |\nabla u|, -\Delta_\infty u\} = -\mu. \tag{109}$$

Proof. Let $\lambda > 0$ be so small that $2\lambda\|u\|_{C(\overline{\Omega})} < 1$. Define w_λ by letting $u = G_\lambda(w_\lambda)$, where

$$G_\lambda(t) = t - \frac{\lambda}{2}t^2, \ t \in \mathbf{R},$$

or equivalently

$$w_\lambda = H_\lambda(u) := \frac{1}{\lambda}\left(1 - \sqrt{1 - 2\lambda u}\right).$$

It is clear that $w_\lambda \to u$ uniformly in $\overline{\Omega}$ as $\lambda \downarrow 0$. Now we want to show that for sufficiently small $\lambda > 0$, $u_\lambda = w_\lambda$ satisfies (109) for some $\mu = \mu(\epsilon, \lambda) > 0$. For simplicity, denote $w = w_\lambda$. To see it, let $(x_0, \phi) \in \Omega \times C^2(\Omega)$ be such that

$$0 = (w - \phi)(x_0) \geq (w - \phi)(x), \ \forall x \in \Omega.$$

Note that H_λ is strictly monotone function, we see that $\phi_\lambda = G_\lambda(\phi) \in C^2(\Omega)$ touches u at x_0 from above. Hence we have

$$\max\{\epsilon - |\nabla\phi_\lambda|,\ -\Delta_\infty\phi_\lambda\}\big|_{x=x_0} \le 0. \tag{110}$$

Note that

$$G'_\lambda(t) = 1 - \lambda t,\ G''_\lambda(t) = -\lambda.$$

Direct calculations imply that at x_0,

$$\epsilon \le |\nabla\phi_\lambda| = G'_\lambda(\phi)|\nabla\phi|, \tag{111}$$

and

$$0 \le \Delta_\infty\phi_\lambda = G'_\lambda(\phi)^3\Delta_\infty\phi + G''_\lambda(\phi)G'_\lambda(\phi)^2|\nabla\phi|^4. \tag{112}$$

Hence we have, at x_0,

$$\begin{aligned}\Delta_\infty\phi &\ge -\frac{G''_\lambda(\phi)}{G'_\lambda(\phi)}|\nabla\phi|^4 \\ &= -\frac{G''_\lambda(\phi)}{G'_\lambda(\phi)^5}|\nabla\phi_\lambda|^4 \\ &= \frac{\lambda}{(1-\lambda\phi)^5}\epsilon^4 \ge (\frac{3}{2})^5\lambda\epsilon^4,\end{aligned} \tag{113}$$

where we have used the fact that $\lambda\phi(x_0) \le \frac{1}{3}$. It follows from (111) that at x_0

$$|\nabla\phi| \ge \frac{3}{2}\epsilon$$

so that

$$\epsilon - |\nabla\phi| \le -\frac{1}{2}\epsilon. \tag{114}$$

It follows from (113) and (114) that u is a viscosity subsolution of (109), with

$$\mu = \min\left\{\frac{1}{2}\epsilon,\ (\frac{3}{2})^5\lambda\epsilon^4\right\}.$$

The proof is complete. □

We have the following theorem, whose proof is similar to theorem 5.8.

Theorem 5.9 *For $\epsilon > 0$, suppose that $u \in C(\overline{\Omega})$ is a viscosity supsolution of the equation (99). Then for any $\lambda > 0$, there exist $\mu = \mu(\epsilon, \lambda) > 0$ and $u_\lambda \in C(\overline{\Omega})$, with $\|u_\lambda - u\|_{L^\infty(\Omega)} \le \lambda$, such that u_λ is a viscosity supersolution of*

$$\min\{|\nabla u| - \epsilon, -\Delta_\infty u\} = \mu. \tag{115}$$

5.4 Jensen's maximum principle

In this section, we will present one of the most important tools in the study of viscosity solutions to elliptic equations, namely, Jensen's maximum principle for seminconvex functions. First, we present Alexandrov's theorem for convex functions.

Definition 5.10 A function $f : \mathbf{R}^n \to \mathbf{R}$ is called convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $0 \le t \le 1$, $x, y \in \mathbf{R}^n$.

Theorem 5.11 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be convex. Then f is locally Lipschitz on $\mathbf{R}^n$, and there is a constant C depending only on n such that*

$$\sup_{B_{\frac{r}{2}}(x)} |f| \le Cr^{-n} \int_{B_r(x)} |f|\, dy, \tag{116}$$

and

$$\|\nabla f\|_{L^\infty(B_{\frac{r}{2}}(x))} \le Cr^{-(n+1)} \int_{B_r(x)} |f|\, dy \tag{117}$$

for any ball $B_r(x) \subseteq \mathbf{R}^n$.

Proof. See also Evans-Gariepy [15]. Let's first assume that $f \in C^2(\mathbf{R}^n)$. Fix $x \in \mathbf{R}^n$. Then for any $y \in \mathbf{R}^n$ and $t \in (0, 1)$,

$$f(x + t(y-x)) \le f(x) + t(f(y) - f(x)).$$

Hence

$$\frac{f(x + t(y-x)) - f(x)}{t} \le f(y) - f(x).$$

Sending t to 0 gives

$$f(y) \ge f(x) + \nabla f(x) \cdot (y - x),\ \forall x, y \in \mathbf{R}^n. \tag{118}$$

Fix $z \in B_{\frac{r}{2}}(x)$, applying (118) to $x = z$ and $y \in B_{\frac{r}{2}}(x)$ and integrating it with respect to $y \in B_{\frac{r}{2}}(x)$ yields

$$f(z) \le \frac{1}{|B_{\frac{r}{2}}(x)|} \int_{B_{\frac{r}{2}}(x)} f(y)\, dy \le Cr^{-n} \int_{B_r(x)} |f|\, dy. \tag{119}$$

Next choose a smooth cutoff function $\eta \in C_0^\infty(\mathbf{R}^n)$, which satisfies

$$0 \le \eta \le 1,\ |\nabla \eta| \le Cr^{-1},\ \eta \equiv 1 \text{ on } B_{\frac{r}{2}}(x),\ \eta \equiv 0 \text{ outside } B_r(x).$$

Since (118) implies

$$f(z) \geq f(y) + \nabla f(y) \cdot (z - y).$$

Multiplying this inequality by $\eta(y)$ and integrating with respect to y over $B_r(x)$, we obtain

$$\begin{aligned} f(z) \int_{B_r(x)} \eta(y)\, dy &\geq \int_{B_r(x)} f(y)\eta(y)\, dy + \int_{B_r(x)} \eta(y) \nabla f(y) \cdot (z - y)\, dy \\ &= \int_{B_r(x)} f(y) \left[\eta(y) - \nabla \cdot (\eta(y)(z - y)) \right] dy \\ &\geq -C \int_{B_r(x)} |f|\, dy. \end{aligned}$$

This inequality implies

$$f(z) \geq -Cr^{-n} \int_{B_r(x)} |f|\, dy. \tag{120}$$

It is clear that (119) and (120) imply (116).

To show (117), observe first that for any $z \in B_{\frac{r}{2}}(x)$, the set

$$S_z = \left\{ y \in B_{\frac{r}{2}}(z) \setminus B_{\frac{r}{4}}(z) : \ \nabla f(z) \cdot (y - z) \geq \frac{1}{2} |\nabla f(z)||y - z| \right\}$$

has

$$|S_z| \geq Cr^n$$

for some positive C depending only on n. Use (119) to get

$$f(y) \geq f(z) + \frac{r}{8} |\nabla f(z)|, \ \forall y \in S_z.$$

Integrating over S_z gives

$$|\nabla f(z)| \leq Cr^{-(n+1)} \int_{B_r(x)} |f(y) - f(z)|\, dy.$$

This inequality and (120) implies (117).

If f is assumed to be convex only, then we define $f^\epsilon = \eta_\epsilon * f$, where $\epsilon > 0$ and η_ϵ is a standard mollifier. It is clear that f^ϵ is smooth. We now claim that f^ϵ is convex. In fact, for any $x, y \in \mathbf{R}^n$ and $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} f^\epsilon(\lambda x + (1 - \lambda) y) &= \int_{\mathbf{R}^n} f(z - (\lambda x + (1 - \lambda) y)) \eta_\epsilon(z)\, dz \\ &\leq \lambda \int_{\mathbf{R}^n} f(z - x) \eta_\epsilon(z)\, dz + (1 - \lambda) \int_{\mathbf{R}^n} f(z - y) \eta_\epsilon(z)\, dz \\ &= \lambda f^\epsilon(x) + (1 - \lambda) f^\epsilon(y), \end{aligned}$$

where we have used both the convexity of f and nonnegativity of η_ϵ.

For f^ϵ, by the estimates (116) and (117) we have

$$\sup_{B_{\frac{r}{2}}(x)} (|f^\epsilon| + r|\nabla f^\epsilon|) \le Cr^{-n} \int_{B_r(x)} |f^\epsilon| \, dy.$$

Sending ϵ to zero, we obtain in the limit the same estimates for f. This completes the proof of the theorem. □

It is well-known that for any convex function $f \in C^2(\mathbf{R}^n)$, the Hessian matrix of f is positive semi-definite:

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \ge 0.$$

The next theorem indicates that a weaker version of the above fact holds.

Theorem 5.12 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be convex. Then there exist signed Radon measures $\mu^{ij} = \mu^{ji}$ such that*

$$\int_{\mathbf{R}^n} f \frac{\partial^2 \phi}{\partial x_i \partial x_j} \, dx = \int_{\mathbf{R}^n} \phi \, d\mu^{ij}, \ 1 \le i, j \le n, \tag{121}$$

for all $\phi \in C_0^2(\mathbf{R}^n)$. Furthermore, μ^{ii} are nonnegative for $1 \le i \le n$.

Proof. See also [?]. For $\epsilon > 0$, let η_ϵ be a standard mollifier. Write $f^\epsilon = \eta_\epsilon * f$. Then f^ϵ is smooth and convex, hence

$$\nabla^2 f^\epsilon \ge 0.$$

For any unit vector $\xi = (\xi_1, \cdots, \xi_n)$ and a nonnegative $\phi \in C_0^2(\mathbf{R}^n)$, we then have

$$\int_{\mathbf{R}^n} f^\epsilon \frac{\partial^2 \phi}{\partial x_i \partial x_j} \xi_i \xi_j \, dx = \int_{\mathbf{R}^n} \phi \frac{\partial^2 f^\epsilon}{\partial x_i \partial x_j} \xi_i \xi_j \, dx \ge 0.$$

Sending ϵ to 0, we obtain

$$L(\phi) \equiv \int_{\mathbf{R}^n} f \frac{\partial^2 \phi}{\partial x_i \partial x_j} \xi_i \xi_j \, dx \ge 0.$$

Hence there exists a signed Radon measure μ^ξ such that

$$L(\phi) = \int_{\mathbf{R}^n} \phi \, d\mu^\xi$$

for all $\phi \in C_0^2(\mathbf{R}^n)$.

Let $e_i, 1 \le i \le n$, be the standard base of $\mathbf{R}^n$. Define $\mu^{ii} = \mu^{e_i},\ \ 1 \le i \le n$. For $i \neq j$, let $\xi = \frac{e_i + e_j}{\sqrt{2}}$. In this case, it is easy to see that

$$\sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial x_k \partial x_l} \xi_k \xi_l = \frac{1}{2} \left[\frac{\partial^2 \phi}{\partial x_i^2} + 2 \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{\partial^2 \phi}{\partial x_j^2} \right].$$

Thus

$$\int_{\mathbf{R}^n} f \frac{\partial^2 \phi}{\partial x_i \partial x_j}\, dx = \int_{\mathbf{R}^n} \phi \, d\mu^{ij}$$

where

$$\mu^{ij} = \mu^{\xi} - \frac{1}{2}\mu^{ii} - \frac{1}{2}\mu^{jj}$$

for $i \neq j$. □

Remark 5.13 If $f : \mathbf{R}^n \to \mathbf{R}$ is convex, then

$$\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_1} \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{R}^n).$$

Proof. Let $V \subset\subset \mathbf{R}^n$ and $\phi \in C_0^2(V, \mathbf{R}^n)$, $|\phi| \le 1$. Then for $1 \le i \le n$,

$$\begin{aligned} \int_{\mathbf{R}^n} \frac{\partial f}{\partial x_k} \mathrm{div}(\phi) &= -\int_{\mathbf{R}^n} f \sum_{i=1}^{n} \frac{\partial^2 \phi^i}{\partial x_i \partial x_k}\, dx \\ &= -\sum_{i=1}^{n} \int_{\mathbf{R}^n} \phi^i \, d\mu^{ik} \\ &\le \sum_{i=1}^{n} \mu^{ik}(V) < +\infty. \end{aligned}$$

This completes the proof. □

By Lebesgue's decomposition theorem, we may write

$$\mu^{ij} = \mu^{ij}_{\mathrm{ac}} + \mu^{ij}_{\mathrm{s}},$$

where

$$\mu^{ij}_{\mathrm{ac}} << \mathcal{L}^n,\ \ \mu^{ij}_{\mathrm{s}} \perp \mathcal{L}^n.$$

Hence there exist $f_{ij} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that

$$\mu^{ij}_{\mathrm{ac}} = f_{ij} \mathcal{L}^n,\ \ 1 \le i, j \le n.$$

Denote

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = f_{ij},\ \ 1 \le i, j \le n,$$

and write

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix},$$

$$[D^2 f] = \begin{pmatrix} \mu^{11} & \cdots & \mu^{1n} \\ \vdots & & \vdots \\ \mu^{n1} & \cdots & \mu^{nn} \end{pmatrix},$$

$$[D^2 f]_{\mathrm{ac}} = \begin{pmatrix} \mu^{11}_{\mathrm{ac}} & \cdots & \mu^{1n}_{\mathrm{ac}} \\ \vdots & & \vdots \\ \mu^{n1}_{\mathrm{ac}} & \cdots & \mu^{nn}_{\mathrm{ac}} \end{pmatrix} = D^2 f \mathrm{L} \mathcal{L}^n,$$

and

$$[D^2 f]_{\mathrm{s}} = \begin{pmatrix} \mu^{11}_{\mathrm{s}} & \cdots & \mu^{1n}_{\mathrm{s}} \\ \vdots & & \vdots \\ \mu^{n1}_{\mathrm{s}} & \cdots & \mu^{nn}_{\mathrm{s}} \end{pmatrix}.$$

Then we have that $D^2 f \in L^1_{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^{n\times n})$, and

$$[D^2 f] = [D^2 f]_{\mathrm{ac}} + [D^2 f]_{\mathrm{s}}.$$

Now we present Alexandrov's theorem on convex functions.

Theorem 5.14 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be convex. Then f has second order derivative for a.e. $x \in \mathbf{R}^n$.*

Proof. First note that for a.e. $x \in \mathbf{R}^n$, the following three conditions hold:
(1) $\nabla f(x)$ exists and

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)} |\nabla f(y) - \nabla f(x)| \, dy = 0.$$

(2)

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)} \left| D^2 f(y) - D^2 f(x) \right| \, dy = 0.$$

(3)

$$\lim_{r\to 0} \frac{\left|[D^2 f]_{\mathrm{s}}\right| (B_r(x))}{r^n} = 0.$$

Fix such a x, we may assume $x = 0$. For $\epsilon > 0$, let $f^\epsilon = \eta_\epsilon * f$. For $r > 0$ and $y \in B_r$, Taylor's theorem implies

$$\begin{aligned} f^\epsilon(y) &= f^\epsilon(0) + \nabla f^\epsilon(0) \cdot y + \int_0^1 (1-s) y^T \cdot D^2 f^\epsilon(sy) \cdot y \, ds \\ &= f^\epsilon(0) + \nabla f^\epsilon(0) \cdot y + \frac{1}{2} y^T \cdot D^2 f(0) \cdot y \\ &\quad + \int_0^1 (1-s) y^T \cdot \left[D^2 f^\epsilon(sy) - D^2 f(0) \right] \cdot y \, ds. \end{aligned}$$

Let $\phi \in C_0^2(B_r)$ be such that $|\phi| \le 1$, multiply the equation above by ϕ and integrate over B_r:

$$
\begin{aligned}
& r^{-n}\int_{B_r}\phi(y)\left(f^\epsilon(y)-f^\epsilon(0)-\nabla f^\epsilon(0)\cdot y-\frac{1}{2}y^T\cdot D^2f(0)\cdot y\right)\,dy \\
= & \int_0^1(1-s)\left(r^{-n}\int_{B_r}\phi(y)y^T\cdot\left[D^2f^\epsilon(sy)-D^2f(0)\right]\cdot y\,dy\right)\,ds \\
= & \int_0^1\frac{(1-s)}{s^2}\left((rs)^{-n}\int_{B_{rs}}\phi(\frac{z}{s})z^T\cdot\left[D^2f^\epsilon(z)-D^2f(0)\right]\cdot z\,dz\right)\,ds.
\end{aligned}
$$

Note

$$
\begin{aligned}
g_\epsilon(s) &\equiv \int_{B_{rs}}\phi(\frac{z}{s})z^T\cdot D^2f^\epsilon(z)\cdot z\,dz \\
&= \int_{B_{rs}}f^\epsilon(z)\sum_{i,j=1}^n\frac{\partial^2}{\partial z_i\partial z_j}\left(\phi(\frac{z}{s})z_iz_j\right)\,dz \\
&\to \sum_{i,j=1}^n\int_{B_{rs}}\phi(\frac{z}{s})z_iz_j\,d\mu^{ij} \\
&= \int_{B_{rs}}\phi(\frac{z}{s})z^T\cdot D^2f(z)\cdot z\,dz+\sum_{i,j=1}^n\int_{B_{rs}}\phi(\frac{z}{s})z_iz_j\,d\mu_{\mathrm{S}}^{ij}.
\end{aligned}
$$

Now we estimate

$$
\begin{aligned}
\frac{|g_\epsilon(s)|}{s^{n+2}} &\le \frac{r^2}{s^n}\int_{B_{rs}}|D^2f^\epsilon(z)|\,dz \\
&= \frac{r^2}{s^n}\int_{B_{rs}}\left|\int_{\mathbf{R}^n}\eta_\epsilon(z-y)\,d[D^2f]\right|\,dz \\
&\le \frac{C}{s^n\epsilon^n}\int_{B_{rs+\epsilon}}\left(\int_{B_{rs}\cap B_\epsilon(y)}dz\right)\,d\|D^2f\| \\
&\le C\frac{\min\{(rs)^n,\epsilon^n\}}{s^n\epsilon^n}\|D^2f\|(B_{rs+\epsilon}) \\
&\le \le C\frac{\min\{(rs)^n,\epsilon^n\}}{s^n\epsilon^n}(rs+\epsilon)^n\le C,
\end{aligned}
$$

where we have used the conditions (2) and (3) in the last step. Hence by Lebesgue's Dominated Convergence Theorem we have

$$
\begin{aligned}
& r^{-n}\int_{B_r}\phi(y)\left[f(y)-f(0)-\nabla f(0)\cdot y-\frac{1}{2}y^T\cdot D^2f(0)\cdot y\right]\,dy \\
\le & Cr^2\int_0^1(rs)^{-n}\int_{B_{rs}}|D^2f(z)-D^2f(0)|\,dz\,ds+Cr^2\int_0^1\frac{|[D^2f]_{\mathrm{S}}|\,(B_{rs})}{(sr)^n}\,ds \\
= & o(r^2) \qquad (122)
\end{aligned}
$$

as $r \to 0$. Take the supremum over all ϕ to get

$$r^{-n} \int_{B_r} |h(y)|\, dy = o(r^2) \quad \text{as } r \to 0 \tag{123}$$

for

$$h(y) = f(y) - f(0) - \nabla f(0) \cdot y - \frac{1}{2} y^T \cdot D^2 f(0) \cdot y.$$

Since f is convex, it is easy to see that h is semiconvex with constant $K = |D^2 f(0)|$. Thus we have

$$\sup_{B_{\frac{r}{2}}} |\nabla h| \leq C r^{-(n+1)} \int_{B_r} |h(y)|\, dy + Cr. \tag{124}$$

Now we claim that

$$\sup_{B_{\frac{r}{2}}} |h| = o(r^2) \quad \text{as } r \to 0. \tag{125}$$

First, by the weak L^1-estimate, (123) implies that

$$\mathcal{L}^n \left(\{ z \in B_r : |h(z)| \geq \epsilon r^2 \} \right) \leq o(r^n) \leq \frac{1}{4} \mathcal{L}^n (B_{\eta r})$$

for any $\eta \in (0, \frac{1}{2})$ satisfying $\eta^{\frac{1}{n}} \leq \frac{1}{2}$. Thus for any $y \in B_{\frac{r}{2}}$, there exists $z \in B_{\eta r}(y)$ such that

$$|h(z)| \leq \epsilon r^2.$$

Consequently,

$$\begin{aligned} |h(y)| &\leq |h(y) - h(z)| + |h(z)| \\ &\leq \epsilon r^2 + \eta r \sup_{B_{\frac{r}{2}}} |\nabla h| \\ &\leq \epsilon r^2 + C \eta r^2 \leq 2 \epsilon r^2, \end{aligned}$$

provided we choose η such that $C\eta = \epsilon$. We have now completed the proof of the theorem. □

Theorem 5.15 *Suppose that $u \in C(\Omega)$ is a semiconvex function with constant K and $0 \in \Omega$ is a local maximum point of u. Then for any $\epsilon > 0$ there exist $p \in \mathbf{R}^n$ with $|p| < \epsilon$ and $x_\epsilon \in B_\epsilon$ such that x_ϵ is a local maximum point of $u(x) + \langle p, x \rangle$, and u is twice differentiable at x_ϵ.*

Before we prove theorem 5.15, we want to study briefly the subdifferential for convex functions.

Definition 5.16 Let $f : \mathbf{R}^n \to \mathbf{R}$ be convex. For $z \in \mathbf{R}^n$, define the subdifferential of f at z by

$$\partial f(z) = \{ p \in \mathbf{R}^n : \ f(z) + p \cdot (x - z) \leq f(x) \ \forall x \in \mathbf{R}^n \}.$$

Remark 5.17 Since f is convex, it is well-known that $\partial f(z) \neq \emptyset$ for any $z \in \mathbf{R}^n$. It is also easy to see that $\partial f(z)$ is a convex, compact subset of $\mathbf{R}^n$ for any $z \in \mathbf{R}^n$. It is also true that f is differentiable at z if $\partial f(z)$ is a singleton.

Proof. To see that f is differentiable at z if $\partial f(z)$ is a singleton, we need the partial continuity property of subdifferentials for convex functions. Let $p = \partial f(z)$. Then

$$\lim_{r \downarrow 0} \sup_{x \in B_r(z)} \{|q - p| : \ q \in \partial f(x)\} = 0. \tag{126}$$

In fact, let $x_i \to z$ and $q_i \in \partial f(x_i)$. Since f is locally Lipschitz, we have that $\{q_i\}$ is a bounded sequence in $\mathbf{R}^n$. We may assume that $q_i \to q$ for some $q \in \mathbf{R}^n$. Note that

$$f(x_i) + q_i \cdot (x - x_i) \leq f(x), \ \forall x \in \mathbf{R}^n.$$

Sending i to ∞, this inequality implies in the limit that

$$f(z) + q \cdot (x - z) \leq f(x), \ \forall x \in \mathbf{R}^n.$$

This means that $q \in \partial f(z)$ and hence $q = p$, since $\partial f(z)$ is a singleton.

Now (126) implies that for any $\epsilon > 0$ there is a small $r > 0$ such that for any $x \in B_r(z)$ we have

$$|q - p| \leq \epsilon, \ \forall q \in \partial f(x),$$

and

$$f(z) + p \cdot (x - z) \leq f(x) \leq f(z) + q \cdot (x - z), \ \forall q \in \partial f(x).$$

Hence

$$f(z) + p \cdot (x - z) \leq f(x) \leq f(z) + p \cdot (x - z) + \epsilon|x - z|.$$

This implies that f is differentiable at z and $\nabla f(z) = \partial f(z)$. □

Now we define the set of points with small derivatives for a semiconvex function.

Definition 5.18 Let $f \in C(\overline{\Omega}) \cap \mathrm{Lip}(\Omega)$ and for $\delta > 0$, define

$$S_\delta(f) = \left\{x \in \Omega \mid \exists p \in \overline{B_\delta} \text{ s.t. } f(z) \leq f(x) + p \cdot (z - x) \ \forall z \in \Omega \right\}.$$

Lemma 5.19 *Assume $w \in C(\overline{\Omega}) \cap Lip(\Omega)$ is a semiconvex function with constant $K > 0$. If w has an interior maximum, then there are constants $c_0 > 0$ and $\delta_0 > 0$ such that*

$$\mathcal{L}^n(S_\delta(w)) \geq c_0 \delta^n, \ \forall \delta < \delta_0. \tag{127}$$

Proof. For $\epsilon > 0$, let η_ϵ be a standard mollifier and $w_\epsilon = \eta_\epsilon * w$ be the ϵ-mollification of w. Then w^ϵ is smooth and semiconvex with the same constant $K > 0$. Define S_δ^ϵ analogously for w_ϵ. We claim that for any $\epsilon_i \downarrow 0$ that

$$\mathcal{L}^n \left(\left[\limsup_{i \to \infty} S_\delta^{\epsilon_i} \right] \setminus S_\delta \right) = 0. \tag{128}$$

In fact, for a.e. $x \in \limsup_{i\to\infty} S_\delta^{\epsilon_i}$, we have

$$w_{\epsilon_i}(x) \to w(x), \ \nabla w_{\epsilon_i}(x) \to \nabla w(x).$$

We may assume that $x \in S_\delta^{\epsilon_i}$ for all i. Then we have

$$w_{\epsilon_i}(z) \le w_{\epsilon_i}(x) + \nabla w_{\epsilon_i}(x) \cdot (z - x) \ \forall z \in \Omega; \ |\nabla w_{\epsilon_i}(x)| \le \delta.$$

Passing to the limit, we obtain

$$w(z) \le w(x) + \nabla w(x) \cdot (z - x) \ \forall z \in \Omega; \ |\nabla w(x)| \le \delta.$$

Thus $x \in S_\delta$ and the claim follows.

In order to prove this lemma, it suffices to prove (127) for S_δ^ϵ with c_0 independent of ϵ. Since w has an interior maximum, it follows that w_ϵ has an interior maximum. Therefore that there are $\epsilon_0 > 0$ and $\delta_0 > 0$ such that

$$\nabla w_\epsilon(S_\delta^\epsilon) = B_\delta \ \text{ if } \delta < \delta_0 \ \text{ and } \epsilon < \epsilon_0. \tag{129}$$

Note that for any $x \in S_\delta^\epsilon$, there exists $p \in \mathbf{R}^n$ with $|p| \le \delta$ such that $w_\epsilon(z) - p\cdot z$ attains its maximum at x and hence $\nabla w_\epsilon(x) = p$ and $\nabla^2 w_\epsilon(x)$ is negative semidefinite. Since $w_\epsilon(z) - p \cdot z$ is seminconvex with constant $K > 0$, this implies

$$-K \le \lambda_i(\nabla^2 w_\epsilon)(x) \le 0, \ \forall 1 \le i \le n,$$

where $\lambda_i(A)$ is the i-th eigenvalue of $A \in \mathcal{S}^{n\times n}$. This implies

$$\left|\det(\nabla^2 w_\epsilon(x))\right| = \left|\Pi_{i=1}^n \lambda_i(\nabla^2 w_\epsilon)\right|(x) \le K^n, \ \forall x \in S_\delta^\epsilon. \tag{130}$$

On the other hand, by the change of variables we have

$$\begin{aligned} K^n \mathcal{L}^n\left(S_\delta^\epsilon\right) &= \int_{S_\delta^\epsilon} \left|\det(\nabla^2 w_\epsilon(x))\right| \\ &= \mathcal{L}^n(\nabla w_\epsilon(S_\delta^\epsilon)) \ge \mathcal{L}^n(B_\delta). \end{aligned} \tag{131}$$

Combining (130) and (131), we obtain

$$\mathcal{L}^n(S_\delta^\epsilon) \ge K^{-n}\mathcal{L}^n(B_\delta) \ge c_0 \delta^n, \ \forall \delta < \delta_0 \ \text{ and } \epsilon < \epsilon_0.$$

This completes the proof of this lemma. □

We also need the following lemma.

Lemma 5.20 *Assume $u \in C(\overline{\Omega})$ is semiconvex with constant $K > 0$ and $x \in \Omega$ is a maximum point of u. Then u is differentiable at x and $\nabla u(x) = 0$.*

Proof. Since u is semiconvex in Ω, it follows that there is $p \in \mathbf{R}^n$ such that,

$$u(z) - u(x) \geq p \cdot (z - x) + O(|z - x|^2), \ \forall z \in \Omega.$$

On the other hand, since x is a maximum point of u, we have

$$u(z) - u(x) \leq 0, \ \forall z \in \Omega.$$

Therefore we have

$$p \cdot (z - x) + O(|z - x|^2) \leq 0,$$

this implies that $p = 0$ so that

$$u(x) + O(|z - x|^2) \leq u(z) \leq u(x), \ \forall z \in \Omega.$$

This implies u is differentiable at x and $\nabla u(x) = 0$. □

Proof of theorem 5.15. By lemma 5.19, we conclude that for any $\epsilon > 0$ there is $c_0 > 0$ such that

$$\mathcal{L}^n(S_\epsilon(w)) \geq c_0 \epsilon^n.$$

Since w is semiconvex, we then have that for a.e. $x \in S_\epsilon(w)$, $\nabla^2 w(x)$ exists. On the other hand, by the definition of $S_\epsilon(w)$ we have that for any $x \in S_\epsilon(w)$ there exists $p \in \overline{B_\epsilon}$ such that x is a maximum point of $w(z) - p \cdot z, \ z \in \Omega$. Hence lemma 5.20 implies that $\nabla(w(z) - p \cdot z)|_{z=x} = 0$ and hence $\nabla w(x)$ exists and $|\nabla w|(x) \leq \epsilon$. Combining these two facts together, we can easily see that for a.e. $x \in S_\epsilon(w)$, $|\nabla w|(x) \leq \epsilon$ and

$$\nabla^2(w(x) - p \cdot x) = \nabla^2 w(x) \leq 0.$$

This completes the proof. □

5.5 Uniqueness of infinity harmonic functions

The main result of this section is the following uniqueness theorem of infinity harmonic functions, due to Jensen [17].

Theorem 5.21 *For any $g \in Lip\ (\partial\Omega)$, there exists a unique viscosity solution $u \in Lip\ (\Omega)$ to*

$$-\Delta_\infty u = 0 \quad in\ \Omega \tag{132}$$

$$u = g \quad on\ \partial\Omega. \tag{133}$$

The uniqueness theorem 5.21 follows from the following comparison principle. We leave it to the reader as an exercise.

Theorem 5.22 *Suppose that $u \in C(\overline{\Omega})$ is a viscosity subsolution of (132), and $v \in C(\overline{\Omega})$ is a viscosity supersolution of (132). Then*

$$\max_{x \in \overline{\Omega}} (u(x) - v(x)) = \max_{x \in \partial\Omega} (u(x) - v(x)). \tag{134}$$

References

[1] G.Aronsson, *Minimization problems for the functional* $\sup_x F(x,\, f(x),\, f'(x))$. Ark. Mat. 6, 1965 33-53 (1965).

[2] G.Aronsson, *Minimization problems for the functional* $\sup_x F(x, f(x), f'(x))$. *II.* Ark. Mat. 6, 1966 409-431 (1966).

[3] G.Aronsson, *Extension of functions satisfying Lipschitz conditions.* Ark. Mat. 6, 1967 551-561 (1967).

[4] G.Aronsson, *Minimization problems for the functional* $\sup_x F(x,\, f(x),\, f'(x))$. *III.* Ark. Mat. 7, 1969 509-512 (1969).

[5] G.Aronsson, *On certain singular solutions of the partial differential equation* $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. Manuscripta Math. 47 (1984), no. 1-3, 133–151.

[6] G. Aronsson, M. Crandall, P. Juutinen, *A tour of the theory of absolutely minimizing functions.* Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439-505 (electronic).

[7] T. Bhattacharya, *Some observations on the local behaviour of infinity-harmonic functions.* Differential Integral Equations 19 (2006), no. 8, 945-960.

[8] G. Barles, J. Busca, *Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term.* Comm. Partial Differential Equations 26 (2001), no. 11-12.

[9] M. Crandall, L. C. Evans, *A remark on infinity harmonic functions.* Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Via del Mar-Valparaiso, 2000), 123-129 (electronic), Electron. J. Differ. Equ. Conf., 6, Southwest Texas State Univ., San Marcos, TX, 2001.

[10] M. Crandall, L. C. Evans, R. Gariepy, *Optimal Lipschitz extensions and the infinity Laplacian.* Calc. Var. Partial Differential Equations 13 (2001), no. 2, 123-139.

[11] M. Crandall, Gunnarsson, P. Wang, *Uniqueness of* ∞*-harmonic functions and the eikonal equation.* Comm. Partial Differential Equations 32 (2007), no. 10-12, 1587-1615.

[12] M. Crandall, H. Ishii, P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1-67.

[13] M. Crandall, C. Y. Wang, Y. Yu, *Derivation of the Aronsson equation for C-1 Hamiltonians.* Trans. AMS (in press).

[14] T. Depauw, C. Y. Wang, *Calibration and isolated singularity of infinity harmonic functions.* In preparation.

[15] L. C. Evans, R. Gariepy, Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[16] L. C. Evans, O. Savin, *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions.* Calc. Var. Partial Differential Equations 32 (2008), no. 3, 325–347.

[17] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient.* Arch. Rational Mech. Anal. 123 (1993), no. 1, 51-74.

[18] R. Jensen, *Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations.* Indiana Univ. Math. J. 38 (1989), no. 3, 629-667.

[19] R. Jensen, P. L. Lions, P. E. Songanidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations.* Proc. Amer. Math. Soc. 102 (1988), no. 4, 975-978.

[20] R. Jensen, C. Y. Wang, Y. Yu, *Uniqueness and nonuniqueness of viscosity solutions to Aronsson's equation.* ARMA (in press).

[21] P. Juutinen, P. Lindqvist, J. Manfredi, *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation.* SIAM J. Math. Anal. 33 (2001), no. 3, 699-717.

[22] P. Juutinen, *Absolutely minimizing Lipschitz extensions on a metric space.* Ann. Acad. Sci. Fenn. Math. 27 (2002), no. 1, 57-67.

[23] J. McShane, *Extension of range of functions.* Bull. Amer. Math. Soc. 40 (1934), 837-842.

[24] O. Savin, *C^1 regularity for infinity harmonic functions in two dimensions.* Arch. Ration. Mech. Anal. 176 (2005), no. 3, 351–361.

[25] O. Savin, C. Y. Wang, Y. Yu, *Asymptotic behavior of infinity harmonic functions near an isolated singularity.* Int. Math. Reas. Notices, Vol. 2008, 23 pages.

[26] J. Serrin, *Local behaviour of solutions of quasilinear equations.* Acta Mathematica 111 (1964): 247302.

[27] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets.* Trans. Amer. Math. Soc. 36 (1934), no. 1, 63-89.