Liquid crystal flows in two dimensions

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Abstract

The paper is concerned with a simplified hydrodynamic equation, proposed by Ericksen and Leslie, modeling the flow of nematic liquid crystals. In dimension two, we establish both interior and boundary regularity theorem for such a flow under smallness conditions. As a consequence, we establish the existence of global weak solutions that are smooth away from at most finitely many singular times in any bounded smooth domain of $\mathbb{R}^2$.

1 Introduction

We consider the following hydrodynamic system modeling the flow of liquid crystal materials in dimension two (see [6] [7] [11] [14] and references therein):

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d) \text{ in } \Omega \times (0, +\infty), \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{1.2}$$

$$d_t + u \cdot \nabla d = \gamma(\triangle d + |\nabla d|^2 d) \quad \text{in } \Omega \times (0, +\infty), \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded smooth domain, $u(x,t) : \Omega \times (0, +\infty) \to \mathbb{R}^2$ represents the velocity field of the flow, $d(x,t) : \Omega \times (0, +\infty) \to S^2$, the unit sphere in $\mathbb{R}^3$, is a unit-vector field that represents the macroscopic molecular orientation of the liquid crystal material, and $P(x,t) : \Omega \times (0, +\infty) \to \mathbb{R}$ represents the pressure

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function. The constants ν, λ, and γ are positive constants that represent viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time for the molecular orientation field. $\nabla\cdot$ denotes the divergence operator, and $\nabla d \odot \nabla d$ denotes the 2×2 matrix whose (i, j)-the entry is given by $\nabla_i d \cdot \nabla_j d$ for $1 \leq i, j \leq 2$.

The above system is a simplified version of the Ericksen-Leslie model, which reduces to the Ossen-Frank model in the static case, for the hydrodynamics of nematic liquid crystals developed during the period of 1958 through 1968 ([7] [6] [11]). It is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow field $u(x, t)$, and the macroscopic description of the microscopic orientation configurations $d(x, t)$ of rod-like liquid crystals. Roughly speaking, the system (1.1)-(1.3) is a coupling between the non-homogeneous Navier-Stokes equation and the transported flow of harmonic maps. In a series of papers, Lin [13] and Lin-Liu [14, 15] initiated the mathematical analysis of (1.1)-(1.3) in 1990's. More precisely, they considered in [14] the Leslie system of variable length, i.e. the Dirichlet energy $\frac{1}{2}\int_\Omega |\nabla d|^2\, dx$ for $d : \Omega \to S^{n-1}$ is replaced by the Ginzburg-Landau energy $\int_\Omega (\frac{1}{2}|\nabla d|^2 + \frac{(1-|d|^2)^2}{4\epsilon^2})\, dx$ for $d : \Omega \to \mathbb{R}^n$ ($\epsilon > 0$), and proved existence of global classical and weak solutions in dimensions two or three. In [15], they proved the partial regularity theorem for suitable weak solutions, similar to the classical theorem by Caffarelli-Kohn-Nirenberg [5] for the Navier-Stokes equation. However, as pointed out in [14, 15], both their estimates and arguments depend on ϵ, and it is a challenging problem to study the convergence as ϵ tends to zero.

In this paper, we are interested in the existence of global weak solutions (u, d) of (1.1)-(1.3) that may enjoy possible regularity under the initial and boundary conditions:

$$(u(x,0), d(x,0)) = (u_0(x), d_0(x)) \quad x \in \Omega, \tag{1.4}$$

$$(u(x,t), d(x,t)) = (0, d_0(x)) \quad (x,t) \in \partial\Omega \times (0, +\infty). \tag{1.5}$$

Throughout this paper, we introduce

$$\mathbf{H} = \text{closure of } C_0^\infty(\Omega, \mathbb{R}^2) \cap \{v : \nabla \cdot v = 0\} \text{ in } L^2(\Omega, \mathbb{R}^2),$$

$$\mathbf{J} = \text{closure of } C_0^\infty(\Omega,\mathbb{R}^2)\cap\{v : \nabla\cdot v = 0\} \text{ in } H_0^1(\Omega,\mathbb{R}^2),$$

$$H^1(\Omega,S^2) = \left\{d\in H^1(\Omega,\mathbb{R}^3):\ d(x)\in S^2 \text{ a.e. } x\in\Omega\right\},$$

We make the assumptions:

$$u_0\in\mathbf{H},\ d_0\in H^1(\Omega,S^2) \text{ and } d_0\in C^{2,\beta}(\partial\Omega,S^2) \text{ for some } \beta\in(0,1). \tag{1.6}$$

Definition 1.1 *For $0 < T \le +\infty$, $u \in L^\infty([0,T],\mathbf{H})\cap L^2([0,T],\mathbf{J})$ and $d \in L^2([0,T],H^1(\Omega,S^2))$ is a weak solution of (1.1)-(1.5), if*

$$\begin{aligned}
&-\int_{\Omega\times[0,T]}\langle u,\psi'\phi\rangle + \int_{\Omega\times[0,T]}[\langle u\cdot\nabla u,\psi\phi\rangle + \nu\langle\nabla u,\psi\nabla\phi\rangle]\\
&\quad = -\psi(0)\int_\Omega\langle u_0,\phi\rangle + \lambda\int_{\Omega\times[0,T]}\langle\nabla d\odot\nabla d,\psi\nabla\phi\rangle,\\
&-\int_{\Omega\times[0,T]}\langle d,\psi'\phi\rangle + \int_{\Omega\times[0,T]}[\langle u\cdot\nabla d,\psi\phi\rangle + \gamma\langle\nabla d,\psi\nabla\phi\rangle]\\
&\quad = -\psi(0)\int_\Omega\langle d_0,\phi\rangle + \gamma\int_{\Omega\times[0,T]}|\nabla d|^2\langle d,\psi\phi\rangle,
\end{aligned}$$

for any $\psi\in C^\infty([0,T])$ with $\psi(T)=0$ and $\phi\in H_0^1(\Omega,\mathbb{R}^3)$. Moreover, (u,d) satisfies (1.5) in the sense of trace.

In this paper, we establish both the regularity and existence of global weak solutions in dimension two. More precisely, we prove

Theorem 1.2 *For $0 < T < +\infty$, assume $u \in L^\infty([0,T],\mathbf{H})\cap L^2([0,T],\mathbf{J})$ and $d\in L^2([0,T],H^1(\Omega,S^2))$ is a weak solution of (1.1)-(1.5), with (1.6). If, in addition, $d\in L^2([0,T],H^2(\Omega))$, then $(u,d)\in C^\infty(\Omega\times(0,T])\cap C_\beta^{2,1}(\overline{\Omega}\times(0,T])$.*

Utilizing Theorem 1.2, the global and local energy inequality, and the global estimate of the pressure function P in §4 below, we establish the existence of global weak solutions that enjoy the partial smoothness property.

Theorem 1.3 *Under the assumption (1.6), there exist a global weak solution $u \in L^\infty([0,\infty),\mathbf{H})\cap L^2([0,\infty),\mathbf{J})$ and $d\in L^\infty([0,\infty),H^1(\Omega,S^2))$ of (1.1)-(1.5) such that the following properties hold:*

(i) There exists $L\in\mathbb{N}$ depending only on (u_0,d_0) and $0<T_1<\cdots<T_L$, $1\le i\le L$, such that

$$(u,d)\in C^\infty(\Omega\times((0,+\infty)\setminus\{T_i\}_{i=1}^L))\cap C_\beta^{2,1}(\overline{\Omega}\times((0,+\infty)\setminus\{T_i\}_{i=1}^L)).$$

(ii) Each singular times T_i, $1 \le i \le L$, can be characterized by

$$\liminf_{t\uparrow T_i} \max_{x\in\overline{\Omega}} \int_{\Omega\cap B_r(x)} (|u|^2 + |\nabla d|^2)(y,t)dy \ge 8\pi, \quad \forall r > 0. \tag{1.7}$$

Moreover, there exist $x_m^i \to x_0^i \in \Omega$, $t_m^i \uparrow T_i$, $r_m^i \downarrow 0$ and a non constant smooth harmonic map $\omega_i : \mathbb{R}^2 \to S^2$ with finite energy such that as $m \to +\infty$,

$$(u_m^i, d_m^i) \to (0, \omega_i) \text{ in } C^2_{\rm loc}(\mathbb{R}^2 \times [-\infty, 0]),$$

where

$$u_m^i(x,t) = r_m^i u(x_m^i + r_m^i x, t_m^i + (r_m^i)^2 t), \;\; d_m^i(x,t) = d(x_m^i + r_m^i x, t_m^i + (r_m^i)^2 t).$$

(iii) Set $T_0 = 0$. Then, for $0 \le i \le L-1$,

$$|d_t| + |\nabla^2 d| \in L^2(\Omega \times [T_i, T_{i+1} - \epsilon]), \;\; |u_t| + |\nabla^2 u| \in L^{\frac{4}{3}}(\Omega \times [T_i, T_{i+1} - \epsilon])$$

for any $\epsilon > 0$, and for any $0 < T_L < T < +\infty$,

$$|d_t| + |\nabla^2 d| \in L^2(\Omega \times [T_L, T]), \;\; |u_t| + |\nabla^2 u| \in L^{\frac{4}{3}}(\Omega \times [T_L, T]).$$

(iv) There exist $t_k \uparrow +\infty$ and a harmonic map $d_\infty \in C^\infty(\Omega, S^2) \cap C^{2,\beta}(\overline{\Omega}, S^2)$ with $d_\infty = d_0$ on $\partial\Omega$ such that $u(\cdot, t_k) \to 0$ in $H^1(\Omega)$, $d(\cdot, t_k) \to d_\infty$ weakly in $H^1(\Omega)$, and there exist $l \in \mathbb{N}$, points $\{x_i\}_{i=1}^l \subset \Omega$, and $\{m_i\}_{i=1}^l \subset \mathbb{N}$ such that

$$|\nabla d(\cdot, t_k)|^2 \, dx \to |\nabla d_\infty|^2 \, dx + \sum_{i=1}^{l} 8\pi m_i \delta_{x_i}. \tag{1.8}$$

(v) If (u_0, d_0) satisfies

$$\int_\Omega (|u_0|^2 + |\nabla d_0|^2) \le 8\pi,$$

then $(u, d) \in C^\infty(\Omega \times (0, +\infty)) \cap C^{2,1}_\beta(\overline{\Omega} \times (0, +\infty))$. Moreover, there exist $t_k \uparrow +\infty$ and $d_\infty \in C^\infty(\Omega, S^2) \cap C^{2,\beta}(\overline{\Omega}, S^2)$ with $d_\infty = d_0$ on $\partial\Omega$ such that $(u(\cdot, t_k), d(\cdot, t_k)) \to (0, d_\infty)$ in $C^2(\overline{\Omega})$.

Remark 1.4 We would like to point out

(i) If d is a constant map, then (1.1)-(1.3) reduces to the Navier-Stokes equation. It is well-known (see Temam [24] and Ladyzhenskaya [9]) that in dimension two, any weak solution $u \in L_t^\infty L_x^2 \cap L_t^2 H^1$ to the Navier-Stokes equation is smooth.

(ii) If $u = 0$, then (1.1)-(1.3) reduces to the heat flow of harmonic maps. The classical theorems by Struwe [23] and Chang [1] assert that there exists a unique global weak solution that is smooth away from finitely many singular points.

(iii) Theorem 1.3 can be viewed as a mixture of the Navier-Stokes equation and the heat flow of harmonic maps in dimension two.

(iv) For smooth initial data (u_0, d_0), it is also a very interesting question to ask whether the short time smooth solution to (1.1)-(1.3) can develop finite time singularity (see Chang-Ding-Ye [2] for the heat flow of harmonic maps).

Remark 1.5 (i) We conjecture that the global weak solution (u, d) in Theorem 1.3 is unique in the class of all weak solutions $(\tilde{u}, \tilde{d})$ of (1.1)-(1.5) that enjoy the following properties: there are $K \in \mathbb{N}$ and $0 < S_1 < \cdots < S_K < +\infty$ such that $\tilde{d}$ satisfies $\tilde{d} \in L^\infty([0, +\infty), H^1(\Omega))$ and

$$\tilde{d} \in \cap_{i=0}^{K-1} \cap_{\epsilon>0} L^2([S_i, S_{i+1} - \epsilon], H^2(\Omega)) \bigcap L^2_{\text{loc}}([S_K, +\infty), H^2(\Omega)).$$

(ii) We also conjecture that there are at most finitely many singular points for the weak solution constructed by Theorem 1.3.

The paper is written as follows. In section 2, we prove both interior and boundary regularity theorems for (1.1)-(1.3) under smallness conditions and Theorem 1.2. In section 3, we employ the contraction map theory to establish the existence of short time smooth solutions to (1.1)-(1.3). In section 4, we show both global and local energy inequalities for smooth solutions to (1.1)-(1.3) and global estimates for the pressure function. In section 5, we prove Theorem 1.3 by estimating the first singular time in terms of energy concentration and performing blow-up analysis near each singularity.

Since the exact values of ν, λ, γ don't play a role, we henceforth assume

$$\nu = \lambda = \gamma = 1.$$

2 Regularity of solutions and proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The proof is based on two Lemmas: (i) the interior regularity under the smallness condition, and (ii) the boundary regularity under the smallness condition. We refer to [3] for interesting results on two dimensional NSE with singular forcing.

We begin with some notations. Throughout the paper, we use $A \lesssim B$ to denote $A \le C_0 B$ for some universal constant $C_0 > 0$. For $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$, $z_0 = (x_0, t_0)$, and $r > 0$, let

$$B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| \le r\} \text{ and } P_r(z_0) = B_r(x_0) \times [t_0 - r^2, t_0]$$

denote the ball in $\mathbb{R}^2$ and the parabolic cylinder in $\mathbb{R}^3$ respectively. Let

$$\partial_p P_r(z_0) \equiv (B_r(x_0) \times \{t_0 - r^2\}) \cup (\partial B_r(x_0) \times [t_0 - r^2, t_0])$$

denote the parabolic boundary of $P_r(z_0)$. For $x_0 = 0$ and $t_0 = 0$, we simply denote

$$B_r = B_r(0),\ P_r = P_r(0,0),\ \partial_p P_r = \partial_p P_r(0,0).$$

For $f \in L^1(P_r(z_0))$, denote by

$$f_{z_0,r} = \frac{1}{|P_r(z_0)|} \int_{P_r(z_0)} f(x,t)\, dxdt \text{ and } f_{x_0,r}(t) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} f(x,t)\, dx$$

as the average of f over $P_r(z_0)$ and $B_r(x_0)$ respectively.
For $1 < p, q < +\infty$, denote $L^{p,q}(P_r(z_0)) = L^q([t_0 - r^2, t_0], L^p(B_r(x_0)))$ and

$$\|f\|_{L^{p,q}(P_r(z_0))} = \left(\int_{t_0 - r^2}^{t_0} \|f(\cdot, t)\|_{L^p(B_r(x_0))}^q \, dt \right)^{\frac{1}{q}}.$$

$W^{1,0}_{p,q}(P_r(z_0)) \equiv L^q([t_0 - r^2, t_0], W^{1,p}(B_r(x_0)))$, with the norm

$$\|f\|_{W^{1,0}_{p,q}(P_r(z_0))} = \|f\|_{L^{p,q}(P_r(z_0))} + \|\nabla f\|_{L^{p,q}(P_r(z_0))}.$$

$W^{2,1}_{p,q}(P_r(z_0)) = \{f \in W^{1,0}_{p,q}(P_r(z_0)) :\ \nabla^2 f, f_t \in L^{p,q}(P_r(z_0))\}$, with the semi-norm

$$[f]_{W^{2,1}_{p,q}(P_r(z_0))} = \||\nabla^2 f| + |f_t|\|_{L^{p,q}(P_r(z_0))},$$

and the norm

$$\|f\|_{W^{2,1}_{p,q}(P_r(z_0))} = \|f\|_{W^{1,0}_{p,q}(P_r(z_0))} + [f]_{W^{2,1}_{p,q}(P_r(z_0))}.$$

For $p = q$, we denote $L^p(\cdot) = L^{p,p}(\cdot)$, $W_p^{1,0}(\cdot) = W_{p,p}^{1,0}(\cdot)$, and $W_p^{2,1}(\cdot) = W_{p,p}^{2,1}(\cdot)$.

By scaling, the first part of Theorem 1.2 follows from the following Lemma.

Lemma 2.1 *For any $\alpha \in (0,1)$, there exists $\epsilon_0 > 0$ such that for $z_0 = (x_0, t_0) \in \mathbb{R}^3$ and $r > 0$, if $(u, d) \in W_2^{1,0}(P_{2r}(z_0))$, $P \in W^{1,\frac{4}{3}}(P_{2r}(z_0))$ is a weak solution to (1.1)-(1.3) and*

$$\int_{P_r(z_0)} (|u|^4 + |\nabla d|^4) \leq \epsilon_0^4, \tag{2.1}$$

then $(u, d) \in C^\alpha(P_{\frac{r}{2}}(z_0), \mathbb{R}^2 \times S^2)$. Moreover,

$$[d]_{C^\alpha(P_{\frac{r}{2}}(z_0))} \leq C(\|u\|_{L^4(P_r(z_0))} + \|\nabla d\|_{L^4(P_r(z_0))}) \tag{2.2}$$

$$[u]_{C^\alpha(P_{\frac{r}{2}}(z_0))} \leq C\left\{\|u\|_{L^4(P_r(z_0))} + \|\nabla d\|_{L^4(P_r(z_0))} + \|\nabla P\|_{L^{\frac{4}{3}}(P_r(z_0))}\right\}. \tag{2.3}$$

Proof. By translation and dilation, we may assume that $z_0 = (0,0)$ and $r = 1$. It then follows from the interior $W_2^{2,1}$-estimate (see [10]) that $d \in W_2^{2,1}(P_{\frac{3}{4}})$ and

$$\|\nabla^2 d\|_{L^2(P_{\frac{3}{4}})} \lesssim \|u\|_{L^4(P_1)} + \|\nabla d\|\|_{L^4(P_1)}. \tag{2.4}$$

Pick any point $z_1 = (x_1, t_1) \in P_{\frac{1}{2}}$ and $0 < R < \frac{1}{4}$. We need

Claim 1.

$$\int_{P_r(z_1)} |\nabla d|^4 \leq \left(\frac{r}{R}\right)^{4\alpha} \int_{P_R} |\nabla d|^4, \ \forall 0 < r \leq R. \tag{2.5}$$

Let $d^1 : P_R(z_1) \to \mathbb{R}^3$ solve

$$\begin{aligned} d_t^1 - \triangle d^1 &= 0 \text{ in } P_R(z_1) \\ d^1 &= d \text{ on } \partial_p P_R(z_1). \end{aligned} \tag{2.6}$$

Then $d^2 = d - d^1 : P_R(z_1) \to \mathbb{R}^3$ solves

$$\begin{aligned} d_t^2 - \triangle d^2 &= -u \cdot \nabla d + |\nabla d|^2 d \text{ in } P_R(z_1) \\ d^2 &= 0 \text{ on } \partial_p P_R(z_1). \end{aligned} \tag{2.7}$$

Since $(-u \cdot \nabla d + |\nabla d|^2 d) \in L^2(P_R(z_1))$, we have that $d_t^2, \nabla^2 d^2 \in L^2(P_R(z_1))$. Hence, multiplying the equation (2.7) by $\triangle d^2$ and integrating over $B_R(x_1)$, we obtain

$$\frac{d}{dt} \int_{B_R(x_1)} \frac{|\nabla d^2|^2}{2} + \int_{B_R(x_1)} |\triangle d^2|^2 = \int_{B_R(x_1)} (u \cdot \nabla d - |\nabla d|^2 d) \cdot \triangle d^2.$$

Integrating over $[t_1-R^2,t_1]$ and applying Hölder inequality yields

$$\begin{aligned}
&\sup_{t_1-R^2\le t\le t_1}\int_{B_R(x_1)}|\nabla d^2|^2+\int_{P_R(z_1)}|\triangle d^2|^2\\
&\lesssim \int_{P_R(z_1)}|u|^2|\nabla d|^2+\int_{P_R(z_1)}|\nabla d|^4\\
&\lesssim [(\int_{P_R(z_1)}|u|^4)^{\frac12}+(\int_{P_R(z_1)}|\nabla d|^4)^{\frac12}](\int_{P_R(z_1)}|\nabla d|^4)^{\frac12}.
\end{aligned}$$

By the Ladyzhenskaya inequality, we have

$$\int_{B_R(x_1)}|\nabla d^2|^4\lesssim \int_{B_R(x_1)}|\nabla d^2|^2(\int_{B_R(x_1)}|\triangle d^2|^2+R^{-2}\int_{B_R(x_1)}|\nabla d^2|^2).$$

Hence, by integrating t over $[t_1-R^2,t_1]$, we have

$$\begin{aligned}
\int_{P_R(z_1)}|\nabla d^2|^4 &\lesssim (\sup_{t_1-R^2\le t\le t_1}\int_{B_R(x_1)}|\nabla d^2|^2)\\
&\quad\cdot(\int_{P_R(z_1)}|\triangle d^2|^2+\sup_{t_1-R^2\le t\le t_1}\int_{B_R(x_1)}|\nabla d^2|^2)\\
&\lesssim (\int_{P_R(z_1)}|u|^4+\int_{P_R(z_1)}|\nabla d|^4)\int_{P_R(z_1)}|\nabla d|^4. \qquad (2.8)
\end{aligned}$$

For d^2, we have that for any $\theta\in(0,1)$,

$$\begin{aligned}
\int_{P_{\theta R}(z_1)}|\nabla d^1|^4 &\lesssim \theta^4\int_{P_R(z_1)}|\nabla d^1|^4\\
&\lesssim \theta^4\{\int_{P_R(z_1)}|\nabla d|^4+\int_{P_R(z_1)}|\nabla d^2|^4\}\\
&\lesssim \theta^4\int_{P_R(z_1)}|\nabla d|^4\\
&\quad+(\int_{P_R(z_1)}|u|^4+\int_{P_R(z_1)}|\nabla d|^4)\int_{P_R(z_1)}|\nabla d|^4. \qquad (2.9)
\end{aligned}$$

Combining (2.8) with (2.9) yields

$$\begin{aligned}
\int_{P_{\theta R}(z_1)}|\nabla d|^4 &\le \{C\theta^4+C(\int_{P_R(z_1)}|u|^4+\int_{P_R(z_1)}|\nabla d|^4)\}\int_{P_R(z_1)}|\nabla d|^4\\
&\le (C\theta^4+C\epsilon_0^4)\int_{P_R(z_1)}|\nabla d|^4. \qquad (2.10)
\end{aligned}$$

For any $\alpha\in(0,1)$, first choose $\theta_0=\theta_0(\alpha)\in(0,\frac12)$ such that $2C\theta_0^4\le\theta_0^{4\alpha}$ and then choose $\epsilon_0\le\theta_0$, we have

$$\int_{P_{\theta_0 R}(z_1)}|\nabla d|^4\le\theta_0^{4\alpha}\int_{P_R(z_1)}|\nabla d|^4. \qquad (2.11)$$

Iteration of this inequality yields that for any $0 < r \leq R$,

$$\int_{P_r(z_1)} |\nabla d|^4 \leq (\frac{r}{R})^{4\alpha} \int_{P_R(z_1)} |\nabla d|^4. \tag{2.12}$$

Claim 2.

$$\int_{P_r(z_0)} |d_t|^2 \leq C(\frac{r}{R})^{2\alpha}(\int_{P_R(z_0)} |\nabla d|^4)^{\frac{1}{2}}, \ \forall 0 < r \leq R. \tag{2.13}$$

Let $\phi \in C_0^\infty(B_r(x_0))$ be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{\frac{r}{2}}(x_0)$, $|\nabla \phi| \leq 2r^{-1}$. Multiplying (1.3) by $d_t\phi^2$ and integrating over $B_r(x_0)$ gives

$$\begin{aligned}
&2\int_{B_r(x_0)} |d_t|^2\phi^2 + \frac{d}{dt}\int_{B_r(x_0)} |\nabla d|^2\phi^2 \\
&= -4\int_{B_r(x_0)} \phi d_t \nabla d \cdot \nabla \phi - 2\int_{B_r(x_0)} (u\cdot\nabla) d d_t \phi^2 \\
&\leq \int_{B_r(x_0)} |d_t|^2\phi^2 + Cr^{-2}\int_{B_r(x_0)} |\nabla d|^2 + C\int_{P_r(z_0)} |u|^2|\nabla d|^2.
\end{aligned}$$

Let $s_0 \in (t_0 - r^2, t_0 - \frac{r^2}{4})$ be such that

$$\int_{B_r(x_0)} |\nabla d(\cdot, s_0)|^2 \leq 2r^{-2}\int_{P_r(z_0)} |\nabla d|^2.$$

Then we obtain

$$\begin{aligned}
\int_{P_{\frac{r}{2}}(z_0)} |d_t|^2 &\leq C[r^{-2}\int_{P_r(z_0)} |\nabla d|^2 + \int_{P_r(z_0)} |u|^2|\nabla d|^2] \\
&\leq C(1 + \|u\|^2_{L^4(P_r(z_0))})\|\nabla d\|^2_{L^4(P_r(z_0))} \\
&\leq C(\frac{r}{R})^{2\alpha}\|\nabla d\|^2_{L^4(P_r(z_0))}.
\end{aligned}$$

Claim 1 and Claim 2 imply that

$$r^{-2}\int_{P_{\frac{r}{2}}(z)} (|\nabla d|^2 + r^2|d_t|^2) \leq Cr^{2\alpha}(\int_{P_1} |\nabla d|^4)^{\frac{1}{2}}, \ \forall z \in P_{\frac{1}{2}}, \ 0 < r \leq \frac{1}{4}. \tag{2.14}$$

Hence the parabolic Morrey's decay Lemma (see [4]) implies that $d \in C^\alpha(P_{\frac{1}{2}})$ and (2.2) holds.

Now we proceed to estimate u as follows. Let $u^1 : P_R(z_1) \to \mathbb{R}^2$ solve

$$\begin{aligned}
u^1_t - \triangle u^1 &= 0 \ \text{ in } P_R(z_1) \\
u^1 &= u \ \text{ on } \partial_p P_R(z_1).
\end{aligned} \tag{2.15}$$

Then $u^2 = u - u^1 : P_R(z_1) \to \mathbb{R}^2$ solves

$$\begin{aligned} u^2_t - \triangle u^2 + \nabla P &= -\nabla \cdot [u \otimes (u - u_{z_1,R}) + \nabla d \odot \nabla d] \ \text{ in } \ P_R(z_1) \\ u^2 &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \ \partial_p P_R(z_1) \end{aligned} \tag{2.16}$$

where $\otimes$ denotes the tensor product. Multiply both sides of (2.16) by u^2 and integrate over $B_R(x_1)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{B_R(x_1)} \frac{|u^2|^2}{2} + \int_{B_R(x_1)} |\nabla u^2|^2 \\ &\le \int_{B_R(x_1)} \left[|\nabla d|^2 + |u||u - u_{z_1,R}|\right] |\nabla u^2| + |\nabla P||u^2| \\ &\le 8\int_{B_R(x_1)} \left[|\nabla d|^4 + |u|^2|u - u_{z_1,R}|^2\right] + \frac{1}{2}\int_{B_R(x_1)} |\nabla u^2|^2 + \int_{B_R(x_1)} |u^2||\nabla P|. \end{aligned}$$

Integrating over $[t_1 - R^2, t_1]$ and applying Hölder inequality, we have

$$\begin{aligned} &\sup_{t_1 - R^2 \le t \le t_1} \int_{B_R(x_1)} |u^2|^2 + \int_{P_R(z_1)} |\nabla u^2|^2 \\ &\lesssim \int_{P_R(z_1)} |\nabla d|^4 + (\int_{P_R(z_1)} |u|^4)^{\frac{1}{2}} (\int_{P_R(z_1)} |u - u_{z_1,R}|^4)^{\frac{1}{2}} \\ &+ \|u\|_{L^4(P_R(z_1))} \|\nabla P\|_{L^{\frac{4}{3}}(P_R(z_1))}. \end{aligned} \tag{2.17}$$

Applying Ladyzhenskaya inequality and Hölder inequality, we have

$$\begin{aligned} \int_{P_R(z_1)} |u^2|^4 &\lesssim [\sup_{t \in [t_1 - R^2, t_1]} \int_{B_R(x_1)} |u^2|^2] \int_{P_R(z_1)} |\nabla u^2|^2 \\ &\le C[\int_{P_R(z_1)} |\nabla d|^4]^2 + C\int_{P_R(z_1)} |u - u_{z_1,R}|^4 \cdot \int_{P_R(z_1)} |u|^4 \\ &\quad + \frac{1}{2}\int_{P_R(z_1)} |u^2|^4 + C(\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}})^3. \end{aligned}$$

Thus

$$\int_{P_R(z_1)} |u^2|^4 \lesssim [\int_{P_R(z_1)} |\nabla d|^4]^2 + \int_{P_R(z_1)} |u - u_{z_1,R}|^4 \cdot \int_{P_R(z_1)} |u|^4 + (\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}})^3. \tag{2.18}$$

Plugging (2.17) into (2.18) also gives

$$\int_{P_R(z_1)} |\nabla u^2|^2 \lesssim [\int_{P_R(z_1)} |\nabla d|^4]^2 + \int_{P_R(z_1)} |u - u_{z_1,R}|^4 \cdot \int_{P_R(z_1)} |u|^4 + (\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}})^3. \tag{2.19}$$

For u^1 solves the heat equation, we have that for any $\theta \in (0, \frac{1}{2})$

$$\int_{P_{\theta R}(z_1)} |u^1 - u^1_{z_1,\theta R}|^4 \lesssim \theta^8 \int_{P_R(z_1)} |u^1 - u^1_{z_1,\theta R}|^4 \lesssim \theta^4 \int_{P_R(z_1)} |u - u_{z_1,\theta R}|^4 + \int_{P_R(z_1)} |u^2|^4,$$

and

$$\int_{P_{\theta R}(z_1)} |\nabla u^1|^2 \lesssim \theta^4 \int_{P_R(z_1)} |\nabla u^1|^2 \lesssim \theta^4 \int_{P_R(z_1)} |\nabla u|^2 + \int_{P_R(z_1)} |\nabla u^2|^2.$$

Putting these inequalities together yields

$$\begin{aligned} \int_{P_{\theta R}(z_1)} |u - u_{z_1,\theta R}|^4 &\lesssim (\theta^8 + \int_{P_R(z_1)} |u|^4) \int_{P_R(z_1)} |u - u_{z_1,R}|^4 \\ &+ [\int_{P_R(z_1)} |\nabla d|^4]^2 + (\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}})^3. \end{aligned} \tag{2.20}$$

$$\begin{aligned} \int_{P_{\theta R}(z_1)} |\nabla u|^2 &\lesssim \theta^4 \int_{P_R(z_1)} |\nabla u|^2 + \int_{P_R(z_1)} |u|^4 \int_{P_R(z_1)} |u - u_{z_1,R}|^4 \\ &+ [\int_{P_R(z_1)} |\nabla d|^4]^2 + (\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}})^3. \end{aligned} \tag{2.21}$$

Now we estimate $\|\nabla P\|_{L^{\frac{4}{3}}}$. On $B_R(x_1)$, write P as $P = P^1 + P^2$, where P^1 solves

$$\begin{aligned} \triangle P^1 &= -\nabla \cdot (u \cdot \nabla u + \nabla \cdot (\nabla d \odot \nabla d)) \text{ in } B_R(x_1) \\ P^1 &= 0 \qquad\qquad \text{on } \partial B_R(x_1) \end{aligned}$$

so that P^2 is a harmonic function on $B_R(x_1)$. For P^1, the Calderon-Zygmund theory implies that $\nabla P^1 \in L^{\frac{4}{3}}(P_R(z_1))$ and

$$\begin{aligned} \|\nabla P^1\|_{L^{\frac{4}{3}}(P_{\frac{R}{2}}(z_1))} &\lesssim \|\nabla u\|_{L^2(P_{\frac{3R}{4}}(z_1))} \|u\|_{L^4(P_{\frac{3R}{4}}(z_1))} \\ &+ \|\nabla d\|_{L^4(P_{\frac{3R}{4}}(z_1))} \|\nabla^2 d\|_{L^2(P_{\frac{3R}{4}}(z_1))} \\ &\lesssim \|u\|_{L^4(P_R(z_1))} \|\nabla u\|_{L^2(P_R(z_1))} \\ &+ (\|u\|_{L^4(P_R(z_1))} + \|\nabla d\|_{L^4(P_R(z_1))}) \|\nabla d\|_{L^4(P_R(z_1))}. \end{aligned} \tag{2.22}$$

For P^2, we have that for $\theta \in (0, \frac{1}{4})$,

$$\begin{aligned}
\int_{P_{\theta R}(z_1)} |\nabla P^2|^{\frac{4}{3}} &\lesssim \theta^2 \int_{P_{\frac{R}{2}}(z_1)} |\nabla P^2|^{\frac{4}{3}} \\
&\lesssim \theta^2 \int_{P_{\frac{R}{2}}(z_1)} (|\nabla P|^{\frac{4}{3}} + |\nabla P^1|^{\frac{4}{3}}) \\
&\lesssim \theta^2 \int_{P_{\frac{R}{2}}(z_1)} |\nabla P|^{\frac{4}{3}} + \|u\|^{\frac{4}{3}}_{L^4(P_R(z_1))} \|\nabla u\|^{\frac{4}{3}}_{L^2(P_R(z_1))} \\
&+ (\|u\|^{\frac{4}{3}}_{L^4(P_R(z_1))} + \|\nabla d\|^{\frac{4}{3}}_{L^4(P_R(z_1))}) \|\nabla d\|^{\frac{4}{3}}_{L^4(P_R(z_1))}. \quad (2.23)
\end{aligned}$$

Putting (2.22) and (2.23) together, we obtain

$$\begin{aligned}
&\int_{P_{\theta R}(z_1)} |\nabla P|^{\frac{4}{3}} \\
\lesssim\ & \theta^2 \int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}} + \|u\|^{\frac{4}{3}}_{L^4(P_R(z_1))} \|\nabla u\|^{\frac{4}{3}}_{L^2(P_R(z_1))} \\
+\ & (\|u\|^{\frac{4}{3}}_{L^4(P_R(z_1))} + \|\nabla d\|^{\frac{4}{3}}_{L^4(P_R(z_1))}) \|\nabla d\|^{\frac{4}{3}}_{L^4(P_R(z_1))}. \quad (2.24)
\end{aligned}$$

Define

$$\Phi(u, z_1, R) = \int_{P_R(z_1)} |u - u_{z_1,\theta R}|^4 + \int_{P_R(z_1)} |\nabla u|^2,$$

$$\Psi(u, z_1, R) = \int_{P_R(z_1)} |u|^4, \quad D(P, z_1, R) = [\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}}]^2,$$

$$\Theta(z_1, R) = \Phi(u, z_1, \theta R) + D(P, z_1, R).$$

Putting (2.20) together with (2.21) and applying Claim 1, we have

$$\Phi(u, z_1, \theta^2 R) \lesssim (\theta^8 + \Psi(u, z_1, \theta R))\Phi(u, z_1, \theta R) + (\theta R)^{8\alpha} + D(P, z_1, \theta R),$$

$$D(P, z_1, \theta R) \lesssim \left\{\theta^6 D(P, z_1, R) + \Psi(u, z_1, R)\Phi(u, z_1, R) + (R^{4\alpha} + \Psi(u, z_1, R))R^{4\alpha}\right\}.$$

Adding these two inequalities, we obtain

$$\begin{aligned}
\Theta(z_1, \theta R) &\leq C[\theta^6 + \Psi(u, z_1, \theta R)]\Theta(z_1, R) + C\Psi(u, z_1, R)\Phi(u, z_1, R) \\
&+ C(R^{4\alpha} + \Psi(u, z_1, R))R^{4\alpha} \quad (2.25)
\end{aligned}$$

provide $P_{\theta^{-1}R}(z_1) \subset P_1$. Since $\Psi(u, z_1, \theta R) \leq \Psi(u, z_1, R)$, we have

$$\Theta(z_1, \theta R) \leq C[\theta^6 + \Psi(u, z_1, R)]\Theta(z_1, \theta^{-1} R) + C(R^{4\alpha} + \Psi(u, z_1, R))R^{4\alpha} \quad (2.26)$$

provide $P_{\theta^{-1}R}(z_1) \subset P_1$. Let $\alpha_1 = \frac{1+\alpha}{4}$ and choose $\theta = \theta(\alpha_1) \in (0, \frac{1}{2})$ so that

$$2C\theta^6 \leq (\theta^2)^{2+2\alpha_1}.$$

If $\epsilon_0 \leq \theta^{\frac{3}{2}}$, then $\int_{P_1} |u|^4 + |\nabla d|^4 \leq \theta^6$. Hence, for any $z_1 \in P_{\frac{1}{2}}$ and $0 < R \leq R_0 = \frac{\theta}{4}$, we have

$$\Psi(u, z_1, R) \leq \theta^4.$$

Hence, for $\rho = \frac{R}{\theta}$ and $\tau = \theta^2$, we have

$$\Theta(z_1, \tau\rho) \leq \tau^{2+2\alpha_1}[\Theta(z_1, \rho) + C\rho^{2+2\alpha_1}], \ \forall z_1 \in P_{\frac{1}{2}}, \ 0 < \rho \leq \frac{1}{4}. \tag{2.27}$$

Iterating (2.27) finitely many times, we obtain

$$\Theta(z_1, R) \leq R^{2+2\alpha_1} M_0, \forall R \in P_{\frac{1}{2}}, 0 < R \leq \frac{1}{4} \tag{2.28}$$

where

$$M_0 \equiv C[\int_{P_1} (|u|^4 + |\nabla d|^4) + (\int_{P_1} |\nabla P|^{\frac{4}{3}})^3].$$

Now we want to show the decay estimate for $\Psi(u, z_1, R)$ as follows. By (2.28), we have

$$|u_{2^{-1}\rho} - u_\rho| \leq C\rho^{-1}(\int_{P_\rho(z_0)} |u - u_\rho|^4)^{\frac{1}{4}} \leq \frac{C}{\rho}\Theta^{\frac{1}{4}}(z_1, \rho) \leq CM_0^{\frac{1}{4}}\rho^{\frac{\alpha_1 - 1}{2}}$$

for any $z_1 \in P_{\frac{1}{2}}$ and $0 < \rho \leq \frac{1}{4}$. Hence, for $R_0 = \frac{1}{4}$, we have

$$|u_{2^{-k}R_0} - u_{R_0}| \leq CM_0^{\frac{1}{4}} \sum_{i=0}^{k-1} (\frac{R_0}{2^i})^{\frac{\alpha_1 - 1}{2}} \leq CM_0^{\frac{1}{4}} \left(\frac{R_0}{2^k}\right)^{\frac{\alpha_1 - 1}{2}}, \ \forall k \in \mathbb{N}$$

so that

$$|u_\rho - u_{R_0}| \leq CM_0^{\frac{1}{4}}\rho^{\frac{\alpha_1 - 1}{2}}, \ \forall \rho \in (0, \frac{1}{4}).$$

This implies, for any $0 < \rho \leq R_0 = \frac{1}{4}$,

$$\begin{aligned}
\Psi(u, z_1, \rho) &\lesssim \Phi(u, z_1, \rho) + \rho^4 |u_{z_1,\rho}|^4 \\
&\lesssim \Phi(u, z_1, \rho) + \rho^4 |u_{z_1,\rho} - u_{z_1,R_0}|^4 + \rho^4 |u_{z_1,R_0}|^4 \\
&\lesssim \Phi(u, z_1, \rho) + M_0\rho^4 + M_0\rho^{2+2\alpha_1} \\
&\lesssim M_0\rho^{2+2\alpha_1}.
\end{aligned} \tag{2.29}$$

Substituting (2.29) into (2.25), we arrive

$$\Theta(z_1, \theta R) \le \theta^{4+4\alpha_1}\Theta(z_1, R) + C(1+M_0)R^{4+4\alpha_1},\ \forall z_1 \in P_{\frac{1}{2}},\ 0 < R \le \frac{1}{4}. \tag{2.30}$$

Iterating (2.30) finitely many times yields

$$\Phi(u, z_1, \rho) \le CM_0\rho^{4+4\alpha_1},\ \forall z_1 \in P_{\frac{1}{2}}, 0 < \rho \le \frac{1}{4}.$$

Using the characterization by Campanato spaces, we conclude that $u \in C^{\alpha_1}(P_{\frac{1}{2}})$ and (2.3) holds with $\alpha = \alpha_1$. By repeating the above argument, we can show that $u \in C^{\alpha}(P_{\frac{1}{2}})$ and (2.3) holds for any $\alpha \in (0, 1)$. □

Now we need to establish the boundary regularity Lemma under the smallness condition. To state it, we need some notations. Denote $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. For $x_0 \in \partial\mathbb{R}^2$, $t_0 \in \mathbb{R}$ and $r > 0$, denote $z_0 = (x_0, t_0)$ let $B_r^+(x_0) = B_r(x_0) \cap \mathbb{R}^2$ be the upper half ball, $P_r^+(z_0) = B_r^+(x_0) \times [t_0 - r^2, t_0]$ denote the half parabolic cylinder. Write $\partial B_r^+(x_0) = \Gamma_r(x_0) \cup S_r(x_0)$, where $\Gamma_r(x_0) = \partial B_r^+(x_0) \cap \partial\mathbb{R}^2$ and $S_r(x_0) = \partial B_r^+(x_0) \cap \mathbb{R}^2_+$, and

$$\partial_p P_r^+(z_0) = (B_r^+(x_0) \times \{t_0 - r^2\}) \cup (\partial B_r^+(x_0) \times (t_0 - r^2, t_0))$$

be its parabolic boundary. When $x_0 = 0$ and $t_0 = 0$, we simply denote $B_r^+ = B_r^+(0)$, $P_r^+ = P_r^+(0,0)$, $\partial B_r^+ = \partial B_r^+(0)$, $\Gamma_r = \Gamma_r(0)$, $S_r = S_r(0)$, and $\partial_p P_r^+ = \partial_p P_r^+(0,0)$.

Lemma 2.2 *For any $\alpha \in (0,1)$, there exists ϵ_0 depending on α and $\|d_0\|_{C^{2,\beta}(\Gamma_{2r}^+(x_0))}$ such that if for $z_0 = (z_0, t_0) \in \partial\mathbb{R}^2 \times \mathbb{R}$ and $r > 0$, $(u, d) \in W_2^{1,0}(P_{2r}^+(z_0))$, $P \in W^{1,\frac{4}{3}}(P_{2r}^+(z_0))$ is a weak solution of (1.1)-(1.3), with $(u, d) = (0, d_0)$ on $\Gamma_{2r}^+(x_0) \times [t_0 - r^2, t_0]$ and*

$$\int_{P_r^+(z_0)} (|u|^4 + |\nabla d|^4) \le \epsilon_0^4, \tag{2.31}$$

then $(u, d) \in C^{\alpha}(P_{\frac{r}{2}}^+(z_0))$. Moreover,

$$[d]_{C^{\alpha}(P^+_{\frac{r}{2}}(z_0))} \lesssim \|u\|_{L^4(P_r^+(z_0))} + \|\nabla d\|_{L^4(P_r^+(z_0))} + \|d_0\|_{C^{2,\beta}(\Gamma_r(x_0))} \tag{2.32}$$

$$[u]_{C^{\beta}(P^+_{\frac{r}{2}}(z_0))} \lesssim \|u\|_{L^4(P_r^+(z_0))} + \|\nabla d\|_{L^4(P_r^+(z_0))} + \|d_0\|_{C^{2,\beta}(\Gamma_r(x_0))} + \|\nabla P\|_{L^{\frac{4}{3}}(P_r^+(z_0))}. \tag{2.33}$$

Proof. By scalings, it suffices to consider $z_0 = (0,0)$ and $r = 1$. The argument is similar to Lemma 2.1, except that we have to estimate P differently. Here we only outline it.

For any $z_1 = (x_1, t_1) \in \Gamma_{\frac{1}{2}} \times [-\frac{1}{4}, 0]$ and $0 < R \leq \frac{1}{4}$, let $d^1 : P_R^+(z_1) \to \mathbb{R}^3$ solve

$$\begin{aligned} d^1_t - \triangle d^1 &= 0 \ \text{ in } P_R^+(z_1) \\ d^1 &= d_0 \ \text{ on } \Gamma_R(x_1) \times [t_1 - R^2, t_1] \\ d^1 &= d \ \text{ on } S_R(x_1) \times [t_1 - R^2, t_1] \cup (B_R^+(x_1) \times \{t_1 - R^2\}). \end{aligned} \tag{2.34}$$

Then $d^2 = d - d^1 : P_R^+(z_1) \to \mathbb{R}^3$ solves

$$\begin{aligned} d^2_t - \triangle d^2 &= -u \cdot \nabla d + |\nabla d|^2 d \ \text{ in } P_R(z_1) \\ d^2 &= 0 \qquad \text{on } \partial_p P_R^+(z_1). \end{aligned} \tag{2.35}$$

Since $(-u \cdot \nabla d + |\nabla d|^2 d) \in L^2(P_R(z_1)^+)$, we have that $d^2_t, \nabla^2 d^2 \in L^2(P_R^+(z_1))$. Multiplying (2.35) by $\triangle d^2$ and integrating over $B_R^+(x_1)$, we obtain

$$\frac{d}{dt} \int_{B_R^+(x_1)} \frac{|\nabla d^2|^2}{2} + \int_{B_R^+(x_1)} |\triangle d^2|^2 = \int_{B_R^+(x_1)} (u \cdot \nabla d - |\nabla d|^2 d) \cdot \triangle d^2.$$

Integrating over $[t_1 - R^2, t_1]$ and applying Hölder inequality yields

$$\begin{aligned} &\sup_{t_1 - R^2 \leq t \leq t_1} \int_{B_R^+(x_1)} |\nabla d^2|^2 + \int_{P_R(z_1)^+} |\triangle d^2|^2 \\ &\lesssim \int_{P_R^+(z_1)} |u|^2 |\nabla d|^2 + \int_{P_R^+(z_1)} |\nabla d|^4 \\ &\lesssim [(\int_{P_R^+(z_1)} |u|^4)^{\frac{1}{2}} + (\int_{P_R^+(z_1)} |\nabla d|^4)^{\frac{1}{2}}](\int_{P_R^+(z_1)} |\nabla d|^4)^{\frac{1}{2}}. \end{aligned}$$

This, combined with the Ladyzhenskaya inequality, yields

$$\begin{aligned} \int_{P_R^+(z_1)} |\nabla d^2|^4 &\lesssim (\sup_{t_1 - R^2 \leq t \leq t_1} \int_{B_R^+(x_1)} |\nabla d^2|^2) \\ &\quad \cdot (\int_{P_R^+(z_1)} |\triangle d^2|^2 + \sup_{t_1 - R^2 \leq t \leq t_1} \int_{B_R^+(x_1)} |\nabla d^2|^2) \\ &\lesssim (\int_{P_R^+(z_1)} |u|^4 + \int_{P_R^+(z_1)} |\nabla d|^4) \int_{P_R^+(z_1)} |\nabla d|^4. \end{aligned} \tag{2.36}$$

Since $d^1|_{\Gamma_R(x_1) \times [t_1 - R^2, t_1]} = d_0 \in C^{2,\beta}$, the boundary regularity for the heat equation implies that $d^1 \in C^{2,\beta}(P^+_{\frac{R}{2}}(z_1))$ and for any $\theta \in (0, \frac{1}{2})$,

$$\begin{aligned}
\int_{P^+_{\theta R}(z_1)} |\nabla d^1|^4 &\lesssim (\theta R)^4 \|\nabla d^1\|^4_{C^0(P^+_{\frac{R}{2}}(z_1))} \\
&\lesssim (\theta R)^4 \|d_0\|^4_{C^{2,\beta}(\Gamma_1\times[-1,0])} + \theta^4 [\int_{P^+_R(z_1)} |\nabla d|^4 + \int_{P^+_R(z_1)} |\nabla d^2|^4] \\
&\lesssim \theta^4 [R^4 \|d_0\|^4_{C^{2,\beta}(\Gamma_1\times[-1,0])} + \int_{P^+_R(z_1)} |\nabla d|^4] \\
&+ \int_{P^+_R(z_1)} (|u|^4 + |\nabla d|^4) \int_{P^+_R(z_1)} |\nabla d|^4. \qquad (2.37)
\end{aligned}$$

Combining (2.36) with (2.37) yields that for any $\theta \in (0, \frac{1}{2})$,

$$\begin{aligned}
\int_{P^+_{\theta R}(z_1)} |\nabla d|^4 &\leq C_0(\theta R)^4 + C[\theta^4 + \int_{P^+_R(z_1)} (|u|^4 + |\nabla d|^4)] \int_{P^+_R(z_1)} |\nabla d|^4 \\
&\leq C_0(\theta R)^4 + C(\theta^4 + C\epsilon_0^4) \int_{P^+_R(z_1)} |\nabla d|^4, \qquad (2.38)
\end{aligned}$$

where $C_0 \lesssim \|d_0\|^4_{C^{2,\beta}(\Gamma_1\times[-1,0])}$.

For any $\alpha \in (0,1)$, choose $\theta_0 = \theta_0(\alpha) \in (0, \frac{1}{2})$ such that $2C\theta_0^4 \leq \theta_0^{4\alpha}$ and $\epsilon_0 \leq \theta_0$, we obtain

$$\int_{P^+_{\theta_0 R}(z_1)} |\nabla d|^4 \leq C_0(\theta_0 R)^4 + \theta_0^{4\alpha} \int_{P^+_R(z_1)} |\nabla d|^4. \qquad (2.39)$$

Iteration of this inequality yields that for any $0 < r \leq R$,

$$\int_{P^+_r(z_1)} |\nabla d|^4 \leq C_0 r^4 + (\frac{r}{R})^{4\alpha} \int_{P^+_R(z_1)} |\nabla d|^4. \qquad (2.40)$$

The estimate of $\int_{P^+_r(z_1)} |d_t|^2$ is similar to Lemma 2.1. Let $\phi \in C_0^\infty(B_r(x_1))$ be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{\frac{r}{2}}(x_1)$, $|\nabla \phi| \leq 2r^{-1}$. Since $d_t = 0$ on $\Gamma_r(x_1) \times [t_1 - r^2, t_1]$, multiplying (1.3) by $d_t\phi^2$ and integrating over $B^+_r(x_1)$ gives

$$\int_{B^+_r(x_1)} |d_t|^2\phi^2 + \frac{d}{dt}\int_{B^+_r(x_1)} |\nabla d|^2\phi^2 \lesssim r^{-2}\int_{B^+_r(x_1)} |\nabla d|^2 + \int_{B^+_r(z_1)} |u|^2|\nabla d|^2.$$

Integrate t from s_0 to t_0, where $s_0 \in (t_0 - r^2, t_0 - \frac{r^2}{4})$ is such that

$$\int_{B^+_r(x_1)} |\nabla d(\cdot, s_0)|^2 \leq 2r^{-2} \int_{P^+_r(z_1)} |\nabla d|^2,$$

we obtain

$$\begin{aligned}
\int_{P^+_{\frac{r}{2}}(z_1)} |d_t|^2 &\leq C[r^{-2}\int_{P^+_r(z_1)} |\nabla d|^2 + \int_{P^+_r(z_1)} |u|^2|\nabla d|^2] \\
&\leq C[r^2 + (\frac{r}{R})^{2\alpha} \|\nabla d\|^2_{L^4(P^+_R(z_1))}]. \qquad (2.41)
\end{aligned}$$

Hence we have

$$r^{-2}\int_{P_r^+(z_1)}(|\nabla d|^2+r^2|d_t|^2)\leq C[r^2+r^{2\alpha}(\int_{P_1^+}|\nabla d|^4)^{\frac{1}{2}}] \tag{2.42}$$

for any $z_1\in\Gamma_{\frac{1}{2}}^+\times[-\frac{1}{4},0]$ and $0<r\leq\frac{1}{4}$. Combining this with Lemma 2.1 and the parabolic Morrey's decay Lemma ([4]), we conclude that $d\in C^\alpha(P_{\frac{1}{2}}^+)$ and (2.32) holds.

To estimate u and P, let $u^1:\mathbb{R}_+^2\times[-1,0]\to\mathbb{R}^2$ and $P^1:\mathbb{R}_+^2\times[-1,0]\to\mathbb{R}$ solve the non homogeneous, non-stationary Stokes equation:

$$\begin{aligned} u_t^1-\triangle u^1+\nabla P^1 &= -[u\cdot\nabla u+\nabla\cdot(\nabla d\odot\nabla d)]\lambda_{P_R^+(z_1)}, \ \mathbb{R}_+^2\times(-1,0) \\ u^1 &= 0, \qquad (\partial\mathbb{R}_+^2\times[-1,0])\cup(\mathbb{R}_+^2\times\{-1\}), \end{aligned} \tag{2.43}$$

where $\lambda_{P_R^+(z_1)}$ is the characteristic function of $P_R^+(z_1)$.

Set $u^2=u-u^1:P_R^+(z_1)\to\mathbb{R}^2$ and $P^2=P-P^1:P_R^+(z_1)\to\mathbb{R}$. Then (u^2,P^2) solves

$$\begin{aligned} u_t^2-\triangle u^2+\nabla P^2 &= 0, \quad P_R^+(z_1) \\ u^2 &= 0, \quad \Gamma_R^+(x_1)\times[t_1-R^2,t_1]. \end{aligned} \tag{2.44}$$

By the boundary $W_q^{2,1}$-estimate of non-stationary Stokes equations (see [17] Lemma 3.1), we conclude that $u^1\in W_{\frac{4}{3}}^{2,1}(\mathbb{R}_+^2\times[-1,0])$ and $P^1\in W_{\frac{4}{3}}^{1,0}(\mathbb{R}_+^2\times[-1,0])$, and

$$\begin{aligned} &\|u^1\|_{W_{\frac{4}{3}}^{2,1}(\mathbb{R}_+^2\times[-1,0])}+\|\nabla P^1\|_{L^{\frac{4}{3}}(\mathbb{R}_+^2\times[-1,0])} \\ \lesssim\ &\big\||u||\nabla u|+|\nabla u||\nabla^2 u|\big\|_{L^{\frac{4}{3}}(P_R^+1(z_1))} \\ \lesssim\ &\|u\|_{L^4(P_R^+(z_1))}\|\nabla u\|_{L^2(P_R^+(z_1))} \\ +\ &\|\nabla^2 d\|_{L^2(P_R^+(z_1))}\|\nabla d\|_{L^4(P_R^+(z_1))} \end{aligned} \tag{2.45}$$

By Sobolev embedding theorem (see [10]), $W_{\frac{4}{3}}^{2,1}(\mathbb{R}_+^2\times[-1,0])\subseteq W_2^{1,0}(\mathbb{R}_+^2\times[-1,0])\cap L^4(\mathbb{R}_+^2\times[-1,0])$, (2.45) yields

$$\begin{aligned} \|\nabla u^1\|_{L^2(\mathbb{R}_+^2\times[-1,0])}+\|u^1\|_{L^4(\mathbb{R}_+^2\times[-1,0])} &\lesssim \|u\|_{L^4(P_R^+(z_1))}\|\nabla u\|_{L^2(P_R^+(z_1))} \\ &+ \|\nabla^2 d\|_{L^2(P_R^+(z_1))}\|\nabla d\|_{L^4(P_R^+(z_1))} \end{aligned} \tag{2.46}$$

For (u^2,P^2), by the boundary $W^{2,1}_{p,q}$-estimate for homogeneous, non-stationary Stokes equation (see [17] Lemma 3.2), we have $u^2\in W^{2,1}_{q,\frac43}P^+_{\frac R2}(z_1)), \nabla P^2\in L^{q,\frac43}(P^+_{\frac R2}(z_1))$ for any $4<q<+\infty$ and

$$
\begin{aligned}
& R^{\frac32-\frac2q}[[u^2]_{W^{2,1}_{q,\frac43}(P^+_{\frac R2}(z_1))}+\|\nabla P^2\|_{L^{q,\frac43}(P^+_{\frac R2}(z_1))}]\\
\lesssim\ & \|\nabla u^2\|_{L^2(P^+_R(z_1))}+\|\nabla P^2\|_{L^{\frac43}(P^+_R(z_1))}\\
\lesssim\ & (\|\nabla u\|_{L^2(P^+_R(z_1))}+\|\nabla P\|_{L^{\frac43}(P^+_R(z_1))})+(\|\nabla u^1\|_{L^2(P^+_R(z_1))}+\|\nabla P^1\|_{L^{\frac43}(P^+_R(z_1))})\\
\lesssim\ & (\|\nabla u\|_{L^2(P^+_R(z_1))}+\|\nabla P\|_{L^{\frac43}(P^+_R(z_1))})+\|u\|_{L^4(P^+_R(z_1))}\|\nabla u\|_{L^2(P^+_R(z_1))}\\
+\ & \|\nabla^2 d\|_{L^2(P^+_R(z_1))}\|\nabla d\|_{L^4(P^+_R(z_1))}.
\end{aligned}
$$

By Sobolev inequality and Hölder inequality, we have that for any $\theta\in(0,\frac14)$,

$$
\begin{aligned}
& \|u^2-u^2_{z_1,\theta R}\|_{L^4(P^+_{\theta R}(z_1))}+\|\nabla P^2\|_{L^{\frac43}(P^+_{\theta R}(z_1))}\\
\le\ & \theta^{\frac32-\frac2q}[R^{\frac32-\frac2q}([\nabla u^2]_{W^{2,1}_{q,\frac43}(P^+_{\frac R2}(z_1))}+\|\nabla P^2\|_{L^{q,\frac43}(P^+_{\frac R2}(z_1))})]\\
\lesssim\ & \theta^{\frac32-\frac2q}[\|\nabla u\|_{L^2(P^+_R(z_1))}+\|\nabla P\|_{L^{\frac43}(P^+_R(z_1))}] \qquad (2.47)\\
+\ & \theta^{\frac32-\frac2q}[\|u\|_{L^4(P^+_R(z_1))}\|\nabla u\|_{L^2(P^+_R(z_1))}+\|\nabla^2 d\|_{L^2(P^+_R(z_1))}\|\nabla d\|_{L^4(P^+_R(z_1))}].
\end{aligned}
$$

To estimate $\|\nabla u^2\|_{L^2}$, let $\phi\in C^1_0(B_{\theta R}(x_1))$ be such that $0\le\phi\le1$, $\phi\equiv1$ on $B_{\frac{\theta R}{2}}(x_1)$, and $|\nabla\phi|\le\frac{2}{\theta R}$. Multiplying the equation of u^2 by $(u^2-u^2_{z_1,\theta R})\phi^2$ and integrating over $B^+_{\theta R}(x_1)$, we obtain

$$
\begin{aligned}
& \frac{d}{dt}\int_{B^+_{\theta R}(x_1)}|u^2-u^2_{z_1,\theta R}|^2\phi^2+\int_{B^+_{\theta R}(x_1)}|\nabla u^2|\phi^2\\
\lesssim\ & (\theta R)^{-2}\int_{B^+_{\theta R}(x_1)}|u^2-u^2_{z_1,\theta R}|^2+\int_{B^+_{\theta R}(x_1)}|\nabla P^2||u^2-u^2_{z_1,\theta R}|.
\end{aligned}
$$

Integrating over $[s_0,t_1]$, where $s_0\in[t_1-(\theta R)^2,t_1-\frac{(\theta R)^2}{2}]$ is such that

$$
\int_{B^+_{\theta R}(x_1)\times\{s_0\}}|u^2-u^2_{z_1,\theta R}|^2\lesssim(\theta R)^{-2}\int_{P^+_{\theta R}(z_1)}|u^2-u^2_{z_1,\theta R}|^2,
$$

we obtain

$$
\begin{aligned}
& \int_{P^+_{\frac{\theta R}{2}}(z_1)} |\nabla u^2|^2 \\
\lesssim\ & (\theta R)^{-2} \int_{P^+_{\theta R}(z_1)} |u^2 - u^2_{z_1,\theta R}|^2 + \int_{P^+_{\theta R}(z_1)} |\nabla P^2||u^2 - u^2_{z_1,\theta R}| \\
\lesssim\ & \|u^2 - u^2_{z_1,\theta R}\|^2_{L^4(P^+_{\theta R}(z_1))} + \|\nabla P^2\|_{L^{\frac{4}{3}}(P^+_{\theta R}(z_1))} \|u^2 - u^2_{z_1,\theta R}\|_{L^4(P^+_{\theta R}(z_1))} \\
\lesssim\ & \|u^2 - u^2_{z_1,\theta R}\|^2_{L^4(P^+_{\theta R}(z_1))} + \|\nabla P^2\|^2_{L^{\frac{4}{3}}(P^+_{\theta R}(z_1))}. \qquad (2.48)
\end{aligned}
$$

Putting together (2.45), (2.46), (2.47), (2.48), applying (2.39), and setting $q = 8$, we arrive

$$
\begin{aligned}
& \|u - u_{z_1,\theta R}\|_{L^4(P^+_{\theta R}(z_1))} + \|\nabla u\|_{L^2(P^+_{\theta R}(z_1))} + \|\nabla P\|_{L^{\frac{4}{3}}(P^+_{\theta R}(z_1))} \\
\lesssim\ & [\theta^{\frac{5}{4}} + \|u\|_{L^4(P^+_R(z_1))}][\|u - u_{z_1,R}\|_{L^4(P^+_R(z_1))} + \|\nabla u\|_{L^2(P^+_R(z_1))} + \|\nabla P\|_{L^{\frac{4}{3}}(P^+_R(z_1))}] \\
+\ & \|\nabla^2 d\|_{L^2(P^+_{\frac{R}{2}}(z_1))} \|\nabla d\|_{L^4(P^+_R(z_1))}. \qquad (2.49)
\end{aligned}
$$

For $\|\nabla^2 d\|_{L^2(P^+_{\frac{R}{2}}(z_1))}$, we have

$$
\|\nabla^2 d\|_{L^2(P^+_{\frac{R}{2}}(z_1))} \lesssim \|u\|_{L^4(P^+_R(z_1))} + \|\nabla d\|_{L^4(P^+_R(z_1))}.
$$

Thus, by choosing $\theta = \theta_0$ sufficiently small and $\epsilon_0 \le \theta_0$, we obtain

$$
\Theta^+(z_1, \theta R) \le \theta \Theta^+(z_1, R) + C_0(\theta + R^\alpha) R^\alpha, \qquad (2.50)
$$

where C_0 depends on $\|d_0\|_{C^{2,\beta}(\Gamma_1)}$, and

$$
\Theta^+(z_1, r) \equiv \|u - u_{z_1,r}\|_{L^4(P^+_r(z_1))} + \|\nabla u\|_{L^2(P^+_r(z_1))} + \|\nabla P\|_{L^{\frac{4}{3}}(P^+_r(z_1))}, 0 < r \le \frac{1}{4}.
$$

The same argument as in Lemma 2.1 can imply

$$
r^{-4} \int_{P^+_r(z_1)} |u - u_{z_1,r}|^4 \le C_1(\Theta^+(z_1, \frac{1}{2}))^4 r^{4\alpha}, \ \forall 0 < r \le \frac{1}{4}.
$$

This, combined with Lemma 2.1, implies that $u \in C^\alpha(P^+_{\frac{1}{2}})$ and (2.33) holds. □

Proof of Theorem 1.2: Since $u \in L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H^1(\Omega))$, it follows from the Ladyzhenskaya's inequality that $u \in L^4(\Omega \times [0,T])$. Since $\nabla d \in L^2([0,T], H^1(\Omega))$ and $|d| = 1$, we have

$$
d \triangle d + |\nabla d|^2 = 0.
$$

Hence $|\nabla d| \in L^4(\Omega \times [0,T])$ and $u \cdot \nabla u + \nabla \cdot (\nabla d \odot \nabla d) \in L^{\frac{4}{3}}(\Omega \times [0,T])$. Hence Lemma 4.4 implies that $\nabla P \in L^{\frac{4}{3}}(\Omega \times [0,T])$. It follows from the absolute continuity of $\int(|u|^4 + |\nabla d|^4)$ that

(i) for any $z_0 = (x_0, t_0) \in \Omega \times (0,T]$, there exists $0 < r_0 \le \min\{\mathrm{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$ such that

$$\int_{P_{r_0}(z_0)} (|u|^4 + |\nabla d|^4) \le \epsilon_0^4,$$

where $\epsilon_0 > 0$ is given by Lemma 2.1. Hence we conclude $(u,d) \in C^\alpha(P_{\frac{r_0}{2}}(z_0))$ for any $\alpha \in (0,1)$.

(ii) for any $z_0 = (x_0, t_0) \in \partial\Omega \times (0,T]$, since $\partial\Omega$ is smooth, it is well known that there exists $r_0 > 0$ depending only on $\partial\Omega$ such that $(\Omega \cap B_{r_0}(x_0)) \times [t_0 - r_0^2, t_0]$ is C^3-close to $P_{r_0}^+$ and

$$\int_{(\Omega \cap B_{r_0}(x_0)) \times [t_0 - r_0^2, t_0]} |u|^4 + |\nabla d|^4) \le \epsilon_0^4,$$

where $\epsilon_0 > 0$ is given by Lemma 2.2. Hence, we can perform the standard boundary flatten technique, which is a small perturbation of the one on $P_{r_0}^+$, so that a slight modification of Lemma 2.2 implies that $(u,d) \in C^\alpha((\Omega \cap B_{\frac{r_0}{2}}(x_0)) \times [t_0 - \frac{r_0^2}{4}, t_0]$. The reader can consult with [17] for such details.

The higher order regularity can be obtained as follows. One can follow the standard hole filling argument for (1.3) of d to show that $\nabla d \in C^\alpha(\Omega \times (0,T])$ and then apply the Schauder theory to show $d \in C^{2,1}_\alpha(\Omega \times (0,T])$. Substitute this regularity of d into (1.1)-(1.2), we can apply the standard $C^{2,1}_\alpha$-regularity theory (see [19]) to show $u \in C^{2,1}_\alpha(\Omega \times (0,T])$. Once we obtain $C^{2,1}_\alpha$-regularity for (u,d), the smoothness of (u,d) on $\Omega \times (0,T]$ can be obtained by the standard boot-strap argument. Similarly, the boundary $C^{2,1}_\beta$-regularity for (u,d) can be obtained. □

3 Existence of short time smooth solutions

In this section, we prove the existence of short time smooth solutions to (1.1)-(1.3) for smooth initial and boundary data. We would like to point out that the proof also works for $\Omega \subset \mathbb{R}^3$. More precisely, we have

Theorem 3.1 *For any $\alpha > 0$, if $u_0 \in C_0^{2,\alpha}(\Omega, \mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$, and $d_0 \in C^{2,\alpha}(\overline{\Omega}, S^2)$, then there exists $T > 0$ depending on $\|u_0\|_{C^{2,\alpha}(\Omega)}$, $\|d_0\|_{C^{2,\alpha}(\Omega)}$ such that there is a unique smooth solution $(u, d) \in C_\alpha^{2,1}(\overline{\Omega} \times [0, T), \mathbb{R}^2 \times S^2)$ to the initial-boundary value problem (1.1), (1.2), (1.3), (1.4), and (1.5).*

Proof. The proof is based on the contraction mapping principle. For $T > 0$ and $K > 0$ to be chosen later, denote $\Omega_T = \Omega \times [0, T]$,

$$\begin{aligned} X &= \{(v, f) \in C_\alpha^{2,1}(\overline{\Omega}_T, \mathbb{R}^2 \times \mathbb{R}^3) \mid \nabla \cdot v = 0, \ (v, f)\big|_{t=0} = (u_0, d_0), \\ &\quad \|v - u_0\|_{C_\alpha^{2,1}(\Omega_T)} + \|f - d_0\|_{C_\alpha^{2,1}(\Omega_T)} \leq K\}. \end{aligned}$$

Equip X with the norm

$$\|(v, f)\|_X := \|v\|_{C_\alpha^{2,1}(\Omega_T)} + \|f\|_{C_\alpha^{2,1}(\Omega_T)}, \ (v, f) \in X.$$

It is easy to see that $(X, \|\cdot\|_X)$ is a Banach space. Define the operator

$$L : X \to C_\alpha^{2,1}(\overline{Q}_T, \mathbb{R}^2 \times \mathbb{R}^3)$$

as follows. For any $(v, f) \in X$, let $(u, d) = L(v, f)$ be the unique solution to the non homogeneous, non-stationary Stokes system:

$$\begin{aligned} u_t - \triangle u + \nabla P &= -v \cdot \nabla v - \nabla \cdot (\nabla d \odot \nabla d), \quad \Omega \times (0, T), && (3.1) \\ \nabla \cdot u &= 0, \quad \Omega \times (0, T) && (3.2) \\ d_t - \triangle d &= |\nabla f|^2 f - v \cdot \nabla f, \quad \Omega \times (0, T) && (3.3) \\ (u, d) &= (u_0, d_0), \quad \Omega \times \{0\} && (3.4) \\ (u, t) &= (0, d_0), \quad \partial\Omega. && (3.5) \end{aligned}$$

We will prove that for $T > 0$ sufficiently small and $K > 0$ sufficiently large, $L : X \to X$ is a contraction map.

Lemma 3.1 *There exist $T > 0$ and $K > 0$ such that $L : X \to X$.*

Proof. For any $(v, f) \in X$, let $(u, d) = L(v, f)$ be the unique solution to (3.1)-(3.5). Let $C_0 > 0$ denote constants depending only on $\|u_0\|_{C^{2,\alpha}}$ and $\|d_0\|_{C^{2,\alpha}}$.

Assume $K \geq C_0$. By the Schauder theory of parabolic systems, we have

$$\|d - d_0\|_{C^{2,1}_\alpha(\Omega_T)} \lesssim \|v \cdot \nabla f\|_{C^\alpha(\Omega_T)} + \||\nabla f|^2 f\|_{C^\alpha(\Omega_T)}. \tag{3.6}$$

For the first term in the right hand side, we have

$$\begin{aligned}
& \|v \cdot \nabla f\|_{C^\alpha(\Omega_T)} \\
\leq \;& \|v \cdot \nabla f - u_0 \cdot \nabla d_0\|_{C^\alpha(\Omega_T)} + \|u_0 \cdot \nabla d_0\|_{C^\alpha(\Omega)} \\
\leq \;& \|(v - u_0) \cdot \nabla f\|_{C^\alpha(\Omega_T)} + \|u_0 \cdot \nabla(f - d_0)\|_{C^\alpha(\Omega_T)} + C_0 \\
\leq \;& 2K[\|v - u_0\|_{C^0(\Omega_T)} + \|v - u_0\|_{C^\alpha(\Omega_T)}] \\
+ \;& C_0[1 + \|\nabla(f - d_0)\|_{C^0(\Omega_T)} + \|\nabla(f - d_0)\|_{C^\alpha(\Omega_T)}].
\end{aligned}$$

Since $v - u_0 = f - d_0 = 0$ at $t = 0$, it is easy to see

$$\|v - u_0\|_{C^0(\Omega_T)} \leq KT, \; \|\nabla(f - d_0)\|_{C^0(\Omega_T)} \leq KT.$$

By the interpolation inequality, we have that for any $0 < \delta < 1$,

$$\|v - u_0\|_{C^\alpha(\Omega_T)} \lesssim \frac{1}{\delta}\|v - u_0\|_{C^0(\Omega_T)} + \delta\|v - u_0\|_{C^{2,1}_\alpha(\Omega_T)} \lesssim (\delta + \frac{T}{\delta})K,$$

and

$$\|\nabla(d - f)\|_{C^\alpha(\Omega_T)} \lesssim \frac{1}{\delta}\|\nabla(d - f)\|_{C^0(\Omega_T)} + \delta\|d - f\|_{C^{2,1}_\alpha(\Omega_T)} \lesssim (\delta + \frac{T}{\delta})K.$$

Putting these inequalities together, we obtain

$$\|v \cdot \nabla f\|_{C^\alpha(\Omega_T)} \leq (C_0K + CK^2)(T + \delta + \frac{T}{\delta}) + C_0 \leq \frac{\sqrt{K}}{4} \tag{3.7}$$

provided $K = 16C_0^2$, $\delta \lesssim \frac{1}{(C_0 + C^2K)\sqrt{K}}$, and $T = \delta^2$.

The second term in the right hand side of (3.6) can be estimated by

$$\begin{aligned}
\||\nabla f|^2 f\|_{C^\alpha(\Omega_T)} \;&\leq\; \big\||\nabla f|^2 f - |\nabla d_0|^2 d_0\big\|_{C^\alpha(\Omega_T)} + \big\|\nabla d_0|^2 d_0\big\|_{C^\alpha(\Omega_T)} \\
&\leq\; \big\||\nabla f|^2 (f - d_0)\big\|_{C^\alpha(\Omega_T)} + \big\|d_0(|\nabla f|^2 - |\nabla d_0|^2)\big\|_{C^\alpha(\Omega_T)} + C_0 \\
&=\; I_1 + I_2 + C_0.
\end{aligned}$$

$$\begin{aligned}
I_1 \;&\leq\; \|f - d_0\|_{C^\alpha(\Omega_T)}\|\nabla f\|^2_{C^0(\Omega_T)} + \|f - d_0\|_{C^0(\Omega_T)}\|\nabla f\|^2_{C^\alpha(\Omega_T)} \\
&\lesssim\; K^2(\|f - d_0\|_{C^0(\Omega_T)} + \|f - d_0\|_{C^\alpha(\Omega_T)}) \\
&\lesssim\; K^2[(1 + \frac{1}{\delta})\|f - d_0\|_{C^0(\Omega_T)} + \delta\|f - d_0\|_{C^{2,1}_\alpha(\Omega_T)}] \\
&\lesssim\; K^3(\frac{T}{\delta} + \delta).
\end{aligned}$$

Similarly, I_2 can be estimated by

$$\begin{aligned}
I_2 &\leq \left\||\nabla f|^2 - |\nabla d_0|^2\right\|_{C^\alpha(\Omega_T)} + \left\||\nabla f|^2 - |\nabla d_0|^2\right\|_{C^0(\Omega_T)} \|d_0\|_{C^\alpha(\Omega_T)} \\
&\leq \|(|\nabla f| + |\nabla d_0|)|\nabla(f - d_0)|\|_{C^\alpha(\Omega_T)} \\
&+ C_0 \|(|\nabla f| + |\nabla d_0|)|\nabla(f - d_0)|\|_{C^0(\Omega_T)} \\
&\lesssim (1 + C_0)K \left(\|\nabla(f - d_0)\|_{C^0(\Omega_T)} + \|\nabla(f - d_0)\|_{C^\alpha(\Omega_T)}\right) \\
&\lesssim (1 + C_0)K^2(T + \delta + \frac{T}{\delta}).
\end{aligned}$$

Hence

$$\left\||\nabla f|^2 f - |\nabla d_0|^2 d_0\right\|_{C^\alpha(\Omega_T)} \lesssim K^3(\frac{T}{\delta} + \delta) + (1 + C_0)K^2(T + \delta + \frac{T}{\delta})$$

provided $K = 16C_0^2$, $\delta \lesssim \frac{1}{(1+C_0)K^{\frac{5}{2}}}$, and $T = \delta^2$. Thus

$$\|d - d_0\|_{C_\alpha^{2,1}(\Omega_T)} \leq \frac{\sqrt{K}}{2}. \tag{3.8}$$

By the Schauder theory for non homogeneous, non-stationary Stokes equations [19], we have

$$\|u - u_0\|_{C_\alpha^{2,1}(\Omega_T)} \lesssim \|v \cdot \nabla v\|_{C^\alpha(\Omega_T)} + \|\nabla \cdot (\nabla d \odot \nabla d)\|_{C^\alpha(\Omega_T)}. \tag{3.9}$$

For the first term of the right hand side of (3.9), we have

$$\begin{aligned}
&\|v \cdot \nabla v\|_{C^\alpha(\Omega_T)} \\
\leq\ & \|(v - u_0) \cdot \nabla v\|_{C^\alpha(\Omega_T)} + \|u_0 \cdot \nabla(v_0 - u_0)\|_{C^\alpha(\Omega_T)} + \|u_0 \cdot \nabla u_0\|_{C^\alpha(\Omega)} \\
\leq\ & K(\|v - u_0\|_{C^0(\Omega_T)} + \|v - u_0\|_{C^\alpha(\Omega_T)}) \\
+\ & C_0(\|\nabla(v - u_0)\|_{C^0(\Omega_T)} + \|\nabla(v - u_0)\|_{C^\alpha(\Omega_T)}) + C_0 \\
\lesssim\ & (C_0K + K^2)(T + \delta + \frac{T}{\delta}) + C_0 \leq \frac{K}{4}
\end{aligned}$$

provided $K = 8C_0$, $\delta \lesssim \frac{1}{(1+C_0)K}$, and $T = \delta^2$.

For the second term in the right hand side of (3.9), it follows from (3.8) that

$$
\begin{aligned}
& \|\nabla\cdot(\nabla d\odot\nabla d)\|_{C^\alpha(\Omega_T)} \\
\le\ & \big\||\nabla^2(d-d_0)||\nabla d|\big\|_{C^\alpha(\Omega_T)}+\big\||\nabla^2 d_0||\nabla(d-d_0)|\big\|_{C^\alpha(\Omega_T)}+\big\||\nabla^2 d_0||\nabla d_0|\big\|_{C^\alpha(\Omega_T)} \\
\le\ & C_0+\|d-d_0\|_{C^{2,1}_\alpha(\Omega_T)}\|d\|_{C^{2,1}_\alpha(\Omega_T)}+C_0\|d-d_0\|_{C^{2,1}_\alpha(\Omega_T)} \\
\le\ & C_0+\frac{\sqrt{K}}{2}(C_0+\frac{\sqrt{K}}{2})+C_0\sqrt{K} \\
\le\ & \frac{K}{2}. \qquad (3.10)
\end{aligned}
$$

Combining (3.8) with (3.10), we have

$$\|u-u_0\|_{C^{2,1}_\alpha(\Omega_T)}+\|d-d_0\|_{C^{2,1}_\alpha(\Omega_T)}\le K.$$

Therefore L maps X to X.

Lemma 3.2 *There exist sufficiently large $K>0$ and sufficiently small $T>0$ such that $L:X\to X$ is a contraction map.*

Proof. For any $(v_i,f_i)\in X$, let $(u_i,d_i)\in X$ be such that $(u_i,d_i)=L(v_i,f_i)$, $i=1,2$. Denote

$$u=u_1-u_2,\ d=d_1-d_2,\ P=P_1-P_2,\ v=v_1-v_2,\ f=f_1-f_2.$$

Then (u,d) solves

$$
\begin{aligned}
u_t-\triangle u+\nabla P &= G,\ \Omega\times(0,T) && (3.11)\\
\nabla\cdot u &= 0,\ \ \Omega\times(0,T) && (3.12)\\
d_t-\triangle d &= H,\ \ \Omega\times(0,T) && (3.13)\\
(u,d)\big|_{t=0} &= (0,0) && (3.14)\\
(u,d) &= 0,\ \ \partial\Omega\times(0,T) && (3.15)
\end{aligned}
$$

where

$$
\begin{aligned}
G &= -(v_1\cdot\nabla v_1-v_2\cdot\nabla v_2)-\nabla\cdot(\nabla d_1\odot\nabla d_1-\nabla d_2\odot\nabla d_2)\\
&= -(v\cdot\nabla v_1+v_2\cdot\nabla v)-\nabla\cdot(\nabla d\odot\nabla d_1+\nabla d_2\odot\nabla d),
\end{aligned}
$$

and

$$
\begin{aligned}
H &= (|\nabla f_1|^2 f_1 - |\nabla f_2|^2 f_2) - (v_1 \cdot \nabla f_1 - v_2 \cdot \nabla f_2) \\
&= |\nabla f_1|^2 f + [\nabla (f_1 + f_2)] \cdot \nabla f f_2 - (v \cdot \nabla f_1 - v_2 \cdot \nabla f).
\end{aligned}
$$

By Lemma 3.1, we have that for $i = 1, 2$,

$$
\|u_i - u_0\|_{C^{2,1}_\alpha(\Omega_T)} + \|d_i - d_0\|_{C^{2,1}_\alpha(\Omega_T)} \le K.
$$

Applying the Schauder theory of parabolic systems, we have

$$
\begin{aligned}
\|d\|_{C^{2,1}_\alpha(\Omega_T)} &\lesssim \|H\|_{C^\alpha(\Omega_T)} \\
&\lesssim \big\| |v \cdot \nabla f_1| + |v_2 \cdot \nabla f| + |\nabla f_1|^2 |f| + |f_2| (|\nabla f_1| + |\nabla f_2|) |\nabla f| \big\|_{C^\alpha(\Omega_T))} \\
&\lesssim K^2 (\|v\|_{C^\alpha(\Omega_T)} + \|f\|_{C^\alpha(\Omega_T)} + \|\nabla f\|_{C^\alpha(\Omega_T)}) \\
&\lesssim K^2 [\delta (\|v\|_{C^{2,1}_\alpha(\Omega_T)} + \|v\|_{C^{2,1}_\alpha(\Omega_T)}) + \frac{1}{\delta} (\|v\|_{C^\alpha(\Omega_T)} + \|f\|_{C^\alpha(\Omega_T)})] \\
&\lesssim K^2 (\delta + \frac{T}{\delta}) [\|v\|_{C^{2,1}_\alpha(\Omega_T)} + \|f\|_{C^{2,1}_\alpha(\Omega_T)}], \qquad (3.16)
\end{aligned}
$$

where we have used

$$
\|v\|_{C^\alpha(\Omega_T)} + \|f\|_{C^\alpha(\Omega_T)} \lesssim (\|v\|_{C^{2,1}_\alpha(\Omega_T)} + \|f\|_{C^{2,1}_\alpha(\Omega_T)}) T. \qquad (3.17)
$$

Applying the Schauder theory for non homogeneous, non-stationary Stokes equations ([19]) to (3.11)-(3.12), we have

$$
\begin{aligned}
\|u\|_{C^{2,1}_\alpha(\Omega_T)} &\lesssim \|G\|_{C^\alpha(\Omega_T)} \\
&\lesssim \big\| |v| |\nabla v_1| + |v_2| |\nabla v| + |\nabla^2 d| |\nabla d_1| + |\nabla^2 d_2| |\nabla d| \big\|_{C^\alpha(\Omega_T)} \\
&\lesssim K \|d\|_{C^{2,1}_\alpha(\Omega_T)} + K (\|v\|_{C^\alpha(\Omega)} + \|\nabla v\|_{C^\alpha(\Omega_T)}) \\
&\lesssim K^3 (\delta + \frac{T}{\delta}) [\|v\|_{C^{2,1}_\alpha(\Omega_T)} + \|f\|_{C^{2,1}_\alpha(\Omega_T)}] + K^2 (\delta + \frac{T}{\delta}) \|v\|_{C^{2,1}_\alpha(\Omega_T)} \\
&\lesssim K^3 (\delta + \frac{T}{\delta}) [\|v\|_{C^{2,1}_\alpha(\Omega_T)} + \|f\|_{C^{2,1}_\alpha(\Omega_T)}]. \qquad (3.18)
\end{aligned}
$$

It follows from (3.16) and (3.18) that

$$
\begin{aligned}
\|L(v_1, f_1) - L(v_2, f_2)\|_X &\lesssim K^3 (\delta + \frac{T}{\delta}) \|(v_1, f_1) - (v_2, f_2)\|_X \\
&\le \frac{1}{2} \|(v_1, f_1) - (v_2, f_2)\|_X
\end{aligned}
$$

provided δ and T are sufficiently small. Therefore, $L : X \to X$ is a contraction map.

It follows from Lemma 3.1 and Lemma 3.2 that if $T > 0$ is sufficiently small, then there exists a unique solution $(u, d) \in C_\alpha^{2,1}(\overline{\Omega} \times [0, T), \mathbb{R}^2 \times S^2)$ to (1.1-1.4), (1.5).

Applying the maximum principle to the equation for $|d|^2$, one can easily see that $|d| = 1$ in $\Omega \times (0, T)$. The proof of Theorem 3.1 is now complete. $\square$

4 Energy inequalities, estimates of pressure function

This section is devoted to both global and local energy inequalities, and the estimate of the pressure function.

First, we have

Lemma 4.1 *For $0 < T < +\infty$, suppose $u \in L^{2,\infty}(\Omega \times [0, T]) \cap W_2^{1,0}(\Omega_T)$, $d \in L^\infty([0, T], H^1(\Omega)) \cap L^2([0, T], H^2(\Omega))$, and $\nabla P \in L^{\frac{4}{3}}(\Omega_T)$ is a weak solution to (1.1)-(1.4), (1.5). Then, for any $0 < t \leq T$, we have*

$$\int_\Omega \left(|u|^2 + |\nabla d|^2\right)(\cdot, t) + 2\int_{\Omega_t} \left(|\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|^2\right) = \int_\Omega \left(|u_0|^2 + |\nabla d_0|^2\right). \tag{4.1}$$

Proof. First, by Ladyzhenskaya's inequality, we have

$$u \in L^4(\Omega_T),\ \nabla d \in L^4(\Omega_T).$$

Multiply (1.1) by u and integrate over Ω. Since $u \in \mathbf{H}$, it is well-known ([24]) that

$$\int_\Omega (u \cdot \nabla u) \cdot u = 0,\ \int_\Omega \nabla P \cdot u = 0.$$

Hence we have

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega \nabla d \odot \nabla d : \nabla u. \tag{4.2}$$

Multiplying (1.3) by $\Delta d + |\nabla d|^2 d$ and integrating over Ω, we obtain

$$\int_\Omega (d_t + u \cdot \nabla d) \cdot \Delta d = \int_\Omega |\Delta d + |\nabla d|^2 d|^2,$$

where we have used the fact that $|d| = 1$ to get

$$(d_t + u \cdot \nabla d) \cdot |\nabla d|^2 d = \frac{1}{2}(|\nabla d|^2 |d|_t^2 + u \cdot \nabla |d|^2 |\nabla d|^2) = 0.$$

Since $d_t = 0$ on $\partial\Omega$, by integration by parts, we have

$$\int_\Omega d_t \cdot \Delta d = -\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla d|^2.$$

Now we claim

$$\int_\Omega \Delta d \cdot (u \cdot \nabla d) = -\int_\Omega \nabla d \odot \nabla d : \nabla u. \tag{4.3}$$

In fact, (4.3) follows from

$$\begin{aligned}\int_\Omega \Delta d \cdot (u\cdot\nabla d) &= \int_\Omega d_{\beta\beta} u^\alpha d_\alpha \\ &= \int_\Omega \left[(d_\beta u^\alpha d_\alpha)_\beta - d_\alpha \cdot d_\beta u^\alpha_\beta - u^\alpha(\frac{|\nabla d|^2}{2})_\alpha\right] \\ &= -\int_\Omega d_\alpha \cdot d_\beta u^\alpha_\beta = -\int_\Omega \nabla d \odot \nabla d : \nabla u.\end{aligned}$$

Hence we obtain

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla d|^2 + \int_\Omega |\Delta d + |\nabla d|^2 d|^2 = -\int_\Omega \nabla d \odot \nabla d : \nabla u. \tag{4.4}$$

It is now easy to see that (4.1) follows by adding (4.2) and (4.4) and integrating from 0 to T. □

In order to prove Theorem 1.3, we also need a local energy inequality of both interior and boundary types for solutions to (1.1)-(1.4), (1.5).

Lemma 4.2 *For $0 < T < +\infty$, suppose $u \in L^{2,\infty}(\Omega\times[0,T]) \cap W_2^{1,0}(\Omega_T)$, $d \in L^\infty([0,T], H^1(\Omega)) \cap L^2([0,T], H^2(\Omega))$, and $\nabla P \in L^{\frac{4}{3}}(\Omega_T)$ is a weak solution to (1.1)-(1.4), (1.5). Then, for any nonnegative $\phi \in C_0^\infty(\Omega)$ and $0 < s < t \le T$,*

$$\begin{aligned}&\int_\Omega \phi(|u|^2 + |\nabla d|^2)(t) + 2\int_s^t\int_\Omega \phi(|\nabla u|^2 + |\triangle d + |\nabla d|^2 d|^2) \\ &\le \int_\Omega \phi(|u|^2 + |\nabla d|^2)(s) \\ &+C\int_s^t\int_\Omega |\nabla\phi|[|u|^3 + |P - P_\Omega||u| + |\nabla u||u| + |\nabla d|^2|u| + |d_t||\nabla d|],\end{aligned} \tag{4.5}$$

where P_Ω is the average of P over Ω.

Proof. Multiplying (1.1) by $u\varphi$ and integrating over Ω implies

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |u|^2\phi + 2\int_\Omega |\nabla u|^2\phi \\ =\;& 2\int_\Omega [-u\cdot\nabla u\cdot u\phi - \nabla u\cdot u\cdot\nabla\phi + (P - P_\Omega)\cdot u\cdot\nabla\phi + \nabla d\odot\nabla d : \nabla(u\phi)].\end{aligned}$$

For the first term in the right hand side, we have, by integration by parts,

$$-2\int_\Omega u\cdot\nabla u\cdot u\phi=\int_\Omega \frac{1}{2}|u|^2u\cdot\nabla\phi.$$

For the last term in the right hand side, we have

$$\int_\Omega \nabla d\odot\nabla d:\nabla(u\phi)=\int_\Omega u\cdot\nabla d\cdot\nabla d\nabla\phi+\int_\Omega \nabla d\odot\nabla d:\nabla u\phi.$$

Putting all these two terms into the identity above yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u|^2\phi+2\int_\Omega |\nabla u|^2\phi &\le 2\int_\Omega \nabla d\odot\nabla d:\nabla u\phi \\ &+ \int_\Omega (|u|^3+2|\nabla u||u|+2|P-P_\Omega||u|+2|\nabla d|^2|u|)|\nabla\phi|.\end{aligned}\tag{4.6}$$

Multiplying (1.3) by $(\triangle d+|\nabla d|^2d)\phi$ and integrating over Ω yields

$$\int_\Omega (d_t+u\cdot\nabla d)\cdot(\triangle d+|\nabla d|^2d)\phi=\int_\Omega |\triangle d+|\nabla d|^2d|^2\phi.\tag{4.7}$$

As in Lemma 4.1, since $|d|=1$, we have

$$\int_\Omega (d_t+u\cdot\nabla d)\cdot|\nabla d|^2d\phi=0.\tag{4.8}$$

On the other hand, by integration by parts, we have

$$\int_\Omega d_t\cdot\Delta d\phi=-\frac{d}{dt}\int_\Omega \frac{|\nabla d|^2}{2}\phi-\int_\Omega d_t\cdot\nabla d\cdot\nabla\phi,\tag{4.9}$$

$$\begin{aligned}&\int_\Omega u\cdot\nabla d\cdot\Delta d\phi \\ =&-\int_\Omega (u^id_i)_j\cdot d_j\phi-\int_\Omega u^id_i\cdot d_j\phi_j \\ =&-\int_\Omega u^i(\frac{|\nabla d|^2}{2})_i\phi-\int_\Omega u^i_jd_i\cdot d_j\phi-\int_\Omega u^id_i\cdot d_j\phi_j \\ =&\int_\Omega \frac{|\nabla d|^2}{2}u\cdot\nabla\phi-\int_\Omega \nabla d\odot\nabla d:\nabla u\phi-\int_\Omega (u\cdot\nabla d)(\nabla\phi\cdot\nabla d).\end{aligned}\tag{4.10}$$

Combining (4.7), (4.8), (4.9), with (4.10), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla d|^2\phi+2\int_\Omega |\triangle d+|\nabla d|^2d|^2\phi \\ =&-2\int_\Omega d_t\cdot\nabla d\cdot\nabla\phi-2\int_\Omega \nabla u:(\nabla d\odot\nabla d)\phi \\ +&\ 2\int_\Omega |\nabla d|^2u\cdot\nabla\phi-2\int_\Omega (u\cdot\nabla d)(\nabla d\cdot\nabla)\phi.\end{aligned}\tag{4.11}$$

Adding (4.7) and (4.11), we have

$$\frac{d}{dt}\int_\Omega (|u|^2+|\nabla d|^2)\phi + 2\int_\Omega (|\nabla u|^2+|\triangle d+|\nabla d|^2 d|^2)\phi$$
$$\leq \int_\Omega [|u|^3+2|\nabla u||u|+2|P-P_\Omega||u|+6|\nabla d|^2|u|+|d_t||\nabla d|]|\nabla\phi|.$$

Integrating this inequality from s to t implies (4.5). This completes the proof. □

We also need the boundary version of the local energy inequality. More precisely,

Lemma 4.3 *For $0<T<+\infty$, suppose $u\in L^{2,\infty}(\Omega\times[0,T])\cap W_2^{1,0}(\Omega_T)$, $d\in L^\infty([0,T],H^1(\Omega))\cap L^2([0,T],H^2(\Omega))$, and $\nabla P\in L^{\frac{4}{3}}(\Omega_T)$ is a weak solution to (1.1)-(1.4), (1.5). There exists $r_0=r_0(\partial\Omega)>0$ such that for any $x_0\in\partial\Omega$ and $0<r\leq r_0$, if $\phi\in C_0^\infty(B_r(x_0))$ is nonnegative and $0<s<t\leq T$, then*

$$\begin{aligned}
&\int_{\Omega\cap B_r(x_0)}\phi(|u|^2+|\nabla d|^2)(t)+2\int_s^t\int_{(\Omega\cap B_r(x_0))}\phi(|\nabla u|^2+|\triangle d+|\nabla d|^2 d|^2)\\
&\leq \int_{\Omega\cap B_r(x_0)}\phi(|u|^2+|\nabla d|^2)(s) \qquad (4.12)\\
&+C\int_s^t\int_{(\Omega\cap B_r(x_0))}|\nabla\phi|[|u|^3+|P-P_\Omega||u|+|\nabla u||u|+|\nabla d|^2|u|+|d_t||\nabla d|],
\end{aligned}$$

where P_Ω is the average of P over Ω.

Proof. Multiply (1.1) by $u\phi$ and integrate over $\Omega\cap B_r(x_0)$. Since $u\phi=0$ on $\partial(\Omega\cap B_r(x_0))$, all the boundary terms vanish in the process of integration by parts. Multiply (1.3) by $(\Delta d+|\nabla d|^2 d)\phi$ and integrate over $\Omega\cap B_r(x_0)$. Since both $d_t\phi=0$ and $u\phi=0$ on $\partial(\Omega\cap B_r(x_0))$, all the boundary terms vanish in the process of integration by parts. The rest of the argument is exactly same as Lemma 4.2. We leave it to the readers. □

In order to justify the assumptions on pressure functions in Lemma 2.1 and Lemma 2.2, we need

Lemma 4.4 *For $0<T<+\infty$, suppose $u\in L^{2,\infty}(\Omega\times[0,T])\cap W_2^{1,0}(\Omega_T)$, $d\in L^\infty([0,T],H^1(\Omega))\cap L^2([0,T],H^2(\Omega))$ is a weak solution to (1.1)-(1.4), (1.5). Then $\nabla P\in L^{\frac{4}{3}}(\Omega_T)$. Moreover, P satisfies the following estimate:*

$$\begin{aligned}
&\max\left\{\|\nabla P\|_{L^{\frac{4}{3}}(\Omega_T)},\|P-P_\Omega\|_{L^{4,\frac{4}{3}}(\Omega_T)}\right\}\\
&\lesssim \|u\|_{L^4(\Omega_T)}\|\nabla u\|_{L^2(\Omega_T)}+\|\nabla d\|_{L^4(\Omega_T)}\|\nabla^2 d\|_{L^2(\Omega_T)}. \qquad (4.13)
\end{aligned}$$

Proof. Write $u = v + w$, where v solves the heat equation:

$$\begin{aligned} v_t - \triangle v &= 0, \ \Omega \times (0, T) \\ v &= 0, \ \ \partial\Omega \times (0, T) \\ v &= u_0, \ \Omega \times \{t = 0\}, \end{aligned}$$

and w solves the non homogeneous, non-stationary Stokes equation:

$$w_t - \triangle w + \nabla P = -u \cdot \nabla u - \nabla \cdot (\nabla d \odot \nabla d), \quad \Omega \times (0, T) \tag{4.14}$$

$$w = 0, \quad \partial\Omega \times (0, T) \tag{4.15}$$

$$w = 0, \quad \Omega \times \{t = 0\}. \tag{4.16}$$

Since $f \equiv -u \cdot \nabla u - \nabla \cdot (\nabla d \odot \nabla d) \in L^{\frac{4}{3}}(\Omega_T)$, the L^p-theory [20] of non homogeneous, non-stationary Stokes equations to (4.14)-(4.16) implies that $\nabla P \in L^{\frac{4}{3}}(\Omega_T)$ and

$$\begin{aligned} \|\nabla P\|_{L^{\frac{4}{3}}(\Omega_T)} &\lesssim \|f\|_{L^{\frac{4}{3}}(\Omega_T)} \\ &\lesssim \||u||\nabla u|\|_{L^{\frac{4}{3}}(\Omega_T)} + \||\nabla d||\nabla^2 d|\|_{L^{\frac{4}{3}}(\Omega_T)} \\ &\lesssim \|u\|_{L^4(\Omega_T)} \|\nabla u\|_{L^2(\Omega_T)} + \|\nabla d\|_{L^4(\Omega_T)} \|\nabla^2 d\|_{L^2(\Omega_T)}. \end{aligned}$$

Since

$$\|P - P_\Omega\|_{L^{4,\frac{4}{3}}(\Omega_T))} \lesssim \|\nabla P\|_{L^{\frac{4}{3}}(\Omega_T)},$$

(4.13) follows. The proof of Lemma 4.4 is complete. □

5 Global weak solutions and proof of Theorem 1.3

In this section, we will establish the existence of global weak solutions to (1.1)-(1.5) that enjoy both the regularity and uniqueness properties described as in Theorem 1.3.

First, we need to recall the following version of Ladyzhenskaya's inequality.

Lemma 5.1 *There exist $C_0 > 0$ and $R_0 > 0$ depending only on Ω such that for any $T > 0$, if $u \in L^{2,\infty}(\Omega_T) \cap W_2^{1,0}(\Omega_T)$, then for $R \in (0, R_0)$,*

$$\int_{\Omega_T} |u|^4 \leq C_0 \sup_{(x,t) \in \overline{\Omega}_T} \int_{\Omega \cap B_R(x)} |u|^2(\cdot, t) \left\{ \int_{\Omega_T} |\nabla u|^2 + \frac{1}{R^2} \int_{\Omega_T} |u|^2 \right\}. \tag{5.1}$$

Proof. See Struwe [23] Lemma 3.1. □

We now derive the life span estimate for smooth solutions in term of Sobolev space norms of initial data.

Lemma 5.2 *Let $\epsilon_0 > 0$ be given by Lemma 2.1 and 2.2. There exist $0 < \epsilon_1 << \epsilon_0$ and $\theta_0 = \theta_0(\epsilon_1, E_0) \in (0, \frac{1}{4})$, here $E_0 = \int_\Omega (|u_0|^2 + |\nabla d_0|^2)$, such that if $(u_0, d_0) \in C^{2,\beta}(\overline{\Omega}, \mathbb{R}^2 \times S^2)$ satisfies*

$$\sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{2R_0}(x)} (|u_0|^2 + |\nabla d_0|^2) \le \epsilon_1^2 \tag{5.2}$$

for some $0 < R_0 \le 1$. Then there exist $T_0 \ge \theta_0 R_0^2$ and a unique solution $(u, d) \in C^\infty(\Omega \times (0, T_0), \mathbb{R}^2 \times S^2) \cap C_\beta^{2,1}(\overline{\Omega} \times [0, T_0), \mathbb{R}^2 \times S^2)$ to (1.1)-(1.5) satisfying

$$\sup_{(x,t)\in\overline{\Omega}_{T_0}} \int_{\Omega\cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t) \le 2\epsilon_1^2. \tag{5.3}$$

Proof. By Theorem 3.1, there exists $T_0 > 0$ such that there exists a unique smooth solution $(u, d) \in C^\infty(\Omega \times (0, T_0), \mathbb{R}^2 \times S^2) \cap C_\delta^{2,1}(\overline{\Omega} \times [0, T_0), \mathbb{R}^2 \times S^2)$ to (1.1)-(1.5). Let $0 < t_0 \le T_0$ be the maximal time such that

$$\sup_{0\le t\le t_0} \sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t) \le 2\epsilon_1^2. \tag{5.4}$$

Since t_0 is the maximal time for (5.4), we have

$$\sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t_0) = 2\epsilon_1^2. \tag{5.5}$$

Now we estimate the lower bound of t_0 as follows. Assume $t_0 \le R_0^2$. For, otherwise, we are done. Set

$$E(t) = \int_\Omega (|u|^2 + |\nabla d|^2)(\cdot, t), \ E_0 = \int_\Omega (|u_0|^2 + |\nabla d_0|^2).$$

Then Lemma 4.1 implies that for any $0 < t \le t_0$,

$$E(t) + \int_{\Omega_t} (|\nabla u|^2 + \left|\triangle d + |\nabla d|^2 d\right|^2) \le E_0. \tag{5.6}$$

Lemma 5.1 implies

$$\begin{aligned} \int_{\Omega_t} |\nabla d|^4 &\le C_0 (\sup_{(x,s)\in\Omega_t} \int_{\Omega\cap B_{R_0}(x)} |\nabla d|^2(\cdot, s)) (\int_{Q_\Omega} |\triangle d|^2 + \frac{1}{R_0^2} \int_{\Omega_t} |\nabla d|^2) \\ &\le C_0 \mathcal{E}_{R_0}^2(t) (\int_{\Omega_t} |\triangle d|^2 + \frac{tE_0}{R_0^2}) \end{aligned} \tag{5.7}$$

where

$$\mathcal{E}^1_{R_0}(t) = \sup_{(x,s)\in\Omega_t} \int_{\Omega\cap B_{R_0}(x)} |u|^2(\cdot, s),$$

$$\mathcal{E}^2_{R_0}(t) = \sup_{(x,s)\in\Omega_t} \int_{\Omega\cap B_{R_0}(x)} |\nabla d|^2(\cdot, s),$$

and

$$\mathcal{E}_{R_0}(t) = \mathcal{E}^1_{R_0}(t) + \mathcal{E}^2_{R_0}(t).$$

By (5.4), we have

$$\mathcal{E}_{R_0}(t) \leq 2\epsilon_1^2, \ \forall 0 \leq t \leq t_0.$$

Hence

$$\int_{\Omega_{t_0}} |\nabla d|^4 \leq C_0\epsilon_1^2 (\int_{\Omega_{t_0}} |\triangle d|^2 + \frac{t_0}{R_0^2} E_0). \tag{5.8}$$

Since

$$|\Delta d|^2 \leq 2(|\Delta d + |\nabla d|^2 d|^2 + |\nabla d|^4),$$

(5.6) and (5.8) imply

$$\begin{aligned} \int_{\Omega_{t_0}} |\triangle d|^2 &\leq 2\int_{\Omega_{t_0}} (|\triangle d + |\nabla d|^2 d|^2 + |\nabla d|^4) \leq 2E_0 + 2\int_{\Omega_{t_0}} |\nabla d|^4 \\ &\leq 2E_0 + C_0\epsilon_1^2 (\int_{\Omega_{t_0}} |\triangle d|^2 + \frac{t_0}{R_0^2} E_0). \end{aligned}$$

Therefore we get

$$(1 - C_0\epsilon_1^2) \int_{\Omega_{t_0}} |\triangle d|^2 \leq C_0 (1 + \frac{\epsilon_1^2 t_0}{R_0^2}) E_0.$$

Choose $0 < \epsilon_1^2 \leq \frac{1}{2C_0}$, we have

$$\int_{\Omega_{t_0}} |\triangle d|^2 \leq C_0 (1 + \frac{\epsilon_1^2 t_0}{R_0^2}) E_0 \leq C_0 (1 + \frac{t_0}{R_0^2}) E_0 \leq C_0 E_0. \tag{5.9}$$

This, combined with (5.8), also gives

$$\int_{\Omega_{t_0}} |\nabla d|^4 \leq C_0\epsilon_1^2 (1 + \frac{t_0}{R_0^2}) E_0 \leq C_0\epsilon_1^2 E_0. \tag{5.10}$$

Similarly, we can estimate $\int_{\Omega_{t_0}} |u|^4$ as follows.

$$\begin{aligned} \int_{\Omega_{t_0}} |u|^4 &\leq C_0 \mathcal{E}^1_{R_0}(t_0) (\int_{\Omega_{t_0}} |\nabla u|^2 + \frac{1}{R_0^2} \int_{\Omega_{t_0}} |u|^2) \\ &\leq C_0 \mathcal{E}^1_{R_0}(t_0) (\int_{\Omega_{t_0}} |\nabla u|^2 + \frac{t_0 E_0}{R_0^2}) \\ &\leq C_0\epsilon_1^2 (1 + \frac{t_0}{R_0^2}) E_0 \leq C_0\epsilon_1^2 E_0. \end{aligned} \tag{5.11}$$

Now we estimate the quantity $\mathcal{E}_{R_0}(t)$ as follows. For any $x \in \overline{\Omega}$, let $\phi \in C_0^\infty(B_{2R_0}(x))$ be a cut-off function of $B_{R_0}(x)$ such that

$$0 \le \phi \le 1,\ \phi \equiv 1 \text{ on } B_{R_0}(x),\ \phi \equiv 0 \text{ outside } B_{2R_0}(x),\ |\nabla\phi| \le \frac{4}{R_0}.$$

Then, by the local energy inequality Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}
&\sup_{0\le t\le t_0}\int_{\Omega\cap B_{R_0}(x)}(|u|^2+|\nabla d|^2) - \mathcal{E}_{2R_0}(0)\\
\le\ & \sup_{0\le t\le t_0}\int_{\Omega\cap B_{2R_0}(x)}(|u|^2+|\nabla d|^2)\phi - \mathcal{E}_{2R_0}(0)\\
\lesssim\ & \int_{\Omega_{t_0}}[|u|^3+|\nabla u||u|+|P-P_\Omega||u|+|\nabla d|^2|u|+|d_t||\nabla d|]|\nabla\phi|\\
\lesssim\ & \left(\frac{t_0}{R_0^2}\right)^{\frac14}\left[\|u\|^3_{L^4(\Omega_{t_0})}+\|\nabla u\|_{L^2(\Omega_{t_0})}\|u\|_{L^4(\Omega_{t_0})}\right]+\|P-P_\Omega\|_{L^{4,\frac43}(\Omega_{t_0})}\|u\|_{L^4(\Omega_{t_0})}\\
&+\left(\frac{t_0}{R_0^2}\right)^{\frac14}\left[\|\nabla d\|^2_{L^4(\Omega_{t_0})}\|u\|_{L^4(\Omega_{t_0})}+\|d_t\|_{L^2(\Omega_{t_0})}\|\nabla d\|_{L^4(\Omega_{t_0})}\right], \qquad (5.12)
\end{aligned}$$

where we have used $t_0 \le R_0^2 \le 1$. Notice

$$\|\nabla u\|_{L^2(\Omega_{t_0})} \le (t_0E_0)^{\frac12} \le E_0^{\frac12}. \qquad (5.13)$$

For d_t, multiplying (1.3) by d_t and integrating over Ω_{t_0}, we obtain

$$\begin{aligned}
\int_{\Omega_{t_0}}|d_t|^2 &\le \int_\Omega|\nabla d_0|^2+4\int_{\Omega_{t_0}}|u|^2|\nabla d|^2\\
&\le E_0+4\|u\|^2_{L^4(\Omega_{t_0})}\|\nabla d\|^2_{L^4(\Omega_{t_0})}\\
&\le C_0E_0. \qquad (5.14)
\end{aligned}$$

Putting (4.13), (5.9), (5.10), (5.11), (5.13) and (5.14) into (5.12), we obtain

$$\begin{aligned}
2\epsilon_1^2 &= \sup_{0\le t\le t_0}\int_{\Omega\cap B_{R_0}(x)}(|u|^2+|\nabla d|^2)\\
&\le \mathcal{E}_{2R_0}(0)+C_0\left(\frac{t_0}{R_0^2}\right)^{\frac14}\epsilon_1^{\frac12}E_0^{\frac34}\\
&\le \epsilon_1^2+C_0\left(\frac{t_0}{R_0^2}\right)^{\frac14}\epsilon_1^{\frac12}E_0^{\frac34}. \qquad (5.15)
\end{aligned}$$

This implies

$$t_0 \ge \frac{\epsilon_1^6}{C_0^4E_0^3}R_0^2 = \theta_0R_0^2, \quad \text{with} \quad \theta_0 \equiv \frac{\epsilon_1^6}{C_0^4E_0^3}.$$

Set $T_0 = t_0$, we have $T_0 \geq \theta_0 R_0^2$ and (5.3) holds. This completes the proof. □

Before we prove Theorem 1.3, we need the following density property of Sobolev maps, whose proof can be found in Schoen-Uhlenbeck [22].

Lemma 5.3 *For $n = 2$ and any given map $f \in H^1(\Omega, S^2) \cap C^{2,\delta}(\partial\Omega, S^2)$ with $0 < \delta < 1$, there exist $\{f_k\} \subset C^\infty(\Omega, S^2) \cap C^{2,\delta}(\overline{\Omega}, S^2)$ such that $f_k = f$ on $\partial\Omega$ for all k, and*

$$\lim_{k\to\infty} \|f_k - f\|_{H^1(\Omega)} = 0.$$

Proof of Theorem 1.3:

Since $u_0 \in \mathbf{H}$, there exists $u_0^k \in C_0^\infty(\Omega, \mathbb{R}^2)$, with $\nabla \cdot u_0^k = 0$, such that

$$\lim_{k\to\infty} \|u_0^k - u_0\|_{L^2(\Omega)} = 0.$$

Since $d_0 \in H^1(\Omega, S^2) \cap C^{2,\beta}(\partial\Omega, S^2)$, Lemma 5.3 implies that there exist $\{d_0^k\} \subset C^\infty(\Omega, S^2) \cap C^{2,\beta}(\overline{\Omega}, S^2)$, with $d_0^k = d_0$ on $\partial\Omega$, such that

$$\lim_{k\to\infty} \|d_0^k - d_0\|_{H^1(\Omega)} = 0.$$

By the absolute continuity of $\int(|u_0|^2 + |\nabla d_0|^2)$, we conclude that there exists $R_0 > 0$ such that

$$\sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{2R_0}(x)} (|u_0|^2 + |\nabla d_0|^2) \leq \frac{\epsilon_1^2}{2}, \tag{5.16}$$

where $\epsilon_1 > 0$ is given by Lemma 5.2. By the strong convergence of $(u_0^k, \nabla d_0^k)$ to $(u_0, \nabla d_0)$ in $L^2(\Omega)$, we have that

$$\sup_{x\in\overline{\Omega}} \int_{\Omega\cap B_{2R_0}(x)} (|u_0^k|^2 + |\nabla d_0^k|^2) \leq \epsilon_1^2, \ \forall k >> 1. \tag{5.17}$$

For simplicity, we assume (5.17) holds for all $k \geq 1$. By Lemma 5.2, there exist $\theta_0 = \theta_0(\epsilon_1, E_0) \in (0,1)$ and $T_0^k \geq \theta_0 R_0^2$ such that there exist solutions $(u^k, d^k) \subset C^\infty(\Omega_{T_0^k}, \mathbb{R}^2 \times S^2) \cap C_\beta^{2,1}(\overline{\Omega}_{T_0^k}, \mathbb{R}^2 \times S^2)$ to (1.1)-(1.3) along with the initial-boundary condition:

$$(u^k, d^k)\big|_{\Omega\times\{0\}} = (u_0^k, d_0^k), \quad (u^k, d^k)\big|_{\partial\Omega\times(0,T_0^k]} = (u_0^k, d_0^k). \tag{5.18}$$

Moreover we have

$$\sup_{(x,t)\in\overline{\Omega}_{T_0^k}} \int_{\Omega\cap B_{R_0}(x)} (|u^k|^2 + |\nabla d^k|^2)(\cdot, t) \leq 2\epsilon_1^2. \tag{5.19}$$

By Lemma 4.1, we have

$$\sup_{0\le t\le T_0^k}\int_\Omega(|u^k|^2+|\nabla d^k|^2)(\cdot,t)+2\int_{\Omega_{T_0^k}}(|\nabla u^k|^2+|\Delta d^k+|\nabla d^k|^2d^k|^2)$$
$$\le\int_\Omega |u_0^k|^2+|\nabla d_0^k|^2)(\le 1+E_0) \tag{5.20}$$

for sufficiently large k. Combining (5.19) and (5.20) with Lemma 5.1, we conclude that

$$\int_{\Omega_{T_0^k}}(|u^k|^4+|\nabla d^k|^4)\le C\epsilon_1^2E_0, \tag{5.21}$$

and

$$\int_{\Omega_{T_0^k}}|d_t^k|^2+\int_{\Omega_{T_0^k}}|\nabla^2d^k|^2\le CE_0. \tag{5.22}$$

By Lemma 4.4, (5.19), (5.21), and (5.22), we have

$$\begin{aligned}\|\nabla P^k\|_{L^{\frac43}(\Omega_{T_0^k})} &\lesssim \|\nabla u^k\|_{L^2(\Omega_{T_0^k})}\|u^k\|_{L^4(\Omega_{T_0^k})}+\|\nabla^2d^k\|_{L^2(\Omega_{T_0^k})}\|\nabla d^k\|_{L^4(\Omega_{T_0^k})}\\ &\le C\epsilon_1^{\frac12}E_0^{\frac34}. \end{aligned}\tag{5.23}$$

Furthermore, (1.1) implies that for any $\phi\in\mathbf{J}$,

$$\langle u_t^k,\phi\rangle=-\int_\Omega\nabla u^k\cdot\nabla\phi+\int_\Omega(u^k\otimes u^k+\nabla d^k\odot\nabla d^k)\cdot\nabla\phi,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product between H^{-1} and H_0^1, we conclude that $u_t^k\in L^2([0,T_0^k],H^{-1}(\Omega))$ and

$$\left\|u_t^k\right\|_{L^2([0,T_0^k],H^{-1}(\Omega))}\le CE_0. \tag{5.24}$$

By Theorem 2.1, we conclude that for any $\delta>0$,

$$\begin{aligned}&\|(u^k,d^k)\|_{C_\beta^{2,1}(\overline\Omega\times[\delta,T_0^k])}\\ \le\ & C(\delta,E_0,\|u^k\|_{L^4(\Omega_{T_0^k})},\|\nabla d^k\|_{L^4(\Omega_{T_0^k})},\|\nabla P^k\|_{L^{\frac43}(\Omega_{T_0^k})},\|d_0\|_{C^{2,\beta}(\partial\Omega)})\\ \le\ & C(\epsilon_1,E_0,\delta,\|d_0\|_{C^{2,\beta}(\partial\Omega)}).\end{aligned}\tag{5.25}$$

Furthermore, for any compact sub domain $K\subset\subset\Omega$ and $\delta>0$,

$$\|(u^k,d^k)\|_{C^l(K\times[\delta,T_0^k])}\le C(\mathrm{dist}(K,\partial\Omega),\delta,l,E_0),\ \forall l\ge1. \tag{5.26}$$

Hence, after passing to possible subsequences, there exist $T_0 \geq \theta_0 R_0^2$, $u \in W_2^{1,0}(\Omega_{T_0}, \mathbb{R}^2)$, and $d \in W_2^{2,1}(\Omega_{T_0}, S^2)$ such that

$$u^k \to u \text{ weakly in } W_2^{1,0}(\Omega_{T_0}, \mathbb{R}^2),\ d^k \to d \text{ weakly in } W_2^{2,1}(\Omega_{T_0}, \mathbb{R}^2),$$

$$\lim_{k\to\infty} \|u^k - u\|_{L^4(\Omega_{T_0})} = 0,$$

$$\lim_{k\to\infty} \left(\|d^k - d\|_{L^4(\Omega_{T_0})} + \|\nabla d^k - \nabla d\|_{L^2(\Omega_{T_0})}\right) = 0,$$

and for any $l \geq 2$, $\delta > 0$, $\gamma < \beta$, and compact $K \subset\subset \Omega$,

$$\lim_{k\to\infty} \|(u^k, d^k) - (u, d)\|_{C^l(K\times[\delta,T_0])} = 0,$$

$$\lim_{k\to\infty} \|(u^k, d^k) - (u, d)\|_{C_\gamma^{2,1}(\overline{\Omega}\times[\delta,T_0])} = 0.$$

It is clear that $(u, d) \in C^\infty(\Omega \times (0, T_0], \mathbb{R}^2 \times S^2) \cap C_\beta^{2,1}(\overline{\Omega} \times (0, T_0], \mathbb{R}^2 \times S^2)$ solves (1.1)-(1.3) in $\Omega \times (0, T_0]$ and satisfies the boundary condition. It follows from (5.24) and (5.22) that we can assume

$$(u, \nabla d)(\cdot, t) \to (u_0, \nabla d_0) \text{ weakly in } L^2(\Omega)$$

as $t \downarrow 0$. In particular,

$$E(0) \leq \liminf_{t\downarrow 0} E(t).$$

On the other hand, (5.20) implies

$$E(0) \geq \limsup_{t\downarrow 0} E(t).$$

This implies $(u, \nabla d)(\cdot, t)$ converges to $(u_0, \nabla d_0)$ strongly in $L^2(\Omega)$. Hence (u, d) satisfies the initial condition (1.4).

Let $T_1 \in (T_0, +\infty)$ be the first singular time of (u, d), i.e.

$$(u, d) \in C^\infty(\Omega \times (0, T_1), \mathbb{R}^2 \times S^2) \cap C_\beta^{2,1}(\overline{\Omega} \times (0, T_1), \mathbb{R}^2 \times S^2),$$

but

$$(u, d) \notin C^\infty(\Omega \times (0, T_1], \mathbb{R}^2 \times S^2) \cap C_\beta^{2,1}(\overline{\Omega} \times (0, T_1], \mathbb{R}^2 \times S^2).$$

Then we must have

$$\limsup_{t\uparrow T_1} \max_{x\in\overline{\Omega}} \int_{\Omega\cap B_R(x)} (|u|^2 + |\nabla d|^2)(\cdot, t) \geq \epsilon_1^2,\ \forall R > 0. \tag{5.27}$$

Now we look for an eternal extension of this weak solution in time. In order to do it, we need to define the new initial data at $t = T_1$.

Claim 3. $(u, d) \in C^0([0, T_1], L^2(\Omega))$.

In fact, for any $\phi \in H_0^2(\Omega, \mathbb{R}^3)$, (1.3) yields

$$
\begin{aligned}
|\langle d_t, \phi\rangle| &= \left|\int_\Omega (\nabla d \cdot \nabla\phi + u \cdot \nabla d \cdot \phi) - \int_\Omega |\nabla d|^2 d \cdot \phi\right| \\
&\lesssim \|\nabla d\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} + (\|u\|_{L^2(\Omega)}\|\nabla d\|_{L^2(\Omega)} + \|\nabla d\|^2_{L^2(\Omega)})\|\phi\|_{C^0(\Omega)} \\
&\lesssim [\|\nabla d\|_{L^2(\Omega)} + (\|u\|_{L^2(\Omega)}\|\nabla d\|_{L^2(\Omega)} + \|\nabla d\|^2_{L^2(\Omega)})]\|\phi\|_{H^2(\Omega)},
\end{aligned}
$$

where we have used the fact $H_0^2(\Omega) \subset C^0(\Omega)$ and $\|\phi\|_{C^0(\Omega)} \lesssim \|\phi\|_{H^2(\Omega)}$. Hence $d_t \in L^2([0, T_1], H^{-2}(\Omega))$. This, combined with $d \in L^2([0, T_1], H^1(\Omega))$, implies $d \in C^0([0, T_1], L^2(\Omega))$.

Similarly, for any $\phi \in H_0^3(\Omega, \mathbb{R}^2)$ with $\nabla \cdot \phi = 0$, (1.1) yields

$$
\begin{aligned}
|\langle u_t, \phi\rangle| &= \left|\int_\Omega (\nabla u \cdot \nabla\phi + u \cdot \nabla u \cdot \phi) - \int_\Omega \nabla d \odot \nabla d : \nabla\phi\right| \\
&\lesssim [\|\nabla u\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\phi\|_{C^0(\Omega)} \\
&\quad + \|\nabla u\|^2_{L^2(\Omega)}\|\nabla\phi\|_{C^0(\Omega)}] \\
&\lesssim [\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}]\|\phi\|_{H^3(\Omega)}]
\end{aligned}
$$

where we have used the fact $H^3(\Omega) \subset C^1(\Omega)$ and $\|\phi\|_{C^1(\Omega)} \lesssim \|\phi\|_{H^3(\Omega)}$. Since u_t is divergence free, We have $u_t \in L^2([0, T_1], H^{-3}(\Omega))$. This, combined with $u \in L^2([0, T_1], H^1(\Omega))$, implies $u \in C^0([0, T_1], L^2(\Omega))$.

By Claim 3, we can define

$$(u(T_1), d(T_1)) = \lim_{t \uparrow T_1}(u(t), d(t)) \text{ in } L^2(\Omega).$$

By the energy inequality, we have that $\nabla d \in L^\infty([0, T_1], L^2(\Omega))$. Hence $\nabla d(t) \to \nabla d(T_1)$ weakly in $L^2(\Omega)$. In particular, $u(T_1) \in \mathbf{H}$ and $d(T_1) \in H^1(\Omega)$. Moreover, since $d(t)|_{\partial\Omega} = d_0$, $d(T_1) = d_0$ on $\partial\Omega$.

Now we can use $(u(T_1), d(T_1))$ and $(0, d_0)|_{\partial\Omega}$ as initial and boundary data in the above procedure to obtain a continuation of (u, d) beyond T_1 as a weak solution of (1.1)-(1.5). At any further singular time, we repeat this procedure. We will prove that there are at most finitely many such singular times, afterwards we will have constructed an eternal weak solution.

We want to show that at any singular time there is at least a loss of energy amount of ϵ_1^2. By (5.27), there exist $t_i \uparrow T_1$ and $x_0 \in \overline{\Omega}$ such that

$$\limsup_{t_i \uparrow T_1} \int_{\Omega \cap B_R(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, t_i) \geq \epsilon_1^2 \text{ for all } R > 0.$$

This implies

$$\begin{aligned}
& \int_\Omega (|u|^2 + |\nabla d|^2)(\cdot, T_1) \\
= & \lim_{R \downarrow 0} \int_{\Omega \setminus B_R(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, T_1) \\
\leq & \lim_{R \downarrow 0} \liminf_{t_i \uparrow T_1} \int_{\Omega \setminus B_R(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, t_i) \\
\leq & \lim_{R \downarrow 0} [\liminf_{t_i \uparrow T_1} \int_\Omega (|u|^2 + |\nabla d|^2)(\cdot, t_i) - \limsup_{t_i \uparrow T_1} \int_{\Omega \cap B_R(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, t_i)] \\
\leq & \liminf_{t_i \uparrow T_1} \int_\Omega (|u|^2 + |\nabla d|^2)(\cdot, t_i) - \epsilon_1^2 \leq E_0 - \epsilon_1^2.
\end{aligned}$$

From this, we see that the number of finite singular times must be bounded by $L = [\frac{E_0}{\epsilon_1^2}]$, here $[\cdot]$ denotes the largest integer part. If $0 < T_L < +\infty$ is the last singular time, then we must have

$$E(t_L) = \int_\Omega (|u|^2 + |\nabla d|^2)(\cdot, T_L) < \epsilon_1^2.$$

Hence, if we use $(u(T_L), d(T_L))$ and $(0, d_0)|_{\partial\Omega}$ as the initial and boundary data to construct a weak solution (u, d) to (1.1)-(1.3) as before, then (u, d) will be an eternal weak solution that we look for.

It is clear that (i), (iii), and (ii) (1.7) of Theorem 1.3 has been established. Now, we want to perform the blow-up analysis at each singular time. It follows from (1.7) that there exist $0 < t_0 < T_1$, $t_m \uparrow T_1$, $r_m \downarrow 0$ such that

$$\epsilon_1^2 = \sup_{x \in \overline{\Omega}, t_0 \leq t \leq t_m} \int_{\Omega \cap B_{r_m}(x)} (|u|^2 + |\nabla d|^2). \tag{5.28}$$

By Lemma 5.2, there exist θ_0, depending only on ϵ_1 and E_0 and $x_m \in \Omega$ such that

$$\begin{aligned}
& \int_{\Omega \cap B_{2r_m}(x_m)} (|u|^2 + |\nabla d|^2)(t_m - \theta_0 r_m^2) \\
\geq & \frac{1}{2} \max_{x \in \overline{\Omega}} \int_{\Omega \cap B_{2r_m}(x)} (|u|^2 + |\nabla d|^2)(t_m - \theta_0 r_m^2) \geq \frac{\epsilon_1^2}{4}.
\end{aligned} \tag{5.29}$$

By Lemma 4.1, (5.28), and the Ladyzhenskaya's inequality, we have

$$\int_{\Omega\times[t_0,t_m]}(|u|^4+|\nabla d|^4)\le C(\epsilon_1,E_0). \tag{5.30}$$

Denote $\Omega_m = r_m^{-1}(\Omega\setminus\{x_m\})$. Define the blow-up sequence $(u_m,d_m):\Omega_m\times[\frac{t_0-t_m}{r_m^2},0]\to(\mathbb{R}^2,S^2)$ by

$$u_m(x,t)=r_mu(x_m+r_mx,t_m+r_m^2t),\ d_m(x,t)=d(x_m+r_mx,t_m+r_m^2t).$$

Then (u_m,d_m) solves (1.1)-(1.3) on $\Omega_m\times[\frac{t_0-t_m}{r_m^2},0]$. Moreover,

$$\int_{\Omega_m\cap B_2(0)}(|u_m|^2+|\nabla d_m|^2)(-\theta_0)\ge\frac{\epsilon_1^2}{2},$$

$$\int_{\Omega_m\cap B_1(x)}(|u_m|^2+|\nabla d_m|^2)(t)\le\epsilon_1^2,\ \forall x\in\Omega_m,\ \frac{t_0-t_m}{r_m^2}\le t\le 0,$$

$$\int_{\Omega_m\times[\frac{t_0-t_m}{r_m^2},0]}|u_m|^4+|\nabla d_m|^2\le C(\epsilon_1,E_0).$$

Assume $x_m\to x_0\in\overline{\Omega}$, we divide into two cases.

Case1. $x_0\in\Omega$. We can assume $r_m<\mathrm{dist}(x_0,\partial\Omega)$ and $\Omega_m\to\mathbb{R}^2$. Also notice that $\frac{t_0-t_m}{r_m^2}\to-\infty$. Hence, by Theorem 1.2, we can assume that there exists a smooth solution $(u_\infty,d_\infty):\mathbb{R}^2\times(-\infty,0]\to\mathbb{R}^2\times S^2$ to (1.1)-(1.3) such that

$$(u_m,d_m)\to(u_\infty,d_\infty)\text{ in }C^2_{\mathrm{loc}}(\mathbb{R}^2\times[-\infty,0]).$$

We want to first show $u_\infty\equiv0$. In fact, for any parabolic cylinder $P_R\subset\mathbb{R}^2\times[-\infty,0]$, since $u\in L^4(\Omega\times[0,T_1])$, we have

$$\int_{P_R}|u_\infty|^4=\lim_{m\to\infty}\int_{P_R}|u_m|^4=\lim_{m\to\infty}\int_{B_{Rr_m}(x_m)}\int_{t_m-R^2r_m^2}^{t_m}|u|^4=0.$$

Next we want to show d_∞ is a nontrivial, smooth harmonic map with finite energy. In fact, since $(\Delta d+|\nabla d|^2d)\in L^2(\Omega\times[0,T_1])$, we have, for any compact $K\subset\mathbb{R}^2$,

$$\begin{aligned}\int_{-2\theta_0}^0\int_K|\Delta d_\infty+|\nabla d_\infty|^2d_\infty|^2 &\le \liminf_m\int_{-2\theta_0}^0\int_{\Omega_m}|\Delta d_m+|\nabla d_m|^2d_m|^2\\ &= \lim_m\int_{t_m-2\theta_0r_m^2}^{t_m}\int_\Omega|\Delta d+|\nabla d|^2d|^2=0.\end{aligned}$$

This implies $(d_\infty)_t + u_\infty \cdot \nabla d_\infty \equiv 0$ on $\mathbb{R}^2 \times [-2\theta_0, 0]$. Hence $(d_\infty)_t = u_\infty \equiv 0$ and $d_\infty \in C^2(\mathbb{R}^2, S^2)$ is a harmonic map. Since

$$\int_{B_2} |\nabla d_\infty|^2 = \lim_m \int_{B_2} (|u_m|^2 + |\nabla d_m|^2)(-\theta_0) \geq \frac{\epsilon_1^2}{4},$$

d_∞ is nontrivial. By the lower semicontinuity, we have for any ball $B_R \subset \mathbb{R}^2$,

$$\int_{B_R} |\nabla d_\infty|^2 \leq \liminf_m \int_{B_R} |\nabla d_m|^2(-\theta_0) = \liminf_m \int_{B_{r_m R}(x_m)} |\nabla d|^2(t_m - \theta_0 r_m^2) \leq E_0$$

so that d_∞ has finite energy. It is well-known ([21], [23]) that d_∞ can be lifted to be a non constant harmonic map from S^2 to S^2. In particular, d_∞ has nontrivial degree and

$$\int_{\mathbb{R}^2} |\nabla d_\infty|^2 \geq 8\pi |\deg(d_\infty)| \geq 8\pi.$$

It follows from the above argument that for any $r > 0$,

$$\limsup_{t \uparrow T_1} \max_{x \in \overline{\Omega}} \int_{\Omega \cap B_r(x)} (|u|^2 + |\nabla d|^2)(t) \geq \int_{\mathbb{R}^2} (|u_\infty|^2 + |\nabla d_\infty|^2) \geq 8\pi.$$

Case 2. $x_0 \in \partial\Omega$. Then either (a) $\lim_m \frac{|x_m - x_0|}{r_m} = \infty$, or (b) $\lim_m \frac{|x_m - x_0|}{r_m} < \infty$. If (a) holds, then $\Omega_m \to \mathbb{R}^2$ and the same argument as Case 1 yields that $(u_m, d_m) \to (0, d_\infty)$ in $C^2_{\text{loc}}(\mathbb{R}^2)$, and $d_\infty \in C^\infty(\mathbb{R}^2, S^2)$ is a nontrivial harmonic map with finite energy. Now we want to show that (b) can't happen. For, otherwise, assume that $\frac{x_m - x_0}{r_m} \to (0, a)$ for some $a \in \mathbb{R}$ and

$$\Omega_m \to \mathbb{R}^2_a = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 \geq a\}.$$

Since $d_m(x) = d(x_m + r_m x)$ for $x \in \partial\Omega_m$, we can show, similar to Case 1, that $u_m \to 0$ in $C^2_{\text{loc}}(\mathbb{R}^2_a)$, and $d_m \to d_\infty$ in $C^2_{\text{loc}}(\mathbb{R}^2_a)$, where $d_\infty : \mathbb{R}^2_a \to S^2$ is a nontrivial harmonic map with finite energy and $d_\infty(\cdot, a) = d(x_0)$ is a constant map. This contradicts the nonexistence theorem by Lemaire [12]. Thus (ii) is established.

To show (iv). By Lemma 4.1, there exists $t_k \uparrow +\infty$ such that for $(u_k, d_k) = (u(\cdot, t_k), d(\cdot, t_k))$,

$$\int_\Omega |u_k|^2 + |\nabla d_k|^2 \leq E_0,$$

$$\lim_{k \to \infty} \int_\Omega (|\nabla u_k|^2 + |\Delta d_k + |\nabla d_k|^2 d_k|^2) = 0.$$

Since $u_k|_{\partial\Omega} = 0$, it is easy to see that $u_k \to 0$ in $H^1(\Omega)$. d_k is a H^1-bounded sequence of approximate harmonic maps from Ω into S^2, with $d_k|_{\partial\Omega} = d_0 \in C^{2,\beta}(\partial\Omega)$ and the L^2 norm of their tension fields $\tau(d_k) = \Delta d_k + |\nabla d_k|^2 d_k$ converging to zero. By the energy identity result by Qing [18] and Lin-Wang [16], we can conclude that there exist a harmonic map $d_\infty \in C^{2,\beta}(\overline{\Omega}, S^2)$ with $d_\infty = d_0$ on $\partial\Omega$ and finitely many points $\{x_i\}_{i=1}^l$, $\{m_i\}_{i=1}^l \subset \mathbb{N}$ such that

$$|\nabla d_k|^2\,dx \to |\nabla d_\infty|^2\,dx + \sum_{i=1}^l 8\pi m_i \delta_{x_i}.$$

This yields (iv).

To show (v). First we claim

(a) There exist no finite time singularities. For, otherwise, (ii) implies that we can blow up near the first singular time T_1 to obtain one nontrivial harmonic map $\omega \in C^\infty(\mathbb{R}^2, S^2)$ and

$$8\pi \le \int_{\mathbb{R}^2} |\nabla\omega|^2 \le \lim_{t\uparrow T_1} \int_\Omega (|u|^2 + |\nabla d|^2)(t) \le \int_\Omega (|u_0|^2 + |\nabla d_0|^2) \le 8\pi.$$

This, combined with Lemma 4.1, yields

$$\int_0^{T_1}\int_\Omega (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) = 0$$

so that $u = d_t \equiv 0$ in $\Omega \times [0, T_1]$ and hence $d(\cdot, t) = d_0 \in C^{2,\beta}(\Omega, S^2)$, $0 \le t \le T_1$, is a harmonic map. This contradicts the fact that T_1 is a singular time.

(b) $\phi(t) \equiv \max_{x\in\overline{\Omega}, \tau\le t}(|u| + |\nabla d|)(x, \tau)$ remains bounded as $t \uparrow +\infty$. For, otherwise, there exist $t_k \uparrow +\infty$ and $x_k \in \overline{\Omega}$ such that

$$\lambda_k = \phi(t_k) = (|u| + |\nabla d|)(x_k, t_k) \to +\infty.$$

Define $\Omega_k = \lambda_k(\Omega \setminus \{x_k\})$ and $(u_k, d_k) : \Omega_k \times [-t_k\lambda_k^2, 0] \to \mathbb{R}^2 \times S^2$ by

$$u_k(x, t) = \lambda_k^{-1} u(x_k + \lambda_k^{-1}x, t_k + \lambda_k^{-2}t),\ d_k(x, t) = d(x_k + \lambda_k^{-1}x, t_k + \lambda_k^{-2}t).$$

Then (u_k, d_k) solves (1.1)-(1.3) on $\Omega_k \times [-t_k\lambda_k^2, 0]$, and

$$1 = (|u_k| + |\nabla d_k|)(0, 0) \ge (|u_k| + |\nabla d_k|)(x, t),\ \forall (x, t) \in \Omega_k \times [-t_k\lambda_k^2, 0].$$

As in the proof of (ii), we can conclude that either (i) $\Omega_k \to \mathbb{R}^2$ and $(u_k, d_k) \to (0, d_\infty)$ in $C^2_{\rm loc}(\mathbb{R}^2)$, where $d_\infty \in C^\infty(\mathbb{R}^2, S^2)$ is a nontrivial harmonic map with finite energy. As in (a), this implies

$$\int_0^\infty \int_\Omega (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) = 0$$

so that $u = d_t \equiv 0$ on $\Omega \times [0, +\infty)$ and hence $d(t) = d_0 \in C^{2,\beta}(\Omega, S^2)$, $0 \le t < +\infty$, is a harmonic map. This implies that $\phi(t)$ is constant for $0 < t < +\infty$ and we get a contradiction. Or (ii) $\Omega_k \to \mathbb{R}^2_a$ for some $a \in \mathbb{R}$ and $(u_k, d_k) \to (0, d_\infty)$ in $C^2_{\rm loc}(\mathbb{R}^2_a)$, where d_∞ is a nontrivial harmonic maps with finite energy and d_∞ =constant on $\partial\mathbb{R}^2_a$, which is impossible by Lemaire's theorem.

Since $\phi(t)$ is a bounded function of $t \in (0, +\infty)$, the higher order regularity (see Theorem 1.2) implies that $\|u(\cdot, t)\|_{C^{2,\beta}(\Omega)} + \|d(\cdot, t)\|_{C^{2,\beta}(\Omega)}$ is a bounded function of $t \in (0, +\infty)$. Then we can choose sequence $t_k \to \infty$ such that

$$\int_\Omega (|u|^2 + |\nabla u|^2)(x, t_k) \le E_0, \int_\Omega (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2)(x, t_k) \to 0,$$

and

$$\|u(\cdot, t_k)\|_{C^{2,\beta}(\Omega)} + \|d(\cdot, t_k)\|_{C^{2,\beta}(\Omega)} \le C.$$

Thus we may assume that there exist a harmonic map $d_\infty \in C^{2,\beta}(\overline{\Omega}, S^2)$, with $d_\infty = d_0$ on $\partial\Omega$, such that

$$(u(\cdot, t_k), d(\cdot, t_k)) \to (0, d_\infty) \text{ in } C^2(\overline{\Omega}, S^2).$$

This proves (v). The proof of Theorem 1.3 is now complete. □

Note of proof. After the completion of this paper, we learned that Professor Min-Chun Hong [8] independently obtained Theorem 1.3 (i) on $\mathbb{R}^2$, i.e. the existence of global weak solutions having finitely many singular times to the Cauchy problem of (1.1)-(1.3) on $\mathbb{R}^2$.

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