Harmonic maps on domains with piecewise Lipschitz continuous metrics

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Abstract

For a bounded domain Ω equipped with a piecewise Lipschitz continuous Riemannian metric g, we consider harmonic map from (Ω, g) to a compact Riemannian manifold $(N, h) \hookrightarrow \mathbb{R}^k$ without boundary. We generalize the notion of stationary harmonic maps and prove their partial regularity. We also discuss the global Lipschitz and piecewise $C^{1,\alpha}$-regularity of harmonic maps from (Ω, g) to manifolds that support convex distance functions.

1 Introduction

Throughout this paper, we assume that Ω is a bounded domain in $\mathbb{R}^n$, separated by a $C^{1,1}$-hypersurface Γ into two subdomains Ω^+ and Ω^-, namely, $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$, and g is a piecewise Lipschitz metric on Ω that is $g \in C^{0,1}(\Omega^+) \cap C^{0,1}(\Omega^-)$ but discontinuous at any $x \in \Gamma$. For example, $\Omega = B_1 \subset \mathbb{R}^n$is the unit ball, $\Gamma = B_1 \cap \{x = (x', 0) \in \mathbb{R}^n\}$, and

$$\bar{g}(x) = \begin{cases} g_0 & x \in B_1^+ = \{x_n > 0\} \cap B_1, \\ k g_0 & x \in B_1^- = \{x_n < 0\} \cap B_1, \end{cases}$$

where $g_0(x) = dx^2$ is the Euclidean metric and $1 \neq k$ is a positive constant.

Let $(N, h) \hookrightarrow \mathbb{R}^k$ be a l-dimensional, smooth compact Riemannian manifold without boundary, isometrically embedded in the Euclidean space $\mathbb{R}^k$.

Motivated by the recent studies on elliptic systems in domains consisting of composite materials (see Li-Nirenberg [17]) and the homogenization theory in calculus of variations (see Avellaneda-Lin [1] and Lin-Yan [18]), we are interested in the regularity issue of stationary harmonic maps from (Ω, g) to (N, h).

In order to describe the problem, let's first recall some notations. Throughout this paper, we use the Einstein convention for summation. For the metric $g = g_{ij}\, dx^i\, dx^j$, let (g^{ij}) denote the inverse matrix of (g_{ij}), $\sqrt{g} = \sqrt{\det(g_{ij})}$, and $dv_g = \sqrt{g}\, dx$ denotes the volume form of g. For $1 < p < +\infty$, define the Sobolev space $W^{1,p}(\Omega, N)$ by

$$W^{1,p}(\Omega, N) = \left\{ u : \Omega \to \mathbb{R}^k \;\middle|\; u(x) \in N \text{ a.e. } x \in \Omega,\;\; E_p(u, g) = \int_\Omega \left(|\nabla u|_g^2\right)^{\frac{p}{2}}\, dv_g < +\infty \right\},$$

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where

$$|\nabla u|_g^2 \equiv g^{ij}\langle\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\rangle$$

is the L^2-energy density of u with respect to g, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^k$. Denote $W^{1,2}(\Omega, N)$ by $H^1(\Omega, N)$.

Now let's recall the concept of stationary harmonic maps.

Definition 1.1. A map $u \in H^1(\Omega, N)$ is called a (weakly) harmonic map, if it is a critical point of $E_2(\cdot, g)$, i.e., u satisfies

$$\Delta_g u + A(u)(\nabla u, \nabla u)_g = 0 \tag{1.1}$$

in the sense of distributions. Here

$$\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x_j}\right)$$

is the Laplace-Beltrami operator on (Ω, g), $A(\cdot)(\cdot,\cdot)$ is the second fundamental form of $(N, h) \hookrightarrow \mathbb{R}^k$, and

$$A(u)(\nabla u, \nabla u)_g = g^{ij}A(u)\left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\right).$$

Definition 1.2. A (weakly) harmonic map $u \in H^1(\Omega, N)$ is called a stationary harmonic map, if, in additions, it is a critical point of $E_2(\cdot, g)$ with respect to suitable domain variations:

$$\frac{d}{dt}\Big|_{t=0}\int_\Omega \left|\nabla u^t\right|_g^2 dv_g = 0, \text{ with } u^t(x) = u(F_t(x)), \tag{1.2}$$

where $F(t, x) := F_t(x) \in C^1([-\delta, \delta], C^1(\Omega, \Omega))$ is a C^1 family of differmorphisms for some small $\delta > 0$ satisfying

$$\begin{cases} F_0(x) = x & \forall x \in \Omega \\ F_t(x) = x & \forall (x, t) \in \partial\Omega \times [-\delta, \delta] \\ F_t\left(\overline{\Omega^\pm}\right) \subset \overline{\Omega^\pm} & \forall t \in [-\delta, \delta]. \end{cases} \tag{1.3}$$

It is readily seen that any minimizing harmonic map from (Ω, g) to (N, h) is a stationary harmonic map. It is also easy to see from Definition 1.2 that a stationary harmonic map on (Ω, g) is a stationary harmonic map on $(\Omega^\pm, g)$ and hence satisfies an energy monotonicity inequality on $\Omega^\pm$, since $g \in C^{0,1}(\Omega^\pm)$. We will show in §2 that a stationary harmonic map on (Ω, g) also satisfies an energy monotonicity inequality in Ω under the condition (1.4) below.

The first result is concerned with both the (partial) Lipschitz regularity and (partial) piecewise $C^{1,\alpha}$-regularity of stationary harmonic maps. In this context, we are able to extend the well-known partial regularity theorem of stationary harmonic maps on domains with smooth metrics, due to Hélein [12], Evans [5], Bethuel [2]. More precisely, we have

Theorem 1.1. *Let $u \in H^1(\Omega, N)$ be a stationary harmonic map on (Ω, g). If, in additions, g satisfies the following jump condition on Γ for $n \geq 3$* [1] *: for any $x \in \Gamma$, there exists a positive constant $k(x) \neq 1$ such that*

$$\lim_{y\in\Omega^+, y\to x} g(y) = k(x) \lim_{y\in\Omega^-, y\to x} g(y), \tag{1.4}$$

then there exists a closed set $\Sigma \subset \Omega$, with $H^{n-2}(\Sigma) = 0$, such that for some $0 < \alpha < 1$,

(i) $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega \setminus \Sigma, N)$, (ii) $u \in C^{1,\alpha}_{\mathrm{loc}}((\Omega^+ \cup \Gamma) \setminus \Sigma, N) \cap C^{1,\alpha}_{\mathrm{loc}}((\Omega^- \cup \Gamma) \setminus \Sigma, N)$.

We would like to remark that when the dimension $n = 2$, since the energy monotonicity inequality automatically holds for H^1-maps, Theorem 1.1 holds for any weakly harmonic map from domains of piecewise $C^{0,1}$-metrics, i.e., any weakly harmonic map on domains with piecewise Lipschitz continuous metrics is both Lipschitz continuous and piecewise $C^{1,\alpha}$ for some $0 < \alpha < 1$.

Through the example constructed by Rivière [19], we know that weakly harmonic maps on domains with smooth metrics may not enjoy partial regularity properties in dimensions $n \geq 3$. Here we consider weakly harmonic maps on domains with piecewise Lipschitz continuous metrics into any Riemannian manifold (N, h), on which $d^2_N(\cdot, p)$ is convex. Such Riemannian manifolds N include those with non-positive sectional curvature K_N, and geodesic convex ball in any Riemannian manifold. In particular, we extend the classical regularity theorems on harmonic maps on domains with smooth metrics, due to Eells-Sampson [8] and Hildebrandt-Kaul-Widman [13], and prove

Theorem 1.2. *Let g be the same as in Theorem 1.1. Assume that on the universal cover $(\widetilde{N}, \widetilde{h})$ of (N, h)*[2]*, the square of distance function $d^2_{\widetilde{N}}(\cdot, p)$ is convex for any $p \in \widetilde{N}$. If $u \in H^1(\Omega, N)$ is a weakly harmonic map, then for some $0 < \alpha < 1$,*

(i) $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, N)$, (ii) $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega^+ \cup \Gamma, N) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega^- \cup \Gamma, N)$.

The idea to prove Theorem 1.1 is motivated by Evans [5] and Bethuel [2]. However, there are several new difficulties that we have to overcome. The first difficulty is to establish an almost energy monotonicity inequality for stationary harmonic maps in Ω, which is achieved by observing that an exact monotonicity inequality holds at any $x \in \Gamma$, see §2 below. The second one is to establish a Hodge decomposition in $L^p(B, \mathbb{R}^n)$, for any $1 < p < +\infty$, on a ball $B (= B_r(0))$ equipped with certain piecewise continuous metrics g, in order to adapt the argument by Bethuel [2]. More precisely, we will show that the following elliptic equation on B:

$$\begin{cases} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial v}{\partial x_j}) = \mathrm{div}(f) & \text{in } B, \\ v = 0 & \text{on } \partial B \end{cases}$$

[1]this condition is needed for both energy monotonicity inequalities for u in dimensions $n \geq 3$ and the piecewise $C^{1,\alpha}$-regularity of u.

[2]Here the covering map $\Pi : \widetilde{N} \to N$ is a Riemannian submersion from $(\widetilde{N}, \widetilde{h})$ to (N, h).

enjoys the $W^{1,p}$-estimate: for any $1 < p < +\infty$,

$$\left\|\nabla v\right\|_{L^p(B)} \le C\left\|f\right\|_{L^p(B)}$$

provided that $(a_{ij}) \in C\left(\overline{B^{\pm}}\right) \cap C\left(B^{\delta}\right)$ for some $\delta > 0$ is uniformly elliptic, and is discontinuous on $\partial B^+ \setminus B^{\delta}$, where $B^{\delta} = \left\{x \in B : \mathrm{dist}(x, \partial B) \le \delta\right\}$.

This fact follows from a recent theorem by Byun-Wang [3], see §3 below. The third one is to employ the moving frame method to establish a decay estimate in suitable Morrey spaces under a smallness condition, which is similar to [14]. To obtain Lipschitz and piecewise $C^{1,\alpha}$-regularity, we compare the harmonic map system with an elliptic system with piecewise constant coefficients and extend the hole-filling argument by Giaquinta-Hildebrandt [10].

The paper is organized as follows. In §2, we derive an almost monotonicity inequality for the renormalized energy. In §3, we show the global $W^{1,p}$ $(1 < p < \infty)$ estimate for elliptic systems with certain piecewise continuous coefficients, and a Hodge decomposition theorem. In §4, we adapt the moving frame method, due to Hélein [12] and Bethuel [2], to establish an ϵ-Hölder continuity. In §5, we establish both Lipschitz and piecewise $C^{1,\alpha}$ regularity for Hölder continuous harmonic maps. In §6, we consider harmonic maps into manifolds supporting convex distance square functions and prove Theorem 1.2.

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2 Energy monotonicity inequality

This section is devoted to the derivation of energy monotonicity inequalities for stationary harmonic maps from (Ω, g) to (N, h). More precisely, we have

Theorem 2.1. *Under the same assumption as in Theorem 1.1, there exist* $C > 0$ *and* $r_0 > 0$ *depending only on* Γ *and* g *such that if* $u \in W^{1,2}(\Omega, N)$ *is a stationary harmonic map on* (Ω, g)*, then for any* $x_0 \in \Omega$*, there holds*

$$s^{2-n}\int_{B_s(x_0)} \left|\nabla u\right|_g^2 dv_g \le e^{Cr} r^{2-n}\int_{B_r(x_0)} \left|\nabla u\right|_g^2 dv_g \tag{2.1}$$

for all $0 < s \le r \le \min\{r_0, \mathrm{dist}(x_0, \partial\Omega)\}$.

Since the metric $g \in C^{0,1}(\Omega^{\pm})$, it is well-known that there are $K > 0$ and $r_0 > 0$ such that (2.1) holds for any $x_0 \in \Omega^{\pm}$ and $0 < s \le r \le \min\{r_0, \mathrm{dist}(x_0, \partial\Omega^{\pm})\}$. In particular, (2.1) holds for any $x_0 \in \Omega \setminus \Gamma^{r_0}$ and $0 < s \le r \le \min\{r_0, \mathrm{dist}(x_0, \partial\Omega)\}$, where

$\Gamma^{r_0} = \{x \in \Omega : \ \mathrm{dist}(x, \Gamma) \le r_0\}$. We will see that to show (2.1) for $x_0 \in \Gamma^{r_0}$, it suffices to consider the case $x_0 \in \Gamma$.

It follows from the assumption on Γ and g, there exists $r_0 > 0$ such that for any $x_0 \in \Gamma$ there exists a $C^{1,1}$-differmorphism $\Phi_0 : B_1 \to B_{r_1}(x_0)$, where $r_1 = \min\{r_0, \mathrm{dist}(x_0, \partial\Omega)\}$, such that

$$\begin{cases} \Phi_0(B_1^{\pm}) = \Omega^{\pm} \cap B_{r_1}(x_0) \\ \Phi_0(\Gamma_1) = \Gamma \cap B_{r_1}(x_0), \text{ where } \Gamma_1 = \{x \in B_1 : x_n = 0\}. \end{cases}$$

Define $\widetilde{u}(x) = u(\Phi_0(x))$ and $\widetilde{g}(x) = (\Phi_0)_*(g)(x), \ x \in B_1$. Then it is readily seen that
(i) $\widetilde{g}$ is piecewise $C^{0,1}$, with the discontinuous set Γ_1, and satisfies (1.4) on Γ_1 [3],
(ii) $\widetilde{u} : (B_1, \ \widetilde{g}) \to (N, \ h)$ is a stationary harmonic map, if $u : (B_{r_1}(x_0), \ g) \to (N, \ h)$ is a stationary harmonic map.

Thus we may assume that $\Omega = B_1$, g is a piecewise $C^{0,1}$-metric which satisfies (1.4) on the set of discontinuity Γ_1, and $u : (B_1, g) \to (N, h)$ is a stationary harmonic map. It suffices to establish (2.1) in $B_{\frac{1}{2}}$. We first derive a stationarity identity for u.

Proposition 2.2. *Let $u \in W^{1,2}(B_1, N)$ be a stationary harmonic map on (B_1, g). Then*

$$\int_{B_1} \left(2g^{ij}\langle \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_j}\rangle Y_i^k - |\nabla u|_g^2 \mathrm{div} Y\right) \sqrt{g}\, dx = \int_{B_1} \frac{\partial}{\partial x_k}\Big(\sqrt{g} g^{ij}\Big) Y^k \langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\rangle\, dx \quad (2.2)$$

holds for all $Y = (Y^1, \cdots, Y^{n-1}, Y^n) \in C_0^1(B_1, \mathbb{R}^n)$ satisfying

$$Y^n(x) \begin{cases} \ge 0 & \text{for } x_n > 0 \\ = 0 & \text{for } x_n = 0 \\ \le 0 & \text{for } x_n < 0, \end{cases} \quad (2.3)$$

where $Y_i^k = \dfrac{\partial Y^k}{\partial x_i}$ and $\mathrm{div}\ Y = \displaystyle\sum_{i=1}^n \frac{\partial Y^i}{\partial x_i}$.

Proof. Let Y satisfy (2.3), it is easy to see that there exists $\delta > 0$ such that $F_t(x) = x + tY(x), \ t \in [-\delta, \delta]$, is a family of differmorphisms from B_1 to B_1 satisfying the condition (1.3). Hence

$$0 = \frac{d}{dt}\Big|_{t=0} \int_{B_1} |\nabla(u(F_t(x))|_g^2\, dv_g = \frac{d}{dt}\Big|_{t=0} \Big(\int_{B_1^+} |\nabla(u(F_t(x))|_g^2\, dv_g + \int_{B_1^-} |\nabla(u(F_t(x))|_g^2\, dv_g \Big).$$

[3]In fact, since $(\Phi_0)_*(g)_{ij}(x) = g_{kl}(\Phi_0(x)) \dfrac{\partial \Phi_0^k}{\partial x_i}(x) \dfrac{\partial \Phi_0^l}{\partial x_j}(x)$, (1.4) implies that for any $x \in \Gamma_1$

$$\lim_{y \in \Omega^+, y \to x} (\Phi_0)_* g(y) = k(\Phi_0(x)) \lim_{y \in \Omega^-, y \to x} (\Phi_0)_* g(y).$$

For $t \in [-\delta, \delta]$, set $G_t = F_t^{-1}$. Direct calculations yield

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \int_{B_1^\pm} |\nabla(u(F_t(x))|_g^2 \, dv_g \\
= &\frac{d}{dt}\Big|_{t=0} \int_{B_1^\pm} \sqrt{g(x)} g^{ij}(x) \langle \frac{\partial u}{\partial y_k}, \frac{\partial u}{\partial y_l} \rangle (x + tY(x))(\delta_{ki} + tY_i^k)(\delta_{lj} + tY_j^l) \, dx \\
= &\int_{B_1^\pm} \sqrt{g} g^{ij} \langle \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_l} \rangle (\delta_{ki} Y_j^l + \delta_{lj} Y_i^k) \, dx \\
&+ \int_{B_1^\pm} \frac{d}{dt}\Big|_{t=0} \Big(g^{ij}(G_t(x)) \sqrt{g(G_t(x))} JG_t(x) \Big) \langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \rangle \, dx \\
= &\int_{B_1^\pm} \Big(2g^{ij} \langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_l} \rangle Y_j^l - g^{ij} \langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \rangle \mathrm{div} Y \Big) \sqrt{g} \, dx \\
&- \int_{B_1^\pm} \frac{\partial}{\partial x_k} \Big(\sqrt{g} g^{ij} \Big) Y^k \langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \rangle \, dx,
\end{aligned}$$

where we have used

$$\begin{cases}
\frac{d}{dt}\Big|_{t=0} JG_t(x) = -\mathrm{div} Y, \\
\frac{d}{dt}\Big|_{t=0} G_t(x) = -Y(x), \\
\frac{d}{dt}\Big|_{t=0} \Big(g^{ij}(G_t(x)) \sqrt{g(G_t(x))} \Big) = -\frac{\partial}{\partial x_k} \Big(\sqrt{g} g^{ij} \Big) Y^k.
\end{cases}$$

This completes the proof. □

Proposition 2.3. *Let $u \in W^{1,2}(B_1, N)$ be a stationary harmonic map on (B_1, g). Then there exists $C > 0$ such that*

(i) for any $x^0 = (x_0^{'}, x_0^n) \in B_{\frac{1}{2}} \setminus \Gamma_1$, there exists $0 < R_0 \leq \min\{\frac{1}{4}, |x_0^n|\}$, such that

$$r^{2-n} \int_{B_r(x^0)} |\nabla u|_g^2 dv_g \leq e^{CR} R^{2-n} \int_{B_R(x^0)} |\nabla u|_g^2 dv_g, \qquad 0 < r \leq R < R_0. \tag{2.4}$$

(ii) for any $x^0 \in B_{\frac{1}{2}} \cap \Gamma_1$, there holds

$$r^{2-n} \int_{B_r(x^0)} |\nabla u|_g^2 dv_g \leq e^{CR} R^{2-n} \int_{B_R(x^0)} |\nabla u|_g^2 dv_g, \qquad 0 < r \leq R \leq \frac{1}{4}. \tag{2.5}$$

In particular, for any $x^0 \in B_{\frac{1}{2}}$, there holds

$$r^{2-n} \int_{B_r(x^0)} |\nabla u|_g^2 dv_g \leq e^{CR} R^{2-n} \int_{B_R(x^0)} |\nabla u|_g^2 dv_g, \quad 0 < r \leq R \leq \frac{1}{4}. \tag{2.6}$$

Proof. (i) By choosing $Y \in C_c^\infty(B_1^+, \mathbb{R}^n)$ or $Y \in C_c^\infty(B_1^-, \mathbb{R}^n)$, we have that u is a stationary harmonic map on (B_1^+, g) and (B_1^-, g). Thus the monotonicity inequality (2.4) is standard.

(ii) For simplicity, consider $x^0 = (0', 0)$. For $\epsilon > 0$ and $0 < r \le \frac{1}{2}$, let $Y_\epsilon(x) = x\eta_\epsilon(x)$, where $\eta_\epsilon(x) = \eta_\epsilon(|x|) \in C_0^\infty(B_1)$ satisfies

$$0 \le \eta_\epsilon \le 1;\ \eta_\epsilon(s) \equiv 1 \text{ for } 0 \le s \le r - \epsilon;\ \eta_\epsilon(s) \equiv 0 \text{ for } s \ge r; \eta'_\epsilon \le 0;\ |\eta'_\epsilon| \le \frac{2}{\epsilon}.$$

Then

$$(Y_\epsilon)_i^j = \delta_{ij}\eta_\epsilon(|x|) + \eta'_\epsilon(|x|)\frac{x^i x^j}{|x|}. \tag{2.7}$$

Substituting Y_ϵ into the right hand side of (2.2), and using

$$\Big|\frac{\partial}{\partial x_k}\Big(\sqrt{g}g^{ij}\Big)\Big| \le C,$$

we have

$$\Big|\int_{B_1}\frac{\partial}{\partial x_k}\Big(\sqrt{g}g^{ij}\Big)Y_\epsilon^k\langle\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\rangle\,dx\Big| \le Cr\int_{B_r}|\nabla u|^2\,dx \le Cr\int_{B_r}|\nabla u|_g^2\,dv_g. \tag{2.8}$$

Substituting (2.7) into the left hand side of (2.2), we obtain

$$\begin{aligned}
&\int_{B_1}\Big(2g^{ij}\langle\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_k}\rangle(Y_\epsilon)_i^k - |\nabla u|_g^2 \mathrm{div} Y_\epsilon\Big)\sqrt{g}\,dx\\
&= (2-n)\int_{B_1}|\nabla u|_g^2\eta_\epsilon(x)\sqrt{g}\,dx - \int_{B_1}|\nabla u|_g^2|x|\eta'_\epsilon(x)\sqrt{g}\,dx\\
&\ + \int_{B_1}2g^{ij}\langle\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k}\rangle\frac{x^k x^j}{|x|}\eta'_\epsilon(x)\sqrt{g}\,dx.
\end{aligned} \tag{2.9}$$

Set the piecewise constant metric $\overline{g}$ by

$$\overline{g}(x', x^n) = \begin{cases}\lim\limits_{y\to 0,\ y^n\ge 0} g(y) & \text{if } x^n \ge 0\\ \lim\limits_{y\to 0,\ y^n< 0} g(y) & \text{if } x^n < 0.\end{cases}$$

Then we have

$$|g(x) - \overline{g}(x)| \le C|x|,\ \forall x \in B_1. \tag{2.10}$$

It follows from (1.4) that we can assume

$$\overline{g}(x) = \begin{cases} g_0 & \text{if } x^n \ge 0\\ kg_0 & \text{if } x^n < 0,\end{cases}$$

for some positive constant $k \ne 1$. Thus we can estimate

$$\begin{aligned}
&\int_{B_1}2g^{ij}\langle\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k}\rangle\frac{x^k x^j}{|x|}\eta'_\epsilon(x)\sqrt{g}\,dx\\
= &\ 2\int_{B_1}\overline{g}^{ij}\langle\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k}\rangle\frac{x^k x^j}{|x|}\eta'_\epsilon(x)\sqrt{g}\,dx + 2\int_{B_1}(g^{ij} - \overline{g}^{ij})\langle\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_k}\rangle\frac{x^k x^j}{|x|}\eta'_\epsilon(x)\sqrt{g}\,dx\\
= &\ I_\epsilon + II_\epsilon.
\end{aligned} \tag{2.11}$$

Since

$$\overline{g}^{ij}\langle\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial x_k}\rangle\frac{x^k x^j}{|x|}\equiv h(x):=\begin{cases}|x||\frac{\partial u}{\partial r}|^2 & \text{if } x^n\geq 0\\ \frac{1}{k}|x||\frac{\partial u}{\partial r}|^2 & \text{if } x^n<0,\end{cases}$$

and $h(x)\geq 0$ for $x\in B_1$, we have

$$I_\epsilon=\int_{B_1}h(x)\eta_\epsilon'(|x|)\sqrt{g}\,dx\leq 0. \tag{2.12}$$

For II_ϵ, by (2.10) we have

$$\Big|II_\epsilon\Big|\leq Cr\int_{B_r}|\nabla u|^2\,dv_g\leq Cr\int_{B_r}|\nabla u|_g^2\,dv_g. \tag{2.13}$$

First substituting (2.12) and (2.13) into (2.11), and then plugging the resulting (2.11) into (2.9), and finally combining (2.9) and (2.8) with (2.2), we obtain, after sending ϵ to zero,

$$(2-n)\int_{B_r}|\nabla u|_g^2dv_g+r\int_{\partial B_r}|\nabla u|_g^2\,\sqrt{g}\,dH^{n-1}\geq -Cr\int_{B_r}|\nabla u|_g^2dv_g.$$

This implies

$$\frac{d}{dr}\left(e^{Cr}r^{2-n}\int_{B_r}|\nabla u|_g^2dv_g\right)\geq 0,$$

which clearly yields (2.5).

To show (2.6), it suffices to consider the case

$$x^0\in B_{1/2}\setminus\Gamma_1,\quad |B_R(x^0)\cap B_1^+|>0 \text{ and } |B_R(x^0)\cap B_1^-|>0.$$

For simplicity, assume $x^0\in B_1^-$. We divide it into two cases:
(i) $d(x^0,\Gamma_1)=|x_n^0|\geq\frac{1}{4}R$:

- If $R\geq r\geq\frac{1}{4}R$, then it is easy to see

$$r^{2-n}\int_{B_r(x^0)}|\nabla u|_g^2dv_g\leq\ 4^{n-2}R^{2-n}\int_{B_R(x^0)}|\nabla u|_g^2dv_g.$$

- If $0<r<\frac{1}{4}R(\leq d(x^0,\Gamma_1))$, we have $B_{\frac{R}{4}}(x^0)\subset B_1^-$ so that (2.4) implies

$$r^{2-n}\int_{B_r(x^0)}|\nabla u|_g^2dv_g\leq e^{CR}\left(\frac{R}{4}\right)^{2-n}\int_{B_{\frac{R}{4}}(x^0)}|\nabla u|_g^2dv_g\leq\ e^{CR}4^{n-2}R^{2-n}\int_{B_R(x^0)}|\nabla u|_g^2dv_g.$$

(ii) $d(x^0,\Gamma_1)=|x_n^0|<\frac{1}{4}R$:

- If $R\geq r\geq\frac{1}{4}R$, then

$$r^{2-n}\int_{B_r(x^0)}|\nabla u|_g^2dv_g\leq\ 4^{n-2}R^{2-n}\int_{B_R(x^0)}|\nabla u|_g^2dv_g.$$

- If $0 < r \leq d(x^0, \Gamma_1) = |x_n^0| < \frac{1}{4}R$, then by setting $\overline{x}^0 = (x_1^0, \cdots, x_{n-1}^0, 0)$ we have

$$B_r(x^0) \subset B_{|x_n^0|}(x^0) \subset B_{2|x_n^0|}(\overline{x}^0) \subset B_{\frac{R}{2}}(\overline{x}^0) \subset B_R(x^0)$$

so that (2.5) yields

$$\begin{aligned} r^{2-n} \int_{B_r(x^0)} |\nabla u|_g^2 dv_g &\leq |x_n^0|^{2-n} \int_{B_{|x_n^0|}(x^0)} |\nabla u|_g^2 dv_g \\ &\leq 2^{n-2} (2|x_n^0|)^{2-n} \int_{B_{2|x_n^0|}(\overline{x}^0)} |\nabla u|_g^2 dv_g \\ &\leq 2^{n-2} e^{CR} \left(\frac{R}{2}\right)^{2-n} \int_{B_{\frac{R}{2}}(\overline{x}^0)} |\nabla u|_g^2 dv_g \\ &\leq e^{CR} R^{2-n} \int_{B_R(x^0)} |\nabla u|_g^2 dv_g. \end{aligned}$$

- If $d(x^0, \Gamma_1)(= |x_n^0|) \leq r < \frac{1}{4}R$, then we have

$$B_r(x^0) \subset B_{2r}(\overline{x}^0) \subset B_{\frac{R}{2}}(\overline{x}^0) \subset B_R(x^0),$$

so that (2.5) yields

$$\begin{aligned} r^{2-n} \int_{B_r(x^0)} |\nabla u|_g^2 dv_g &\leq 2^{n-2} (2r)^{2-n} \int_{B_{2r}(\overline{x}^0)} |\nabla u|_g^2 dv_g \\ &\leq 2^{n-2} e^{CR} \left(\frac{R}{2}\right)^{2-n} \int_{B_{\frac{R}{2}}(\overline{x}^0)} |\nabla u|_g^2 dv_g \\ &\leq e^{CR} R^{2-n} \int_{B_R(x^0)} |\nabla u|_g^2 dv_g. \end{aligned}$$

Therefore (2.6) is proven. □

3 $W^{1,p}$-estimate for elliptic equations with certain piecewise continuous coefficients

In this section, we will show the global $W^{1,p}$-estimate for elliptic equations with certain piecewise continuous coefficients, for $1 < p < +\infty$. As a corollary, we will establish the Hodge decomposition Theorem 3.2 for certain piecewise continuous metrics g, which is a key ingredient to prove Theorem 1.1 and may also have its own interest.

For a ball $B = B_r(0) \subset \mathbb{R}^n$, denote $B^\epsilon = \{x \in B : \operatorname{dist}(x, \partial B) \leq \epsilon\}$ for $\epsilon > 0$. Let $(a_{ij}(x))_{1 \leq i,j \leq n}$ be bounded measurable, uniformly elliptic on B, i.e., there exists $0 < \lambda \leq \Lambda < +\infty$ such that

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i^\alpha \xi_\beta^j \leq \Lambda |\xi|^2, \quad \text{a.e. } x \in B, \ \forall \xi \in \mathbb{R}^n. \tag{3.1}$$

Theorem 3.1. *Assume (a_{ij}) satisfies (3.1), and there exists $\epsilon > 0$ such that $(a_{ij}) \in C\left(\overline{B^{\pm}}\right) \cap C\left(B^{\epsilon}\right)$ and is discontinuous on $\partial B^+ \setminus B^{\epsilon}$. For $1 < p < +\infty$, let $f \in L^p(B, \mathbb{R}^n)$. Then there exists a unique weak solution $v \in W_0^{1,p}(B, \mathbb{R}^n)$ to*

$$\begin{cases} \sum\limits_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) = \sum\limits_{i} \frac{\partial f_i}{\partial x_i} & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.2}$$

and

$$\|\nabla v\|_{L^p(B)} \le C\,\|f\|_{L^p(B)} \tag{3.3}$$

for some $C > 0$ depending only on p and (a_{ij}).

Proof. By our assumption, it is easy to verify that for any $\delta > 0$, there exists $R = R(\delta) > 0$ such that the coefficient function (a_{ij}) satisfies the (δ, R)-vanishing of codimension 1 conditions (2.5) and (2.6) of Byun-Wang [3] page 2652. In fact, we have a stronger property:

$$\lim_{r\downarrow 0}\ \max_{x_0=(x_0',x_0^n)\in\overline{B}} \left\|a_{ij}(x',x^n) - a_{ij}(x_0',x^n)\right\|_{L^\infty\left(B_r((x_0',x_0^n))\right)} = 0.$$

Thus the conclusion of Theorem 3.1 follows by direct application of [3] Theorem 2.2, page 2653. □

As an immediate consequence of Theorem 3.1, we have the following Hodge decomposition on B equipped with suitable piecewise continuous metrics g.

Theorem 3.2. *Let $\bar{g}$ be a piecewise continuous metric on B such that $\bar{g} \in C\left(\overline{B^{\pm}}\right) \cap C\left(B^{\delta}\right)$ for some $\delta > 0$, and is discontinuous on $\partial B^+ \setminus B^{\delta}$. Then for any $1 < p < +\infty$, $F = (F_1, \cdots, F_n) \in L^p(B, \mathbb{R}^n)$, there exist $G \in W_0^{1,p}(B)$ and $H \in L^p(B, \mathbb{R}^n)$ such that*

$$F = \nabla G + H,\ \ 0 = \mathrm{div}_{\bar{g}} H\ (:= \frac{1}{\sqrt{\bar{g}}}\frac{\partial}{\partial x_i}(\sqrt{\bar{g}}\bar{g}^{ij}H_j))\text{ in } B, \tag{3.4}$$

and there exists $C = C(p, n, \bar{g}) > 0$ such that

$$\|\nabla G\|_{L^p(B)} + \|H\|_{L^p(B)} \le C\,\|F\|_{L^p(B)}\,. \tag{3.5}$$

Proof. Set $a_{ij} = \sqrt{\bar{g}}\bar{g}^{ij}$ on B for $1 \le i, j \le n$. It is easy to verify that (a_{ij}) satisfies the conditions of Theorem 3.1. Thus Theorem 3.1 yields that there exists a unique solution $G \in W_0^{1,p}(B)$ to

$$\begin{cases} \frac{\partial}{\partial x_i}\left(\sqrt{\bar{g}}\bar{g}^{ij}\frac{\partial G}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\sqrt{\bar{g}}\bar{g}^{ij}F_j\right), & \text{in } B \\ G = 0 & \text{on } \partial B, \end{cases} \tag{3.6}$$

and

$$\|\nabla G\|_{L^p(B)} \le C\left\|\sqrt{\bar{g}}\bar{g}^{ij}F_j\right\|_{L^p(B)} \le C\,\|F\|_{L^p(B)}\,.$$

Set $H = F - \nabla G$. Then we have

$$\mathrm{div}_{\bar g} H = \frac{1}{\sqrt{\bar g}} \frac{\partial}{\partial x_i}\left(\sqrt{\bar g}\bar g^{ij}\left(F_j - \frac{\partial G}{\partial x_j}\right)\right) = 0 \text{ on } B,$$

and

$$\|H\|_{L^p(B_{\frac12})} \le \|F\|_{L^p(B_{\frac12})} + \|\nabla G\|_{L^p(B)} \le C\,\|F\|_{L^p(B)}\,.$$

This completes the proof. □

4 Hölder continuity

In this section, we will prove that any stationary harmonic maps on (B_1, g), with a piecewise Lipschitz continuous metric $g \in C^{0,1}(B_1^\pm \cup \Gamma_1)$, is Hölder continuous under a smallness condition of $\int_{B_1} |\nabla u|_g^2\,dv_g$. The idea is based on suitable modifications of the original argument by Bethuel [2] (see also Ishizuka-Wang [14]), thanks to the energy monotonicity inequality and the Hodge decomposition theorem established in previous sections. More precisely, we have

Theorem 4.1. *There exist $\epsilon_0 > 0$ and $\alpha_0 \in (0,1)$ depending only on n, g such that if the metric $g \in C^{0,1}(B_1^\pm \cup \Gamma_1)$ satisfies the condition (1.4) on Γ_1, and $u \in W^{1,2}(B_1, N)$ is a stationary harmonic map on (B_1, g) satisfying*

$$r_0^{2-n} \int_{B_{r_0}(x_0)} |\nabla u|_g^2\,dv_g \le \epsilon_0^2 \tag{4.1}$$

for some $x_0 \in B_{\frac12}$ and $0 < r_0 \le \frac14$, then $u \in C^{\alpha_0}(B_{\frac{r_0}{2}}(x_0), N)$ and

$$\Big[u\Big]_{C^{\alpha_0}(B_{\frac{r_0}{2}}(x_0))} \le C(r_0, \epsilon_0). \tag{4.2}$$

Proof of Theorem 4.1. The proof is based on suitable modifications of [2] and [14]. First, observe that if $x_0 = (x_0', x_0^n) \in B^\pm$, it follows from the monotonicity inequality (2.6) that we may assume (4.1) holds for some $0 < r_0 < |x_0^n|$. Then the ϵ_0-regularity theorem by Bethuel [2] (see [14] for domains with $C^{0,1}$ metrics) implies that for some $0 < \alpha_0 < 1$, $u \in C^{\alpha_0}(B_{\frac{r_0}{2}}(x_0))$ and (4.2) holds. Hence it suffices to consider the case $x_0 = (x_0', 0) \in \Gamma_{\frac12}$. By translation and scaling, we may assume $x_0 = (0,0)$ and proceed as follows.

Step 1. As in [2] [12] [14], assume that there exists an orthonormal frame on $u^*TN\Big|_{B_1}$. For $0 < \theta < \frac12$ to be determined later, let $\{e_\alpha\}_{\alpha=1}^l \subset W^{1,2}(B_{2\theta}, \mathbb{R}^k)$ be a Coulomb gauge orthonormal frame of $u^*TN\Big|_{B_{2\theta}}$:

$$\begin{cases} \mathrm{div}_g(\langle \nabla e_\alpha, e_\beta\rangle) = 0 \quad \text{in } B_{2\theta} \quad (1 \le \alpha, \beta \le l), \\ \sum\limits_{\alpha=1}^{l} \int_{B_{2\theta}} |\nabla e_\alpha|_g^2 dv_g \le C \int_{B_{2\theta}} |\nabla u|_g^2 dv_g. \end{cases} \tag{4.3}$$

For $1 \le \alpha \le l$, consider $\langle \nabla ((u - u_{2\theta})\eta), e_\alpha \rangle$, where $u_{2\theta} = \fint_{B_{2\theta}} u$ is the average of u on $B_{2\theta}$, and $\eta \in C_0^\infty(B_1)$ satisfies

$$0 \le \eta \le 1; \quad \eta = 1 \text{ in } B_\theta; \quad \eta = 0 \text{ outside } B_{\frac{7}{4}\theta}; \quad |\nabla \eta| \le \frac{2}{\theta}.$$

Let g_0 be the standard metric on $\mathbb{R}^n$. We define a new metric $\widetilde{g}$ on $B_{2\theta}$ by letting

$$\widetilde{g}(x) = \eta(x)g(x) + (1 - \eta(x))g_0(x), \ x \in B_{2\theta}.$$

Then it is easy to see that

$$\widetilde{g} \equiv g \text{ on } B_\theta, \ \widetilde{g} \equiv g_0 \text{ outside } B_{\frac{7}{4}\theta}, \text{ and } \widetilde{g} \in C(\overline{B_{2\theta}^{\pm}}) \cap C(B_{2\theta} \setminus B_{\frac{7}{4}\theta}).$$

In particular, $\widetilde{g}$ satisfies the condition of Theorem 3.2. Hence, by Theorem 3.2, we have that for $1 < p < \frac{n}{n-1}$, there exist $\phi_\alpha \in W_0^{1,p}(B_{2\theta})$ and $\psi_\alpha \in L^p(B_{2\theta})$ such that

$$\begin{cases} \langle \nabla ((u - u_{2\theta})\eta), e_\alpha \rangle = \nabla \phi_\alpha + \psi_\alpha, \quad \operatorname{div}_{\widetilde{g}}(\psi_\alpha) = 0 \ \text{ in } B_{2\theta}, \\ \|\nabla \phi_\alpha\|_{L^p(B_{2\theta})} + \|\psi_\alpha\|_{L^p(B_{2\theta})} \lesssim \|\nabla ((u - u_{2\theta})\eta)\|_{L^p(B_{2\theta})} \lesssim \|\nabla u\|_{L^p(B_{2\theta})}. \end{cases} \tag{4.4}$$

Since u satisfies the harmonic map equation (1.1), we have

$$\operatorname{div}_g (\langle \nabla u, e_\alpha \rangle) = g^{ij} \nabla_i u \langle \nabla_j e_\alpha, e_\beta \rangle \, e_\beta \quad \text{in } B_{2\theta}. \tag{4.5}$$

Thus we obtain

$$\Delta_g \phi_\alpha = g^{ij} \nabla_i u \langle \nabla_j e_\alpha, e_\beta \rangle \, e_\beta \quad \text{in } B_\theta. \tag{4.6}$$

Set $\phi_\alpha = \phi_\alpha^{(1)} + \phi_\alpha^{(2)}$, where $\phi_\alpha^{(1)}$ solves

$$\begin{cases} \Delta_g \phi_\alpha^{(1)} = 0, & \text{in } B_\theta, \\ \phi_\alpha^{(1)} = \phi_\alpha, & \text{on } \partial B_\theta, \end{cases} \tag{4.7}$$

and $\phi_\alpha^{(2)}$ solves

$$\begin{cases} \Delta_g \phi_\alpha^{(2)} = g^{ij} \nabla_i u \langle \nabla_j e_\alpha, e_\beta \rangle \, e_\beta, & \text{in } B_\theta, \\ \phi_\alpha^{(2)} = 0, & \text{on } \partial B_\theta. \end{cases} \tag{4.8}$$

Step 2. Estimation of $\phi_\alpha^{(1)}$: It is well-known (cf. [11]) that $\phi_\alpha^{(1)} \in C^{\alpha_0}(B_\theta)$ for some $\alpha_0 \in (0, 1)$, and for any $0 < r \le \frac{\theta}{2}$

$$\left[\phi_\alpha^{(1)}\right]^p_{C^{\alpha_0}(B_{\frac{r}{2}})} \lesssim \theta^{p-n} \int_{B_\theta} |\nabla \phi_\alpha^{(1)}|^p \, dx \le C\theta^{p-n} \int_{B_{2\theta}} |\nabla u|^p \, dx, \tag{4.9}$$

and

$$(\tau\theta)^{p-n} \int_{B_{\tau\theta}} |\nabla \phi_\alpha^{(1)}|^p \le C\tau^{p\alpha} \left\| \nabla u \right\|_{M^{p,p}(B_1)}, \ \forall 0 < \tau < 1, \tag{4.10}$$

where $M^{p,p}(\cdot)$ denotes the Morrey space:

$$M^{p,p}(E) := \left\{ f : E \to \mathbb{R} : \|f\|^p_{M^{p,p}(E)} = \sup_{B_r(x) \subset \mathbb{R}^n} \left\{ r^{p-n} \int_{B_r(x) \cap E} |f|^p \, dx \right\} < +\infty \right\}, \ E \subset \mathbb{R}^n.$$

Step 3. Estimation of $\phi_\alpha^{(2)}$: First, denote by $\mathcal{H}^1(\mathbb{R}^n)$ the Hardy space on $\mathbb{R}^n$ and BMO(E) the BMO space on E for any open set $E \subset \mathbb{R}^n$. By (4.13) of [14] page 435, for $p' = \dfrac{p}{p-1} > n$, there exists $h \in W_0^{1,p'}(B_\theta)$, with $\|\nabla h\|_{L^{p'}(B_\theta)} = 1$, such that

$$\left\|\nabla \phi_\alpha^{(2)}\right\|_{L^p(B_\theta)} \leq C \int_{B_\theta} \langle \nabla \phi_\alpha^{(2)}, \nabla h\rangle_g dv_g.$$

Hence by the equation (4.8), (4.4), and the duality between $\mathcal{H}^1$ and BMO, we have

$$\begin{aligned}
\left\|\nabla \phi_\alpha^{(2)}\right\|_{L^p(B_\theta)} &\leq C \int_{B_\theta} \sqrt{g} g^{ij} \langle \nabla_i u \langle \nabla_j e_\alpha, e_\beta\rangle\rangle (e_\beta h)\, dx \\
&= -C \int_{B_\theta} \sqrt{g} g^{ij} \langle \nabla_j e_\alpha, e_\beta\rangle\rangle \nabla_i (e_\beta h) u\, dx \\
&\leq C \left\| \sqrt{g} g^{ij} \langle \nabla_j e_\alpha, e_\beta\rangle\rangle \nabla_i (e_\beta h)\right\|_{\mathcal{H}^1(\mathbb{R}^n)} \big[u\big]_{\mathrm{BMO}(B_{2\theta})} \\
&\lesssim \| \sqrt{g} g^{ij} \langle \nabla_j e_\alpha, e_\beta\rangle\|_{L^2(B_\theta)} \|\nabla (e_\beta h)\|_{L^2(B_\theta)} \, [u]_{\mathrm{BMO}(B_{2\theta})} \\
&\lesssim \|\nabla u\|_{L^2(B_{2\theta})} \|\nabla u\|_{M^{p,p}(B_1)} \cdot \theta^{\frac{n}{p} - \frac{n}{2}},
\end{aligned} \tag{4.11}$$

where we have used:
(i) Since $\mathrm{div}_g(\langle \nabla e_\alpha, e_\beta\rangle) = 0$ in B_θ and $h \in W_0^{1,p'}(B_\theta)$, we have $\sqrt{g} g^{ij} \langle \nabla_j e_\alpha, e_\beta\rangle\rangle \nabla_i (e_\beta h) \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\left\| \sqrt{g} g^{ij} \langle \nabla_j e_\alpha, e_\beta\rangle \nabla_i (e_\beta h)\right\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \left\| \sqrt{g} g^{ij} \langle \nabla_j e_\alpha, e_\beta\rangle\right\|_{L^2(B_\theta)} \left\|\nabla_i (e_\beta h)\right\|_{L^2(B_\theta)},$$

(ii) Since $p' > n$, the Sobolev embedding implies $h \in C^{1-\frac{n}{p'}}(B_\theta)$ and

$$\|h\|_{L^\infty(B_\theta)} \leq C \theta^{1-\frac{n}{p'}},$$

so that

$$\|\nabla (e_\beta h)\|_{L^2(B_\theta)} \leq \|\nabla e_\beta\|_{L^2(B_\theta)} \|h\|_{L^\infty(B_\theta)} + \|\nabla h\|_{L^p(B_\theta)} \theta^{\frac{n}{p} - \frac{n}{2}} \leq C \theta^{\frac{n}{p} - \frac{n}{2}},$$

(iii) By Poincaré inequality, it holds

$$[u]_{\mathrm{BMO}(B_{2\theta})} \leq C \|\nabla u\|_{M^{p,p}(B_1)}.$$

Putting the estimates of $\phi_\alpha^{(1)}$ and $\phi_\alpha^{(2)}$ together, we obtain

$$\left((\tau\theta)^{p-n} \int_{B_{\tau\theta}} |\nabla \phi_\alpha|^p dx \right)^{\frac{1}{p}} \leq C \left[\tau^{\alpha_0} + \tau^{1-\frac{n}{p}} \epsilon_0 \right] \|\nabla u\|_{M^{p,p}(B_1)}, \ \forall 0 < \tau < 1. \tag{4.12}$$

Step 4. Estimation of ψ_α: Since $\mathrm{div}_{\widetilde{g}}(\psi_\alpha) = 0$ on $B_{2\theta}$, we have

$$\begin{aligned}
\int_{B_{2\theta}} |\psi_\alpha|^2_{\widetilde{g}} dv_{\widetilde{g}} &= \int_{B_{2\theta}} \langle(\psi_\alpha + \nabla\phi_\alpha), \psi_\alpha\rangle_{\widetilde{g}}\, dv_{\widetilde{g}} \\
&= \int_{B_{2\theta}} \langle\langle\nabla((u - u_{2\theta})\eta), e_\alpha\rangle, \psi_\alpha\rangle_{\widetilde{g}} dv_{\widetilde{g}} \\
&= -\int_{B_{2\theta}} (u - u_{2\theta})\eta\langle\nabla e_\alpha, \psi_\alpha\rangle_{\widetilde{g}} dv_{\widetilde{g}} \\
&\lesssim \left\| \sqrt{\widetilde{g}}\, \widetilde{g}^{ij}\nabla_i e_\alpha \psi^j_\alpha \right\|_{\mathcal{H}^1} [(u - u_{2\theta})\eta]_{\mathrm{BMO}} \\
&\lesssim \|\psi_\alpha\|_{L^2(B_{2\theta})} \|\nabla e_\alpha\|_{L^2(B_{2\theta})} \left[(u - u_{2\theta})\eta\right]_{\mathrm{BMO}} \\
&\lesssim \|\nabla u\|_{L^2(B_{2\theta})} \|\psi_\alpha\|_{L^2(B_{2\theta})} \|\nabla u\|_{M^{p,p}(B_1)},
\end{aligned}$$

where we have used the fact

$$\left[(u - u_{2\theta})\eta\right]_{\mathrm{BMO}} \le C\,[u]_{\mathrm{BMO(B_{2\theta})}} \le C\, \|\nabla u\|_{M^{p,p}(B_1)}\,.$$

This, combined with Hölder's inequality, implies

$$\left(\theta^{p-n} \int_{B_\theta} |\psi_\alpha|^p\right)^{\frac{1}{p}} \le\; C\epsilon_0\, \|\nabla u\|_{M^{p,p}(B_1)}\,. \tag{4.13}$$

Step 5. Decay estimation of ∇u: Putting (4.12) and (4.13) together, we have that for some $0 < \alpha_0 < 1$,

$$\left((\tau\theta)^{p-n} \int_{B_{\tau\theta}} |\nabla u|^p\right)^{\frac{1}{p}} \le C\Big(\epsilon_0 + \tau^{\alpha_0} + \tau^{1-\frac{n}{p}}\epsilon_0\Big)\Big\|\nabla u\Big\|_{M^{p,p}(B_1)} \tag{4.14}$$

holds for any $0 < \tau < 1$ and $0 < \theta < \frac{1}{2}$. Now we claim that for some $\alpha_0 \in (0, 1)$, it holds

$$\|\nabla u\|_{M^{p,p}(B_{\frac{\tau}{4}})} \le C\Big(\epsilon_0 + \tau^{\alpha_0} + \tau^{1-\frac{n}{p}}\epsilon_0\Big)\, \|\nabla u\|_{M^{p,n-p}(B_1)}\,,\ \forall 0 < \tau < 1. \tag{4.15}$$

To show (4.15), let $B_s(y) \subset B_{\frac{\tau}{4}}$. We divide it into three cases:
(a) $y \in B_{\frac{\tau}{4}} \cap B^\pm$ and $s < |y_n|$. As remarked in the begin of proof, we have that for some $0 < \alpha_0 < 1$,

$$\begin{aligned}
\left(s^{p-n} \int_{B_s(y)} |\nabla u|^p\right)^{\frac{1}{p}} &\le C\left(\frac{s}{|y_n|}\right)^{\alpha_0} \left(|y_n|^{p-n} \int_{B_{|y_n|}(y)} |\nabla u|^p\right)^{\frac{1}{p}} \\
&\le C\left(\frac{s}{|y_n|}\right)^{\alpha_0} \left((2|y_n|)^{p-n} \int_{B_{2|y_n|}(y',0)} |\nabla u|^p\right)^{\frac{1}{p}} \\
&\le C\left(\left(\frac{\tau}{2}\right)^{p-n} \int_{B_{\frac{\tau}{2}}(y'0)} |\nabla u|^p\right)^{\frac{1}{p}} \text{ (since } |y_n| \le \frac{\tau}{4}) \\
&\le C(\epsilon_0 + \tau^{\alpha_0} + \tau^{1-\frac{n}{p}}\epsilon_0)\|\nabla u\|_{M^{p,p}(B_1)} \text{ (by (4.14)).}
\end{aligned}$$

(b) $y \in B_{\frac{\tau}{4}} \cap B^{\pm}$ and $s \geq |y_n|$. Then we have $B_s(y) \subset B_{|y_n|+s}(y',0) \subset B_{2s}(y',0)$. Hence

$$\left(s^{p-n}\int_{B_s(y)}|\nabla u|^p\right)^{\frac{1}{p}} \leq 2^{\frac{n-p}{p}}\left((2s)^{p-n}\int_{B_{2s}(y',0)}|\nabla u|^p\right)^{\frac{1}{p}}$$
$$\leq C\Big(\epsilon_0+\tau^{\alpha_0}+\tau^{1-\frac{n}{p}}\epsilon_0\Big)\|\nabla u\|_{M^{p,p}(B_1)} \text{ (by (4.14)).}$$

(c) $y \in B_{\frac{\tau}{4}} \cap \Gamma_1$, i.e. $y_n = 0$. Then it follows directly from (4.14) that

$$\left(s^{p-n}\int_{B_s(y)}|\nabla u|^p\right)^{\frac{1}{p}} \leq C\Big(\epsilon_0+\tau^{\alpha_0}+\tau^{1-\frac{n}{p}}\epsilon_0\Big)\|\nabla u\|_{M^{p,p}(B_1)}\,.$$

Combining (a), (b) and (c) together and taking supremum over all $B_s(y) \subset B_{\frac{\tau}{4}}$, we obtain (4.15).

It is now clear that by first choosing sufficiently small τ and then sufficiently small ϵ_0, we have

$$\|\nabla u\|_{M^{p,p}(B_{\frac{\tau}{4}})} \leq \frac{1}{2}\|\nabla u\|_{M^{p,p}(B_1)}\,.$$

Iterating this inequality finitely many time yields that there exists $\alpha_1 \in (0,1)$ such that for any $x \in B_{\frac{1}{4}}$ and $0 < r \leq \frac{1}{2}$, it holds

$$r^{p-n}\int_{B_r(x)}|\nabla u|^p\,dx \leq C\,r^{p\alpha_1}\,\|\nabla u\|^p_{M^{p,p}(B_1)}\,.$$

This immediately implies $u \in C^{\alpha_1}(B_{\frac{1}{2}})$. The proof is now completed. □

5 Lipschitz and piecewise $C^{1,\alpha}$-estimates

In this section, we will first establish both Lipschitz and piecewise $C^{1,\alpha}$-regularity for stationary harmonic maps on domains with piecewise $C^{0,1}$-metrics, under the smallness condition of renormalized energies. Then we will sketch a proof of Theorem 1.1.

Theorem 5.1. *There exist $\epsilon_0 > 0$ and $\beta_0 \in (0,1)$ depending only on n, g such that if the metric $g \in C^{0,1}(B_1^{\pm} \cup \Gamma_1)$ satisfies the condition (1.4) on Γ_1, and $u \in W^{1,2}(B_1, N)$ is a stationary harmonic map on (B_1, g) satisfying*

$$r_0^{2-n}\int_{B_{r_0}(x_0)}|\nabla u|_g^2\,dv_g \leq \epsilon_0^2 \tag{5.1}$$

for some $x_0 \in B_{\frac{1}{2}}$ and $0 < r_0 \leq \frac{1}{4}$, then $u \in C^{1,\beta_0}\Big(B_{\frac{r_0}{2}}(x_0) \cap \overline{B^{\pm}}, N\Big)$, and $u \in C^{0,1}\Big(B_{\frac{r_0}{2}}(x_0), N\Big)$.

Proof. The proof is based on both the hole filling argument and freezing coefficient method. It is divided into two steps.

Step 1. $u \in C^{\alpha}(B_{\frac{3r_0}{4}}(x_0), N)$ for any $0 < \alpha < 1$. To see this, recall Theorem 4.1 implies that there exists $0 < \alpha_0 < 1$ such that $u \in C^{\alpha_0}(B_{\frac{7r_0}{8}}(x_0))$ and for any $y \in B_{\frac{7r_0}{8}}(x_0)$, it holds

$$s^{2-n} \int_{B_s(y)} |\nabla u|^2 \, dx \leq C \left(\frac{s}{r}\right)^{2\alpha_0} r^{2-n} \int_{B_r(y)} |\nabla u|^2 \, dx, \ 0 < s \leq r < \frac{r_0}{8}, \tag{5.2}$$

and

$$\text{osc}_{B_r(y)} u \leq C r^{\alpha_0}, \ 0 < r < \frac{r_0}{8}. \tag{5.3}$$

For $y \in B_{\frac{7r_0}{8}}(x_0)$ and $0 < r < \frac{r_0}{8}$, let $v : B_r(y) \to \mathbb{R}^k$ solve

$$\begin{cases} \Delta_g v = 0 & \text{in } B_r(y) \\ v = u & \text{on } \partial B_r(y). \end{cases} \tag{5.4}$$

Then by the maximum principle and (5.3), we have

$$\text{osc}_{B_r(y)} v \leq \text{osc}_{\partial B_r(y)} u \leq C r^{\alpha_0}.$$

Moreover, since $g \in C^{0,1}(B_1^{\pm} \cup \Gamma_1)$, it is well-known (cf. [17] Theorem 1.1) that $v \in C^{0,1}(B_{\frac{r}{2}}(y), \mathbb{R}^k)$ and $v \in C^{1,\beta}(B_{\frac{r}{2}}(y) \cap \overline{B^{\pm}}, \mathbb{R}^k)$ for any $0 < \beta < 1$.

Now multiplying both the equations (1.1) and (5.4) by $(u - v)$ and subtracting each other and then integrating over $B_r(y)$, we obtain

$$\int_{B_r(y)} |\nabla(u - v)|^2 \, dx \lesssim \int_{B_r(y)} |\nabla u|^2 |u - v| \lesssim r^{n-2+3\alpha_0}.$$

Since

$$\int_{B_{\frac{r}{2}}(y)} |\nabla v|^2 \, dx \leq C \left\| \nabla v \right\|^2_{L^\infty(B_{\frac{r}{2}}(y))} r^n,$$

we obtain that if $0 < \alpha_0 \leq \frac{2}{3}$, then

$$\left(\frac{r}{2}\right)^{2-n} \int_{B_{\frac{r}{2}}(y)} |\nabla u|^2 \, dx \leq C\Big(\|\nabla v\|^2_{L^\infty(B_{\frac{r}{2}}(y))} r^2 + r^{3\alpha_0} \Big) \leq C r^{3\alpha_0}.$$

This, combined with Morrey's decay lemma, yields $u \in C^{\frac{3\alpha_0}{2}}(B_{\frac{7r_0}{8}}(x_0))$. Repeating this argument, we can show that $u \in C^{\alpha}(B_{\frac{3r_0}{4}}(x_0))$ for any $0 < \alpha < 1$, and

$$r^{2-n} \int_{B_r(y)} |\nabla u|^2 \, dx \leq C r^{2\alpha}, \ \forall y \in B_{\frac{3r_0}{4}}(x_0), \ 0 < r < \frac{r_0}{4}. \tag{5.5}$$

Step 2. There exists $0 < \beta_0 < 1$ such that $u \in C^{1,\beta_0}\Big(B_{\frac{r_0}{2}}(x_0) \cap \overline{B^{\pm}}, N\Big)$. The proof is divided into two cases.

Case I. $x_0 = (x_0', x_0^n) \in B_1^{\pm}$. We may assume $0 < r_0 < |x_0^n|$ so that $B_{r_0}(x_0) \subset B^{\pm}$. For $B_r(x) \subset B_{r_0}(x_0)$, let $v : B_r(x) \to \mathbb{R}^k$ solve

$$\begin{cases} \Delta_g v = 0 & \text{in } B_r(x), \\ v = u & \text{on } \partial B_r(x). \end{cases} \tag{5.6}$$

Then by Step 1, we have that for any $\frac{2}{3} < \alpha < 1$,

$$\int_{B_r(x)} |\nabla(u-v)|^2\,dx \le C\int_{B_r(x)} |\nabla u|^2|u-v|\,dx \le C\,r^{3\alpha+n-2}. \tag{5.7}$$

Moreover, since $g \in C^{0,1}(B_{r_0}(x_0))$, we have that for any $0<\beta<1$, $v \in C^{1,\beta}(B_{\frac{r}{2}}(x))$ and

$$\fint_{B_s(x)} \left|\nabla v - (\nabla v)_{B_s(x)}\right|^2\,dx \le C(\frac{s}{r})^{2\beta}\fint_{B_r(x)}\left|\nabla u-(\nabla u)_{B_r(x)}\right|^2\,dx,\quad 0<s\le\frac{r}{2}. \tag{5.8}$$

Henceforth, we will denote $\fint_E f = \frac{1}{|E|}\int_E f\,dx$. Combining (5.7) and (5.8)[4], we obtain that for any $0<\theta<1$,

$$\begin{aligned}\fint_{B_{\theta r}(x)}\left|\nabla u-(\nabla u)_{B_{\theta r}(x)}\right|^2\,dx &\le 2\Big[\fint_{B_{\theta r}(x)}|\nabla u-\nabla v|^2\,dx+\fint_{B_{\theta r}(x)}\left|\nabla v-(\nabla v)_{B_{\theta r}(x)}\right|^2\,dx\Big]\\ &\le C\Big[\theta^{2\beta}\fint_{B_r(x)}\left|\nabla u-(\nabla u)_{B_r(x)}\right|^2\,dx+\theta^{-n}r^{3\alpha-2}\Big].\end{aligned}$$

For $\frac{3\alpha-2}{2}<\beta_0<\beta$, let $0<\theta_0<1$ be such that $C\theta_0^{2\beta}=\theta_0^{2\beta_0}$. Then we have

$$\fint_{B_{\theta_0 r}(x)}\left|\nabla u-(\nabla u)_{B_{\theta_0 r}(x)}\right|^2\,dx\le\theta_0^{2\beta_0}\fint_{B_r(x)}\left|\nabla u-(\nabla u)_{B_r(x)}\right|^2\,dx+Cr^{3\alpha-2}. \tag{5.9}$$

Iterating (5.9) m-times, $m\ge1$, yields

$$\begin{aligned}\fint_{B_{\theta_0^m r}(x)}\left|\nabla u-(\nabla u)_{B_{\theta_0 r}(x)}\right|^2\,dx &\le (\theta_0^m)^{2\beta_0}\fint_{B_r(x)}\left|\nabla u-(\nabla u)_{B_r(x)}\right|^2\,dx\\ &\quad +C(\theta_0^m r)^{3\alpha-2}\sum_{j=1}^m\theta_0^{j(2\beta_0-(3\alpha-2))}\\ &\le (\theta_0^m)^{3\alpha-2}\Big[\fint_{B_r(x)}\left|\nabla u-(\nabla u)_{B_r(x)}\right|^2\,dx+Cr^{3\alpha-2}\Big].\end{aligned} \tag{5.10}$$

This clearly implies that $\nabla u\in C^{\frac{3\alpha-2}{2}}(B_{r_0}(x_0))$.

Case II. $x_0=(x_0',0)\in\Gamma_1$. For simplicity, we assume $x_0'=0$. Define the piecewise constant metric $\bar g$ on B_1 by letting

$$\bar g(x)=\begin{cases}\lim_{t\downarrow 0^+}g(0',t) & x\in B_1^+\\ \lim_{t\uparrow 0^-}g(0',t) & x\in B_1^-.\end{cases}$$

Then we have

$$|g(x)-\bar g(x)|\le C|x|,\ x\in B_1. \tag{5.11}$$

Moreover, by suitable dilations and rotations of the coordinate system, (1.4) implies that there exists a positive constant $k\ne1$ such that

$$\bar g(x)=(1+(k-1)\chi_{B_1^-}(x))g_0,\ x\in B_1,$$

[4]note that (5.8) trivially holds for $\frac{r}{2}\le s\le r$.

where $\chi_{B_1^-}$ is the characteristic function of B_1^-.

For $0 < r < \frac{r_0}{2}$, let $v : B_r(0) \to \mathbb{R}^k$ solve

$$\begin{cases} \Delta_{\bar{g}} v = 0 & \text{in } B_r(0), \\ v = u & \text{on } \partial B_r(0). \end{cases} \tag{5.12}$$

Then we have

$$\text{osc}_{B_r(0)} v \le \text{osc}_{B_r(0)} u \le C r^{\alpha}, \quad \int_{B_r(0)} |\nabla v|^2 \, dx \le C \int_{B_r(0)} |\nabla u|^2 \le C r^{n-2+2\alpha}.$$

Multiplying (1.1) and (5.12) by $(u-v)$ and integrating over $B_r(0)$, we obtain

$$\begin{aligned} & \int_{B_r(0)} |\nabla (u-v)|^2 \, dx \\ \le \; & \int_{B_r(0)} g^{ij} (u-v)_i (u-v)_j \sqrt{g} \, dx \\ \le \; & C \int_{B_r(0)} |\nabla u|^2 |u-v| \, dx + \int_{B_r(0)} | \sqrt{g} g^{ij} - \sqrt{\bar{g}} \bar{g}^{ij} | |v_i| |(u-v)_j| \, dx \\ \le \; & C \text{osc}_{B_r(0)} v \int_{B_r(0)} |\nabla u|^2 \, dx + C r^2 \int_{B_r(0)} |\nabla v|^2 + \frac{1}{2} \int_{B_r(0)} |\nabla (u-v)|^2 \, dx \\ \le \; & C r^{n-2+3\alpha} + C r^{n+\alpha} + \frac{1}{2} \int_{B_r(0)} |\nabla (u-v)|^2 \, dx. \end{aligned}$$

This implies

$$\int_{B_r(0)} |\nabla (u-v)|^2 \, dx \le C r^{n-2+3\alpha}. \tag{5.13}$$

It is well-known that $v \in C^{\infty}\Big(\overline{B_s^{\pm}(0)}\Big)$ for any $0 < s < r$. In fact, (5.12) is equivalent to:

$$\frac{\partial}{\partial x_i} \Big((1 + (k^{\frac{n}{2}} - 1) \chi_{B_1^-}) \frac{\partial v}{\partial x_i} \Big) = 0, \text{ in } B_r(0), \tag{5.14}$$

we conclude

(i) $\frac{\partial v}{\partial x_n}$ satisfies the jump property on Γ_1:

$$\lim_{x_n \downarrow 0^+} \frac{\partial v}{\partial x_n}(x', x_n) = k^{\frac{n}{2}} \lim_{x_n \uparrow 0^-} \frac{\partial v}{\partial x_n}(x', x_n), \quad \forall (x', 0) \in \Gamma_1 \cap B_r(0).$$

(ii) $\nabla^{\alpha} v \in C^0(B_r(0))$ for any $\alpha = (\alpha_1, \cdots, \alpha_{n-1}, 0) \in \mathbb{N}^n$.

(iii) $\nabla v \in L^{\infty}(B_s(0))$ for any $0 < s < r$, and

$$\|\nabla v\|^2_{L^{\infty}(B_{\frac{r}{2}}(0))} \le C r^{2-n} \int_{B_r(0)} |\nabla u|^2. \tag{5.15}$$

For $f : B_r(0) \to \mathbb{R}^k$, set

$$\widetilde{D} f := \Big(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n-1}}, (1 + (k^{\frac{n}{2}} - 1) \chi_{B_1^-}) \frac{\partial f}{\partial x_n} \Big), \tag{5.16}$$

and let $\left(\widetilde{D}f\right)_s = \fint_{B_s(0)} \widetilde{D}f\,dx$ denote the average of $\widetilde{D}f$ over $B_s(0)$. Then we have that for any $0 < \beta < 1$,

$$\fint_{B_s(0)} \left|\widetilde{D}v - (\widetilde{D}v)_s\right|^2\,dx \le C\left(\frac{s}{r}\right)^{2\beta} \fint_{B_r(0)} \left|\widetilde{D}u - (\widetilde{D}u)_r\right|^2\,dx,\ \forall 0 < s \le r. \tag{5.17}$$

Combining (5.13) with (5.17) yields that for any $0 < \theta < 1$,

$$\fint_{B_{\theta r}(0)} \left|\widetilde{D}u - (\widetilde{D}u)_{\theta r}\right|^2\,dx \le C\theta^{2\beta} \fint_{B_r(0)} \left|\widetilde{D}u - (\widetilde{D}u)_r\right|^2\,dx + C\theta^{-n} r^{3\alpha-2}. \tag{5.18}$$

As in Case I, iteration of (5.18) yields that for any $0 < s \le r$, it holds

$$\fint_{B_s(0)} \left|\widetilde{D}u - (\widetilde{D}u)_s\right|^2\,dx \le C\left(\frac{s}{r}\right)^{3\alpha-2} \fint_{B_r(0)} \left|\widetilde{D}u - (\widetilde{D}u)_r\right|^2\,dx + Cs^{3\alpha-2}. \tag{5.19}$$

This, combined with Case I, further implies that for any $B_r(x) \subset B_{r_0}(x_0)$ and $0 < s \le r$,

$$\fint_{B_s(x)} \left|\widetilde{D}u - (\widetilde{D}u)_{x,s}\right|^2\,dx \le C\left(\frac{s}{r}\right)^{3\alpha-2} \fint_{B_r(x)} \left|\widetilde{D}u - (\widetilde{D}u)_{x,r}\right|^2\,dx + Cs^{3\alpha-2}, \tag{5.20}$$

where $(\widetilde{D}u)_{x,s}$ denotes the average of $\widetilde{D}u$ over $B_s(x)$. It is readily seen that (5.20) yields $\nabla u \in C^{1,\frac{3\alpha-2}{2}}(B_{\frac{r_0}{2}}(x_0) \cap \overline{B_1^{\pm}})$ and $u \in C^{0,1}(B_{\frac{r_0}{2}}(x_0))$. This completes the proof. □

Now we sketch the proof of Theorem 1.1.

Proof of Theorem 1.1 . Define the singular set

$$\Sigma = \left\{x \in \Omega :\ \varliminf_{r\to 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2\,dx \ge \epsilon_0^2\right\}.$$

Then by a simple covering argument we have $H^{n-2}(\Sigma) = 0$ (see, for example, Evans-Gariepy [7]). For any $x_0 \in \Omega \setminus \Sigma$, there exists $0 < r_0 < \mathrm{dist}(x_0, \partial\Omega)$ such that

$$r_0^{2-n} \int_{B_{r_0}(x)} |\nabla u|^2\,dx \le \epsilon_0^2.$$

Hence by Theorem 2.1, Theorem 4.1, and Theorem 5.1, we have

$$u \in C^{1,\alpha}\Big(B_{\frac{r_0}{2}}(x_0) \cap \overline{\Omega^{\pm}}, N\Big) \text{ and } u \in C^{0,1}\Big(B_{\frac{r_0}{2}}(x_0), N\Big),$$

for some $0 < \alpha < 1$. In particular, we have

$$\lim_{r\downarrow 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2\,dx = 0,\ \forall x \in B_{\frac{r_0}{2}}(x_0)$$

so that $B_{\frac{r_0}{2}}(x_0) \cap \Sigma = \emptyset$ and hence Σ is closed. This completes the proof. □

6 Harmonic maps to manifolds supporting convex distance functions

In this section, we consider weakly harmonic maps u from (Ω, g), with g the piecewise Lipschitz continuous metric as in Theorem 1.1, to (N, h), whose universal cover $(\widetilde{N}, \widetilde{h})$ supports a convex distance function square $d^2_{\widetilde{N}}(\cdot, p)$ for any $p \in \widetilde{N}$. We will establish both the global Lipschitz continuity and piecewise $C^{1,\alpha}$-regularity for such harmonic maps u. This can be viewed as a generalization of the well-known regularity theorem by Eells-Sampson [8] and Hildebrandt-Kaul-Widman [13].

The crucial step is the following theorem on Hölder continuity.

Theorem 6.1. *Assume that the metric g is bounded measurable on Ω, i.e. there exist two constants $0 < \lambda < \Lambda < +\infty$ such that $\lambda \mathbb{I}_n \le g(x) \le \Lambda \mathbb{I}_n$ for a.e. $x \in \Omega$. Assume also that the universal cover $(\widetilde{N}, \widetilde{h})$ of (N, h) supports a convex distance function square $d^2_{\widetilde{N}}(\cdot, p)$ for any $p \in \widetilde{N}$. If $u \in H^1(\Omega, N)$ is a weakly harmonic map, then there exists $\alpha \in (0, 1)$ such that $u \in C^\alpha(\Omega, N)$.*

Proof. Here we sketch a proof that is based on slight modifications of that by Lin [16]. Similar ideas have been used by Evans in his celebrated work [6] and Caffarelli [4] for quasilinear systems under smallness conditions. First, by lifting $u : \Omega \to N$ to a harmonic map $\widetilde{u} : \Omega \to \widetilde{N}$, we may simply assume $(N, h) = (\widetilde{N}, \widetilde{h})$ and $d^2_N(\cdot, p)$ is convex on N for any $p \in N$.

We first claim that

$$\Delta_g d^2(u, p) \ge 0. \tag{6.1}$$

In fact, by the chain rule of harmonic maps (cf. Jost [15]), we have

$$\Delta_g d^2(u, p) = \nabla_u d^2(u, p)(\Delta_g u) + \nabla^2_u d^2(u, p)(\nabla u, \nabla u)_g.$$

Since $\Delta_g u \perp T_u N$, $\nabla_u d^2(u, p) \in T_u N$, the first term in the right hand side vanishes. By the convexity of d^2_N, the second term in the right hand side satisfies

$$\nabla^2_u d^2(u, p)(\nabla u, \nabla u)_g \ge 0.$$

Since $u \in H^1(\Omega, N)$, by suitably choosing $p \in N$ and applying Poincaré inequality and Harnack's inequality, (6.1) implies $u \in L^\infty_{\text{loc}}(\Omega, N)$.

For a set $E \subset N$, let $\mathrm{diam}_N(E)$ denote the diameter of E with respect to the distance function $d_N(\cdot, \cdot)$. For any ball $B_r(x) \subset \Omega$, we want to show that $u \in C^\alpha(B_{\frac{r}{2}}(x))$ for some $0 < \alpha < 1$. To do it, denote

$$C_r := \mathrm{diam}_N\left(u(B_r(x))\right) < +\infty.$$

We may assume $C_r > 0$ (otherwise, u is constant on $B_r(x)$ and we are done). Now we want to show that there exists $0 < \delta_0 = \delta_0(N) \le \frac{1}{2}$ such that

$$\mathrm{diam}_N\left(u(B_{\delta_0 r}(x))\right) \le \frac{1}{2} C_r. \tag{6.2}$$

Since $u_r(y) = u(x + ry) : B_1(0) \to N$ is a harmonic map $(B_1(0), g_r)$, with $g_r(y) = g(x + ry)$, we may, for simplicity, assume $x = 0$ and $r = 2$. For any $0 < \epsilon < \frac{1}{2}$, since $u(B_1) \subset N$ is a bounded set, there exists $m = m(\epsilon) \geq 1$ such that $u(B_1)$ is covered by m balls $B^1, \cdots, B^m$ of radius ϵC_1. Now we have

Claim: *There exists a sufficiently small* $\epsilon > 0$ *such that* $u(B_{\frac{1}{2}})$ *can be covered by at most* $(m-1)$ *balls among* $B^1, \cdots, B^m$.

To see this, let $x_i \in B_1$ such that $B^i \subset B_{2\epsilon C_1}(p_i)$, $p_i = u(x_i)$, for $1 \leq i \leq m$. Let $1 \leq m' \leq m$ be the maximum number of points in $\{p_i\}_{i=1}^m$ such that the distance between any two of them is at least $\frac{1}{32}C_1$. Thus we may assume $B_{\frac{1}{16}C_1}(p_i), 1 \leq i \leq m'$, covers $u(B_1)$. Then there exists $i_0 \in \{1, \cdots, m'\}$ such that

$$\frac{1}{4}C_1^2 \leq \sup_{x \in B_2} d_N^2(u(x), p_{i_0}) \leq C_1^2, \tag{6.3}$$

and

$$H^n\left(u^{-1}(B^N(p_{i_0}, \frac{1}{16}C_1)) \cap B_1\right) \geq c_0, \tag{6.4}$$

for some universal constant $c_0 > 0$, where $B^N(p_{i_0}, R)$ is the ball in N with center p_{i_0} and radius R.

In fact, since

$$B_1 \subset \bigcup_{i=1}^{m'} u^{-1}\left(B^N(p_i, \frac{1}{16}C_1)\right),$$

we have

$$\sum_{i=1}^{m'} H^n\left(u^{-1}(B^N(p_i, \frac{1}{16}C_1)) \cap B_1\right) \geq H^n(B_1).$$

Hence there exists $i_0 \in \{1, \cdots, m'\}$ such that

$$H^n\left(u^{-1}(B^N(p_{i_0}, \frac{1}{16}C_1))\right) \geq c_0 := \frac{1}{m'}H^n(B_1).$$

This implies (6.4). By the triangle inequality, (6.3) also holds.

Define

$$f(x) := \sup_{z \in B_1} d_N^2(u(z), p_{i_0}) - d_N^2(u(x), p_{i_0}), \ x \in B_1.$$

It is clear that $f \geq 0$ in B_1, and (6.1) implies

$$\Delta_g f \leq 0, \text{ in } B_1.$$

By Moser's Harnack inequality, we have

$$\begin{aligned}
\inf_{B_{\frac{1}{2}}} f &\geq C \fint_{B_1} f \geq C \int_{B_{\frac{1}{2}}} f \geq C \int_{B_{\frac{1}{2}} \cap u^{-1}(B^N(p_{i_0}, \frac{1}{16}C_1))} f \\
&\geq C\left(\sup_{B_1} d_N^2(u, p_{i_0}) - \sup_{B_1 \cap u^{-1}(B^N(p_{i_0}, \frac{1}{16}C_1))} d_N^2(u, p_{i_0})\right) H^n\left(B_{\frac{1}{2}} \cap u^{-1}(B^N(p_{i_0}, \frac{1}{16}C_1))\right) \\
&\geq C\left(\frac{1}{4}C_1^2 - \frac{1}{256}C_1^2\right) c_0 := \theta_0^2 C_1^2 \qquad (6.5)
\end{aligned}$$

for some universal constant $\theta_0 > 0$. This implies

$$\sup_{z\in B_1} d_N(u(z), p_{i_0}) - \sup_{z\in B_{\frac{1}{2}}} d_N(u(z), p_{i_0}) \geq \theta_0 C_1 = (1-\theta_0)C_1. \tag{6.6}$$

Now we argue that the claim follows from (6.6). For, otherwise, we would have that $u(B_{\frac{1}{2}}) \cap B_{2\epsilon C_1}(p_j) \neq \emptyset$ for all $1 \leq j \leq m$. Let $z_0 \in B_1$ be such that

$$\epsilon C_1 + d_N(u(z_0), p_{i_0}) \geq \sup_{B_1} d_N(u(z), p_{i_0}).$$

Since $u(B_1) \subset \cup_{i=}^m B_{2\epsilon C_1}(p_i)$, there exists $p_{i_1} \in \{p_1, \cdots, p_m\}$ such that $u(z_0) \in B_{2\epsilon C_1}(p_{i_1})$. Since $u(B_{\frac{1}{2}}) \cap B_{2\epsilon C_1}(p_{i_1}) \neq \emptyset$, there exists $z_1 \in B_{\frac{1}{2}}$ such that $u(z_1) \in B_{2\epsilon C_1}(p_{i_1})$. Therefore we have $d_N(u(z_1), u(z_0)) \leq 2\epsilon C_1$. Therefore we have

$$\begin{aligned}
\sup_{z\in B_1} d_N(u(z), p_{i_0}) - \sup_{z\in B_{\frac{1}{2}}} d_N(u(z), p_{i_0}) &\leq \epsilon C_1 + d_N(u(z_0), p_{i_0}) - d_N(u(z_1), p_{i_0}) \\
&\leq \epsilon C_1 + d_N(u(z_0), u(z_1)) \leq 3\epsilon C_1,
\end{aligned}$$

this contradicts (6.6) provide that $\epsilon > 0$ is chosen to be sufficiently small.

From this claim, we have either

(i) $\mathrm{diam}_N(u(B_{\frac{1}{2}})) \leq \frac{1}{2}C_1$. Then (6.2) holds with $\delta_0 = \frac{1}{2}$, or

(ii) $\mathrm{diam}_N(u(B_{\frac{1}{2}})) > \frac{1}{2}C_1$.

Then we consider $v(x) = u(\frac{1}{2}x) : B_1 \to N$ so that

- v is a harmonic map on $(B_1, g_{\frac{1}{2}})$, with the metric $g_{\frac{1}{2}}(x) = g(\frac{1}{2}x)$.
- $\frac{1}{2}C_1 < \mathrm{diam}_N(v(B_1)) \leq C_1$.
- $v(B_1)$ is covered by at most $(m-1)$ balls $B_1, \cdots, B^{m-1}$ of radius ϵC_1.

Thus the claim is applicable to v so that $u(B_{\frac{1}{4}}) = v(B_{\frac{1}{2}})$ can be covered by at most $(m-2)$ balls among $B^1, \cdots, B^{m-1}$.

If $\mathrm{diam}_N(v(B_{\frac{1}{2}})) \leq \frac{1}{2}C_1$, we are done. Otherwise, we can repeat the above argument. It is clear that the process can at most be repeated m-times, and the process will not be stopped at step $k_0 \leq m$ unless $\mathrm{diam}_N u(B_{2^{-k_0}}) \leq \frac{1}{2}C_1$. Thus (6.2) is proven.

It is readily seen that iteration of (6.2) implies Hölder continuity. □

Proof of Theorem 1.2. First, by Theorem 6.1, and the argument from §4, we can show that for some $0 < \alpha < 1$,

$$\int_{B_r(x)} |\nabla u|^2 \, dx \leq C r^{n-2+2\alpha}, \ \forall B_r(x) \subset \Omega.$$

Then we can follow the same proof of Theorem 5.1 to show that $u \in C^{0,1}(\Omega)$ and $u \in C^{1,\alpha}(\Omega^\pm \cup \Gamma, N)$. □

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