Solutions to HW#2

1. $v(x) = u(Ox)$, $O^T O = I_n$, we have

$$\frac{\partial v}{\partial x_i} = \frac{\partial u}{\partial y_j} \frac{\partial (Ox)^j}{\partial x_i} = \frac{\partial u}{\partial y_j} \frac{\partial}{\partial x_i}\left(O_{jk} x_k\right)$$

$$= \frac{\partial u}{\partial y_j} O_{jk} \delta_{ik} = \frac{\partial u}{\partial y_j} O_{ji}$$

$$\frac{\partial^2 v}{\partial x_i^2}(x) = \sum_{j\,l=1}^{n} \frac{\partial^2 u}{\partial y_l \partial y_j}(Ox)\; O_{ji}\, O_{li}$$

$$\Delta v(x) = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} = \sum_{i=1}^{n} \sum_{j\,l=1}^{n} \frac{\partial^2 u}{\partial y_l \partial y_j}(Ox)\; O_{ji}\, O_{li}$$

$$= \sum_{j\,l=1}^{n} \frac{\partial^2 u}{\partial y_l \partial y_j}(Ox) \underbrace{\sum_{i=1}^{n} O_{ji}\, O_{li}}_{=\delta_{jl}}$$

$$= \sum_{j=1,\, l=1}^{n} \frac{\partial^2 u}{\partial y_l \partial y_j}(Ox)\; \delta_{jl}$$

$$= \sum_{l=1}^{n} \frac{\partial^2 u}{\partial y_l^2} = \Delta u(Ox) = 0$$

$\square$

P2

2. For $0<\rho<r$, we have

$$\rho^{1-n}\int_{B_\rho} -\Delta u = -\rho^{1-n}\int_{\partial B_\rho}\frac{\partial u}{\partial \nu} \quad \begin{pmatrix}\text{computed in}\\ \text{classes}\end{pmatrix}$$

$$= -\frac{d}{d\rho}\left(\rho^{1-n}\int_{\partial B_\rho} u\right)$$

$$\Rightarrow \; -\rho^{1-n}\int_{B_\rho} f = \frac{d}{d\rho}\left(\rho^{1-n}\int_{\partial B_\rho} u\right)$$

Integrating this identity for $0<\rho<r$, we obtain

$$r^{1-n}\int_{\partial B_r} u - n\alpha(n)u(0)$$

$$= -\int_0^r\left(\rho^{1-n}\int_{B_\rho} f\right)d\rho$$

$$= -\int_0^r\left(\int_{B_\rho} f\right)d\left(\frac{\rho^{2-n}}{2-n}\right)$$

$$= -\left(\frac{r^{2-n}}{2-n} \int_{B_r} f - \underbrace{\cancel{\lim_{\rho \downarrow 0} \frac{\rho^{2-n}}{2-n} \int_{B_\rho} f}}_{=0} \right)$$

$$+ \int_0^r \frac{\rho^{2-n}}{2-n} \int_{\partial B_\rho} f$$

$$= \frac{r^{2-n}}{n-2} \int_{B_r} f - \frac{1}{n-2} \int_{B_\rho} |x|^{2-n} f(x)$$

$$= \frac{1}{n-2} \int_{B_r} \left(r^{2-n} - |x|^{2-n} \right) f(x)\, dx$$

putting all these quantities together, we obtain the desired formula. $\square$

3. a) Since

$$0 \geq r^{1-n} \int_{B(x,r)} -\Delta v = -r^{1-n} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}$$

$$= -\frac{d}{dr}\left(r^{1-n} \int_{\partial B(x,r)} u \right)$$

$$\Rightarrow \quad \frac{d}{dr}\left(r^{1-n} \int_{\partial B(x,r)} u \right) \geq 0$$

$$\Rightarrow \quad r^{1-n} \int_{\partial B(x,r)} u \geq \lim_{\rho \to 0} \rho^{1-n} \int_{\partial B(x,\rho)} u$$

$$= n\alpha(n)\, u(x)$$

$$\Rightarrow \quad u(x) \leq \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u, \quad \forall\, r > 0$$

This yields the mean value inequality. □

b) Assume U is connected. If (P5)

$\exists\, x_0 \in U$ s.t. $\max_{\bar{U}} u = u(x_0)$,

then set

$$\boxed{E} := \{x \in U \mid u(x) = u(x_0) = \max_{\bar{U}} u\}$$

Then 1) $x_0 \in E$

2) E is closed

3) MV inequality $\Rightarrow$ E is also open:

$\forall x_* \in E, \Longrightarrow \exists\, r_* > 0$ s.t. $u(x) \equiv u(x_*)$

for all $x \in B(x_*, r_*)$.

Since U is connected, we conclude

that $E = U \Longrightarrow u \equiv u(x_0)$.

Hence $\max_{\bar{U}} u = \max_{\partial U} u$ (in any case) c). $v_{x_i} = \varphi'(u)\, u_{x_i}$

$$v_{x_i x_i} = \varphi''(u)\, u_{x_i}^2 + \varphi'(u)\, u_{x_i x_i}$$

$$\Rightarrow \Delta v = \varphi''(u)\, |\nabla u|^2 + \varphi'(u)\, \Delta u$$

$$= \varphi''(u)\, |\nabla u|^2 + \varphi'(u) \cdot 0$$

$$= \varphi''(u)\, |\nabla u|^2 \geq 0 \quad (\because \varphi'' \geq 0).$$

$\Rightarrow$ v is subharmonic.

d) $\Delta v = \Delta(|Du|^2) = 2\langle D\Delta u, Du\rangle + 2|D^2 u|^2$

$$= 0 + 2|D^2 u|^2 \geq 0$$

5. Poisson's formula implies (P8)

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} \, d\sigma(y)$$

Since

$$r - |x| = |y| - |x| \le |x-y| \le |x| + |y| = r + |x|,$$

we have

$$\frac{1}{(r+|x|)^n} \le \frac{1}{|x-y|^n} \le \frac{1}{(r-|x|)^n}$$

$$\overset{u>0}{\Longrightarrow} \quad \frac{r^2 - |x|^2}{n\alpha(n) r} \frac{1}{(r+|x|)^n} \int_{\partial B(0,r)} u(y) \, d\sigma(y)$$

$$\le u(x)$$

$$\le \frac{r^2 - |x|^2}{n\alpha(n) r} \frac{1}{(r-|x|)^n} \int_{\partial B(0,r)} u(y) \, d\sigma(y).$$

By MVP, we have $\int_{\partial B(0,r)} u(y) \, d\sigma(y) = |\partial B(0,r)| \, u(0)$.

Hence we obtain

$$\frac{r^2-|x|^2}{n\alpha(n)r}\,\frac{1}{(r+|x|)^n}\,n\alpha(n)\,r^{n-1}\,u(0)$$

$$\leq u(x)$$

$$\leq \frac{r^2-|x|^2}{n\alpha(n)r}\,\frac{1}{(r-|x|)^n}\,n\alpha(n)\,r^{n-1}\,u(0)$$

This gives the desired estimate.