Solution to HW #3

1. $u(x', x_n) = \dfrac{2x_n}{n\alpha(n)} \displaystyle\int_{\partial \mathbb{R}^n_+} \dfrac{g(y)}{|x-y|^n}\, dy$

$$\frac{u(0,\lambda) - g(0)}{\lambda} = \frac{2}{n\alpha(n)} \int_{\mathbb{R}^{n-1}} \frac{g(y)}{(\lambda^2 + |y|^2)^{n/2}}\, dy$$

$$= \frac{2}{n\alpha(n)} \left[\int_{|y|\le 1} \frac{|y|}{(\lambda^2+|y|^2)^{n/2}}\, dy + \int_{\mathbb{R}^{n-1} \setminus \{|y|\le 1\}} \frac{g(y)}{(\lambda^2+|y|^2)^{n/2}}\, dy \right]$$

$$= \frac{2}{n\alpha(n)} [I + II].$$

$$II = \left| \int_{|y|\ge 1} \frac{g(y)}{(\lambda^2 + |y|^2)^{n/2}}\, dy \right| \le \int_{|y|\ge 1} \frac{|g(y)|}{(\lambda^2+|y|^2)^{n/2}}\, dy$$

$$\le C \int_1^{+\infty} \frac{1}{\cancel{|y|^n}\, r^n} \cancel{dy}\, r^{n-2}\, dr$$

$$= C \int_1^{+\infty} r^{-2}\, dr < +\infty.$$

$$I = \int_{|y|\le 1} \frac{|y|}{(\lambda^2+|y|^2)^{n/2}}\, dy \doteq \int_0^1 \frac{r}{(\lambda^2+r^2)^{n/2}}\, r^{n-2}\, dr$$

$$= \int_0^1 \frac{r^{n-1}}{(\lambda^2+r^2)^{n/2}} dr \xrightarrow{\lambda \searrow 0} \int_0^1 \frac{r^{n-1}}{r^n} dr = +\infty.$$

Hence $\boxed{Du(0) \text{ doesn't exist}}$

2. Note $\begin{cases} \Delta v = 0 & \text{for } x_n > 0 \\ \Delta v = 0 & \text{for } x_n < 0 \end{cases}$

Since $\Delta v = \Delta_{x'} v + \Delta_{x_n} v = -(\Delta_{x'} u)(x', -x_n) - (\Delta_{x_n} u)(x', -x_n)$

$$= -(\Delta u)(x', -x_n) = 0, \text{ for } x_n < 0.$$

Hence by mean value property, we have

$\forall (x', x_n) = x$ with $x_n \neq 0$, $\exists\, r_x > 0$ s.t.

$$v(x) = \fint_{B_{r_x}(x)} v(y', y_n)\, dy, \quad 0 < r_x \leq |x_n| \quad (1)_x$$

On the other hand, for $x_n = 0$

$$0 = v(x', 0) = \left(\fint_{B_r^+(x', 0)} v + \fint_{B_r^-(x', 0)} v \right) = \fint_{B_r(x', 0)} v$$

$\Rightarrow v$ satisfies MVP for any $(x', x_n) \in \mathbb{R}^n$.

P3

$\underline{n=2}$:

$\underline{3.}$ $\Delta\left(\frac{1}{8\pi} r^2 \log r\right)$

$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{1}{8\pi} r^2 \log r\right)$

$= \frac{1}{8\pi}\left[\left(r^2 \log r\right)'' + \frac{1}{r}\left(r^2 \log r\right)'\right]$

$(r^2 \log r)' = 2r \log r + r^2 \frac{1}{r} = 2r \log r + r$

$(r^2 \log r)'' = 2 \log r + 2 + 1 = 3 + 2 \log r$

$= \frac{1}{8\pi}\left[3 + 2\log r + 2 \log r + 1\right]$

$= \frac{1}{2\pi}\left[1 + \log r\right]$

$\Delta^2\left(\frac{1}{8\pi} r^2 \log r\right) = \frac{1}{2\pi} \Delta(1 + \log r)$

$= \frac{1}{2\pi} \Delta \log r = 0.$

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4. $\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$

$\Delta^2 u = 0 \Rightarrow$ Set $v = \Delta u$ Then $\underline{\Delta v = 0}$

$$\begin{cases} \Delta v = 0 \Rightarrow \underline{v(r) = c_1 r^{2-n}} + c_2 \\ \Delta u = v \end{cases}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}u = c_1 r^{2-n} + c_2$$

or $\quad u'' + \frac{n-1}{r}u' = c_1 r^{2-n} + c_2$

$$\begin{aligned}(r^{n-1}u')' &= r^{n-1}u'' + (n-1)r^{n-2}u' \\ &= r^{n-1}\left(u'' + \frac{n-1}{r}u'\right) \\ &= \underline{c_1 r + c_2 r^{n-1}}\end{aligned}$$

$$\Rightarrow \quad r^{n-1}u' = c_1 r^2 + c_2 r^n + c_3$$

$$\Rightarrow \quad u' = c_1 r^{3-n} + c_2 r + c_3 r^{1-n}$$

$$\boxed{u(r) = c_1 r^{4-n} + c_2 r^2 + \boxed{c_3 r^{2-n}} + c_4}$$

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5 a) $L = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + c$

$$Lu(r) = 0 \Longrightarrow u'' + \frac{2}{r}u' + cu = 0$$

Set $V(r) = ru$ $\Rightarrow$ $v' = ru' + u$

$$v'' = ru'' + 2u'$$

$$\Rightarrow \underbrace{ru'' + 2u'}_{} + cru = 0$$

$$v'' + cv = 0$$

$$v(r) = c_1\cos(\sqrt{c}\,r) + c_2\sin(\sqrt{c}\,r)$$

$$\Rightarrow \boxed{u(r) = \frac{c_1\cos(\sqrt{c}\,r) + c_2\sin(\sqrt{c}\,r)}{r}.}$$

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b) Set $c_1 = -\frac{1}{4\pi}$, $c_2 = 0$. we have $K(x,\xi) = \frac{-\cos(\sqrt{c}\,r)}{4\pi r}$, $r = |x-\xi|$.

Need to show $\boxed{L_x K(x,\xi) = \delta_\xi(x)}$

This is equivalent to

$$\int_{\mathbb{R}^3} K(x,\xi)\, L_x\varphi(x)\,dx = \varphi(\xi), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3)$$

$$LHS = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 \setminus B_\varepsilon(\xi)} K(x,\xi)\, L_x \varphi(x)\, dx$$

For simplicity, we assume $\xi = 0$.

$$\int_{\mathbb{R}^3 \setminus B_\varepsilon(0)} K(x,0)\, L_x \varphi(x) = \int_{\mathbb{R}^3 \setminus B_\varepsilon(0)} L_x K(x,0)\, \varphi(x)$$

$$+ \int_{\partial B_\varepsilon(0)} \left\{ K(x,0) \frac{\partial \varphi}{\partial \nu} - \frac{\partial K}{\partial \nu}(x,0)\varphi(x) \right\}$$

$$= \int_{\partial B_\varepsilon(0)} \left\{ K(x,0) \frac{\partial \varphi}{\partial \nu} - \frac{\partial K}{\partial \nu}(x,0)\varphi(x) \right\} d\sigma$$

Since $L_x K(x,0) = 0$ for all $x \in \mathbb{R}^3 \setminus B_\varepsilon(0)$. Here ν is the unit outward normal of $\partial B_\varepsilon(0)$.

$$\int_{\partial B_\varepsilon(0)} \left(K(x,0) \frac{\partial \varphi}{\partial \nu} - \frac{\partial K}{\partial \nu}(x.0)\, \varphi(x) \right) d\sigma$$

$$= \int_{\partial B_\varepsilon(0)} \left\{ \frac{\partial K}{\partial r} \varphi - K \frac{\partial \varphi}{\partial r} \right\} \qquad \left(\frac{\partial}{\partial \nu} = -\frac{d}{dr} \right)$$

It is straightforward to estimate

$$\left|\int_{\partial B_\varepsilon(0)} -K\frac{\partial\varphi}{\partial r}\right| \le \int_{\partial B_\varepsilon(0)} |K(x,0)|\left|\frac{\partial\varphi}{\partial r}\right|$$

$$\le \frac{\cos(\sqrt{c}\,\varepsilon)}{4\pi\varepsilon}\,\|\varphi\|_{C^1}\,|\partial B_\varepsilon|$$

$$= 4\pi\varepsilon^2\,\frac{\cos(\sqrt{c}\,\varepsilon)}{4\pi\varepsilon}\,\|\varphi\|_{C^1}$$

$$= \varepsilon\cos(\sqrt{c}\,\varepsilon)\,\|\varphi\|_{C^1} \to 0.$$

$$\frac{\partial K}{\partial r} = \frac{d}{dr}\left(-\frac{\cos(\sqrt{c}\,r)}{4\pi r}\right)$$

$$= \frac{\sqrt{c}\sin(\sqrt{c}\,r)}{4\pi r} + \cos(\sqrt{c}\,r)\frac{1}{4\pi r^2}$$

$$\int_{\partial B_\varepsilon(0)}\frac{\partial K}{\partial r}\varphi = \left(\frac{\sqrt{c}\sin(\sqrt{c}\,\varepsilon)}{4\pi\varepsilon} + \cos(\sqrt{c}\,\varepsilon)\frac{1}{4\pi\varepsilon^2}\right)\int_{\partial B_\varepsilon(0)}\varphi$$

$$\xrightarrow{\varepsilon\downarrow 0} \varphi(0).$$

Hence $LHS = \varphi(0) = RHS$

$\square$

c) For simplicity, assume $\xi = 0$.

Choose $k \in \mathbb{R}$ so that

$$G(r) = -\frac{\cos(\sqrt{c}\,r)}{4\pi r} + k\,\frac{\sin(\sqrt{c}\,r)}{r} = 0 \quad \text{when } r = \rho:$$

$$\left[\text{or} \quad k = \frac{\cos(\sqrt{c}\,\rho)}{4\pi \sin(\sqrt{c}\,\rho)} \quad \left(\text{Note: } \sin(\sqrt{c}\,\rho) \neq 0\right)\right]$$

Note $LG(r) = 0$ as long as $r \neq 0$ (from (a)).

Consider $B_\rho(0) \setminus B_\varepsilon(0)$ for some $\overset{\text{small}}{\varepsilon} > 0$, we have

$$\int_{B_\rho(0)\setminus B_\varepsilon(0)} (LG\,u - G\,Lu) = 0$$

$$\| $$

$$\int_{\partial(B_\rho(0)\setminus B_\varepsilon(0))} \left(\frac{\partial G}{\partial \nu}\,u - G\,\frac{\partial u}{\partial \nu}\right) \quad \text{by (Green's identity)}$$

Since $G = 0$ on $\partial B_\rho(0)$, we have

$$\int_{\partial B_\rho(0)} \frac{\partial G}{\partial r} u = \int_{\partial B_\varepsilon(0)} \left(G \frac{\partial u}{\partial r} - \frac{\partial G}{\partial r} u \right)$$

Observe

$$\left| \int_{\partial B_\varepsilon(0)} G \frac{\partial u}{\partial r} \right| \leq \|\nabla u\|_{L^\infty} |G(\varepsilon, 0)| \left(4\pi\varepsilon^2\right)$$

$$\leq C \left(\frac{\cos(\sqrt{c}\,\varepsilon)}{4\pi\varepsilon} + k \frac{\sin(\sqrt{c}\,\varepsilon)}{\varepsilon} \right) 4\pi\varepsilon^2 \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

Therefore we obtain

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(0)} -\frac{\partial G}{\partial r} \cdot u = \int_{\partial B_\rho(0)} \frac{\partial G}{\partial r} u$$

At $\partial B_\rho(0)$:

$$\left.\frac{\partial G}{\partial r}\right|_{r=\rho} = \frac{\sqrt{c}\sin(\sqrt{c}\,r)}{4\pi r} + k\frac{\sqrt{c}\cos(\sqrt{c}\,r)}{r} + \underbrace{\left(-\frac{\cos\sqrt{c}\,r}{4\pi} \cancel{+ k \frac{\sin(\sqrt{c}\,r)}{1}} \right)}_{\frac{d}{dr}\left(\frac{1}{r}\right)\big|_{r=\rho}} \quad = 0$$

$$= \frac{\sqrt{c}}{\rho} \left(\frac{\sin(\sqrt{c}\,\rho)}{4\pi} + k\cos(\sqrt{c}\,\rho) \right)$$

$$= \frac{\sqrt{c}}{\rho} \left(\frac{\sin^2(\sqrt{c}\,\rho)}{4\pi\sin(\sqrt{c}\,\rho)} + \frac{\cos^2(\sqrt{c}\,\rho)}{4\pi\sin(\sqrt{c}\,\rho)} \right) = \frac{\sqrt{c}\,\rho}{4\pi\rho^2\sin(\sqrt{c}\,\rho)}$$

$$+\frac{\partial G}{\partial r}\Big|_{r=\varepsilon} = \frac{\sqrt{c}\sin(\sqrt{c}\varepsilon)}{4\pi\varepsilon} + k\frac{\sqrt{c}\cos(\sqrt{c}\varepsilon)}{\varepsilon}$$

$$+\left(-\frac{\cos(\sqrt{c}\varepsilon)}{4\pi} + k\frac{\sin(\sqrt{c}\varepsilon)}{1}\right)\left(-\frac{1}{\varepsilon^2}\right)$$

$$\Rightarrow \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)}\left(-\frac{\partial G}{\partial r}\right)u$$

$$= u(0)\left[4\pi\cancel{\varepsilon^2}\left(\frac{1}{\cancel{\varepsilon^2}}\right)\left(-\frac{\cos(\sqrt{c}\varepsilon)}{4\pi} + k\frac{\sin\sqrt{c}\varepsilon}{1}\right)\right. \to \underline{u(0)}$$

$$\Rightarrow \boxed{u(0) = \frac{\sqrt{c}\rho}{\sin(\sqrt{c}\rho)}\frac{1}{4\pi\rho^2}\int_{\partial B_\rho(0)} u(x)\,d\sigma(x)}$$