

Notes on the regularity of harmonic map systems

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Abstract

In this note, we provide an alternative proof of $C^{1,\alpha}$ -regularity of continuous weak solutions to the system of harmonic map or heat flow of harmonic maps by Riesz potential estimates between Morrey spaces.

1 Introduction

It is well-known that the higher order regularity theory asserts any *continuous* weak solution to both the system of harmonic maps and heat flow of harmonic maps is smooth (see [9] [10]). The classical technique in the literature ([9] [10]) is the *hole filling* argument, which enables to establish (i) C^0 -regularity implies C^α -regularity for all $\alpha \in (0, 1)$, and (ii) C^α -regularity implies $C^{1,\beta}$ -regularity for some $\beta \in (0, 1)$. See Schoen [12] for another proof. In this note, we will give an alternative proof of (ii) by the application of Riesz potential estimates between Morrey spaces. Such an estimate was due to Adams [1] in \mathbb{R}^n with the Euclidean metric. Here we extend such an estimate to \mathbb{R}^n with the parabolic metric, in order to handle the equation of harmonic map heat flows.

Now we recall the notions of Riesz potential and Morrey space on \mathbb{R}^n . Let $d(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a distance function on \mathbb{R}^n . Let \hat{n} denote the Hausdorff dimension of \mathbb{R}^n with respect to the distance d . On (\mathbb{R}^n, d) , we define

Definition 1.1 For any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$, and $0 < \beta \leq \hat{n}$, the Riesz potential of f of order β is defined by

$$I_\beta(f)(x) \equiv \int_{\mathbb{R}^n} \frac{|f(y)|}{d(x, y)^{\hat{n}-\beta}} dy, \quad x \in \mathbb{R}^n. \quad (1.1)$$

For $x \in \mathbb{R}^n$ and $r > 0$, let $B_r^d(x) = \{y \in \mathbb{R}^n : d(y, x) \leq r\}$ be the ball with center x and radius r with respect to the distance d .

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Definition 1.2 For $\Omega \subset \mathbb{R}^n$, $1 \leq p < +\infty$, and $0 \leq \lambda \leq \hat{n}$, the Morrey space $M^{p,\lambda}(\Omega)$ is defined to be the collection of functions $f \in L^p(\Omega)$ satisfying

$$\|f\|_{M^{p,\lambda}(\Omega)} \equiv \sup_{r>0, x \in \Omega} \left(r^{\lambda-\hat{n}} \int_{\Omega \cap B_r^d(x)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty. \quad (1.2)$$

For the purpose of this note, we also need to recall the weak Morrey space.

Definition 1.3 For $\Omega \subset \mathbb{R}^n$ and $1 \leq p < +\infty$, the weak L^p space $L^{p,*}(\Omega)$, is defined to be the collection of measurable functions f on Ω such that

$$\|f\|_{L^{p,*}(\Omega)}^p \equiv \sup_{t>0} t^p |\{x \in \Omega \mid |f(x)| > t\}| < +\infty. \quad (1.3)$$

For $0 \leq \lambda \leq \hat{n}$, the weak Morrey space $M_*^{p,\lambda}(\Omega)$ is defined to be the set of functions $f \in L^{p,*}(\Omega)$ satisfying

$$\|f\|_{M_*^{p,\lambda}(\Omega)}^p \equiv \sup_{x \in \Omega, r>0} \left\{ r^{\lambda-\hat{n}} \|f\|_{L^{p,*}(\Omega \cap B_r^d(x))}^p \right\} < +\infty. \quad (1.4)$$

For the Euclidean distance $d(x, y) = |x - y|$ on \mathbb{R}^n , we have $\hat{n} = n$ and simply denote $B_r(x)$ as $B_r^d(x)$.

In this note, we consider the higher order regularity issues to the following two systems:

(i) A map $u \in H^1(\Omega, \mathbb{R}^m)$ (for some $m \geq 1$) solves

$$-\Delta u = Q(u)(\nabla u, \nabla u) \quad \text{in } \Omega, \quad (1.5)$$

where $Q \in C^\infty(\mathbb{R}^m \times (\mathbb{R}^{nm})^2, \mathbb{R}^m)$ has the quadratic growth in the last two variables:

$$|Q(p)(v, w)| \leq C(R)|v||w|, \quad \forall p \in \mathbb{R}^m \text{ with } |p| \leq R, \text{ and } v, w \in \mathbb{R}^{nm} \quad (1.6)$$

for some positive constant $C = C(R)$ for any $R > 0$. It is readily seen that (1.5) includes the equation of harmonic maps from Ω to a Riemannian manifold $(N, h) \subset \mathbb{R}^m$.

(ii) A map $u \in H^1(\Omega \times [0, T], \mathbb{R}^m)$ ($T > 0$) solves

$$u_t - \Delta u = Q(u)(\nabla u, \nabla u) \quad \text{in } \Omega \times (0, T), \quad (1.7)$$

where Q enjoys the property (1.6). It is also readily seen that (1.7) includes the heat flow of harmonic maps from Ω to a Riemannian manifold $(N, h) \subset \mathbb{R}^m$.

In this note, we provide an alternative approach, based on the Riesz potential estimate between Morrey spaces, to the following two Theorems. Both Theorems are classical results in the literature (see [9] [10] [11]).

Theorem 1.4 *There exists $\epsilon_0 > 0$ such that if $u(x) \in H^1(\Omega, \mathbb{R}^m)$ is a weak solution of (1.5)-(1.6), and satisfies, for $x_0 \in \Omega$ and $0 < r_0 < \text{dist}(x_0, \partial\Omega)$,*

$$\text{osc}_{B_{r_0}(x_0)} u \leq \epsilon_0, \quad (1.8)$$

then $u \in C^\infty(B_{\frac{r_0}{2}}(x_0), \mathbb{R}^m)$.

Theorem 1.5 *There exists $\epsilon_0 > 0$ such that if $u(x) \in H^1(\Omega \times [0, T], \mathbb{R}^m)$ is a weak solution of (1.7) -(1.6) and satisfies, for $(x_0, t_0) \in \Omega \times (0, T)$ and $0 < r_0 < \min\{\text{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$,*

$$\text{osc}_{B_{r_0}(x_0) \times [t_0 - r_0^2, t_0 + r_0^2]} u \leq \epsilon_0, \quad (1.9)$$

then $u \in C^\infty(B_{\frac{r_0}{2}}(x_0) \times [t_0 - \frac{r_0^2}{4}, t_0 + \frac{r_0^2}{4}], \mathbb{R}^m)$.

For both minimizing harmonic maps and stationary harmonic maps (or stationary heat flow of harmonic maps), the small oscillation condition (1.8) (or (1.9)) can be deduced by the small renormalized energy condition (see [8] [13] [4] [2] [7] [3] [6]). In general, for weak harmonic maps (or weak heat flow of harmonic maps), (1.8) (or (1.9)) can only be guaranteed if $K_N \leq 0$ or $u(B_{r_0}(x_0))$ is contained in a geodesic convex ball (see [5] [10] [11]).

2 Proof of Theorem 1.4

In this section, we will apply the Riesz potential estimate between Morrey spaces to prove Theorem 1.4. This can also serve as a guideline to prove Theorem 1.5. The proof of the following Lemma can be found in Adams [1].

Lemma 2.1 *Let $d(x, y) = |x - y|$ be the Euclidean distance on \mathbb{R}^n .*

(i) For any $\beta > 0$, $0 < \lambda \leq n$, $1 < p < \frac{\lambda}{\beta}$, if $f \in L^p(\mathbb{R}^n) \cap M^{p, \lambda}(\mathbb{R}^n)$, then $I_\beta(f) \in L^{\tilde{p}}(\mathbb{R}^n) \cap M^{\tilde{p}, \lambda}(\mathbb{R}^n)$, where $\tilde{p} = \frac{\lambda p}{\lambda - p\beta}$. Moreover,

$$\|I_\beta(f)\|_{L^{\tilde{p}}(\mathbb{R}^n)} \leq C \|f\|_{M^{p, \lambda}(\mathbb{R}^n)}^{\frac{\beta p}{\lambda}} \|f\|_{L^p(\mathbb{R}^n)}^{1 - \frac{\beta p}{\lambda}} \quad (2.1)$$

$$\|I_\beta(f)\|_{M^{\tilde{p}, \lambda}(\mathbb{R}^n)} \leq C \|f\|_{M^{p, \lambda}(\mathbb{R}^n)}. \quad (2.2)$$

*(ii) For $0 < \beta < \lambda \leq n$, if $f \in L^1(\mathbb{R}^n) \cap M^{1, \lambda}(\mathbb{R}^n)$, then $I_\beta(f) \in L^{\frac{\lambda}{\lambda - \beta}, * }(\mathbb{R}^n) \cap M_*^{\frac{\lambda}{\lambda - \beta}, \lambda}(\mathbb{R}^n)$. Moreover,*

$$\|I_\beta f\|_{L^{\frac{\lambda}{\lambda - \beta}, * }(\mathbb{R}^n)} \leq C \|f\|_{M^{1, \lambda}(\mathbb{R}^n)}^{\frac{\beta}{\lambda}} \|f\|_{L^1(\mathbb{R}^n)}^{1 - \frac{\beta}{\lambda}} \quad (2.3)$$

$$\|I_\beta(f)\|_{M_*^{\frac{\lambda}{\lambda - \beta}, \lambda}(\mathbb{R}^n)} \leq C \|f\|_{M^{1, \lambda}(\mathbb{R}^n)}. \quad (2.4)$$

Proof of Theorem 1.4:

Claim 1. $\nabla u \in M^{2,2-2\alpha}(B_{\frac{r_0}{2}}(x_0))$ for any $\alpha \in (0, 1)$.

To see it, let $B_r(x) \subset B_{r_0}(x_0)$ be any ball. Let $v : B_r(x) \rightarrow \mathbb{R}^m$ solve

$$\Delta v = 0 \text{ in } B_r(x), \quad v = u \text{ on } \partial B_r(x). \quad (2.5)$$

Multiplying both (1.5) and (2.5) by $u - v$, integrating over $B_r(x)$, we get

$$\int_{B_r(x)} |\nabla(u - v)|^2 \leq C \int_{B_r(x)} |\nabla u|^2 |u - v| \leq C\epsilon_0 \int_{B_r(x)} |\nabla u|^2,$$

where the last inequality follows from the maximum principle of v :

$$\sup_{B_r(x)} |u - v| \leq \text{osc}_{B_r(x)} u \leq \epsilon_0.$$

Hence, for any $\theta \in (0, 1)$,

$$\begin{aligned} (\theta r)^{2-n} \int_{B_{\theta r}(x)} |\nabla u|^2 &\leq (\theta r)^{2-n} \int_{B_{\theta r}(x)} |\nabla v|^2 + \theta^{2-n} r^{2-n} \int_{B_r(x)} |\nabla(u - v)|^2 \\ &\leq C(\theta^2 + \epsilon_0 \theta^{2-n}) r^{2-n} \int_{B_r(x)} |\nabla u|^2, \end{aligned}$$

where we have used the following inequality for v :

$$(\theta r)^{2-n} \int_{B_{\theta r}(x)} |\nabla v|^2 \leq C\theta^2 r^{2-n} \int_{B_r(x)} |\nabla v|^2.$$

For any $\alpha \in (0, 1)$, first choose $\theta = \theta_0(\alpha) \in (0, 1)$ such that $C\theta_0^2 \leq \frac{1}{2}\theta_0^{2\alpha}$ and then choose ϵ_0 such that $C\theta_0^{2-n}\epsilon_0 \leq \frac{1}{2}\theta_0^{2\alpha}$. We obtain

$$(\theta_0 r)^{2-n} \int_{B_{\theta_0 r}(x)} |\nabla u|^2 \leq \theta_0^{2\alpha} r^{2-n} \int_{B_r(x)} |\nabla u|^2.$$

Iterating this inequality finitely many times yields

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq C r^{2\alpha} r_0^{2-n} \int_{B_{r_0}(x_0)} |\nabla u|^2, \quad \forall B_r(x) \subset B_{\frac{r_0}{2}}(x_0).$$

This implies Claim 1.

Claim 2. $\nabla u \in L^q(B_{\frac{r_0}{2}}(x_0))$ for any $1 < q < +\infty$.

For simplicity, assume $x_0 = 0$ and $r_0 = 4$. Let $\eta \in C_0^\infty(B_4(0))$ be such that $0 \leq \eta(x) \leq 1$, $\eta(x) \equiv 1$ on $B_2(0)$, and $|\nabla \eta| + |\nabla^2 \eta| \leq 8$. Set $v = \eta u$. Then v satisfies

$$-\Delta v(x) = H(x), \quad H(x) = Q(u)(\nabla u, \nabla u)\eta - (u - u_2)\Delta \eta - 2\nabla u \nabla \eta, \quad (2.6)$$

where $u_2 = \frac{1}{|B_2(0)|} \int_{B_2(0)} u$ is the average of u over B_2 . Let $G(x)$ be the fundamental solution of Δ on \mathbb{R}^n . Then we have

$$v(x) = \int_{\mathbb{R}^n} G(x - y)H(y) dy$$

so that

$$\begin{aligned}
|\nabla v(x)| &= \left| \int_{\mathbb{R}^n} \nabla G(x-y)H(y) dy \right| \\
&\leq C(n) \int_{\mathbb{R}^n} \frac{|H(y)|}{|x-y|^{n-1}} dy \\
&= C(n)I_1(|H|)(x)
\end{aligned}$$

for some positive constant $C(n)$.

Since $H = 0$ outside $B_4(0)$, it is not difficult to see that $H \in M^{1,2-2\alpha}(\mathbb{R}^n)$ and

$$\begin{aligned}
\|H\|_{M^{1,2-2\alpha}(\mathbb{R}^n)} &\leq C \|H\|_{M^{1,2-2\alpha}(B_4(0))} \\
&\leq C \sup_{B_r(x) \subset B_4(0)} \left\{ r^{2-2\alpha-n} \int_{B_r(x)} |H(y)| dy \right\} \\
&\leq C [\text{osc}_{B_4(0)} u + \|\nabla u\|_{M^{2,2-2\alpha}(B_4(0))}] \leq C.
\end{aligned}$$

Hence Lemma 2.1 implies that for any $0 < \alpha < \frac{1}{2}$, $|\nabla v| \in L^{\frac{2-\alpha}{1-2\alpha},*}(\mathbb{R}^n)$. Thus $\nabla u \in L^{\frac{2-\alpha}{1-2\alpha},*}(B_2(0))$. Since $L^{\frac{2-\alpha}{1-2\alpha},*}(B_2(0)) \subset L^q(B_2(0))$ for any $q < \frac{2-2\alpha}{1-2\alpha}$, we have $\nabla u \in L^q(B_2(0))$ for any $q < \frac{2-2\alpha}{1-2\alpha}$. Since

$$\lim_{\alpha \uparrow \frac{1}{2}} \frac{2-2\alpha}{1-2\alpha} = +\infty,$$

$\nabla u \in L^q(B_2(0))$ for any $1 \leq q < \infty$. Applying the $W^{2,q}$ -estimate of the equation (1.5), we conclude that $u \in W^{2,q}(B_2(0))$ for any $q \in (1, +\infty)$. In particular, the Sobolev embedding theorem implies $\nabla u \in C^\delta(B_2(0))$ for any $\delta \in (0, 1)$. Now, we can apply the Schauder theory to show that $u \in C^\infty(B_2(0))$. \square

3 Proof of Theorem 1.5

In this section, we will extend Adams' Riesz potential estimate to \mathbb{R}^{n+1} , equipped with the parabolic distance δ , and then apply it to prove Theorem 1.5.

First, let's recall the notion of parabolic distance δ on \mathbb{R}^{n+1} .

$$\delta((x, t), (y, s)) = \max \left\{ |x - y|, \sqrt{|t - s|} \right\}, \quad \forall (x, t), (y, s) \in \mathbb{R}^{n+1}.$$

Denote $P_r(x, t) = B_r^\delta(x, t)$, the ball center at (x, t) with radius r with respect to the distance δ . Notice that $P_r(x, t) = B_r(x) \times [t - r^2, t + r^2]$. For short, write $P_r = P_r(0, 0)$.

Notice that the Hausdorff dimension of \mathbb{R}^{n+1} with respect to the parabolic distance δ is $n + 2$. Hence the Riesz potential of order $\beta \in [0, n + 2]$ on \mathbb{R}^{n+1} is given by

$$\tilde{I}_\beta(f) = \iint_{\mathbb{R}^{n+1}} \frac{|f(y, s)|}{\delta((x, t), (y, s))^{n+2-\beta}} dy ds, \quad f \in L^p(\mathbb{R}^{n+1}),$$

and the Morrey space $\widetilde{M}^{p,\lambda}(\Omega_T)$ (or the weak Morrey space $\widetilde{M}_*^{p,\lambda}(\Omega_T)$ respectively) on $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, for $1 \leq p < +\infty$ and $0 \leq \lambda \leq n+2$, is the collection of functions f on Ω_T such that

$$\|f\|_{\widetilde{M}^{p,\lambda}(\Omega_T)}^p \equiv \sup_{r>0, z \in \Omega_T} r^{\lambda-(n+2)} \|f\|_{L^p(\Omega_T \cap P_r(z))}^p < \infty$$

$$\left(\text{or } \|f\|_{\widetilde{M}_*^{p,\lambda}(\Omega_T)}^p \equiv \sup_{r>0, z \in \Omega_T} r^{\lambda-(n+2)} \|f\|_{L^{p,*}(\Omega_T \cap P_r(z))}^p < \infty \text{ respectively} \right).$$

Now we are ready to prove the Riesz potential estimate on $(\mathbb{R}^{n+1}, \delta)$, namely,

Theorem 3.1 (i) For any $\beta > 0$, $0 < \lambda \leq n+2$, $1 < p < \frac{\lambda}{\beta}$, if $f \in L^p(\mathbb{R}^{n+1}) \cap \widetilde{M}^{p,\lambda}(\mathbb{R}^{n+1})$, then $\widetilde{I}_\beta(f) \in L^{\tilde{p}}(\mathbb{R}^{n+1}) \cap \widetilde{M}^{\tilde{p},\lambda}(\mathbb{R}^{n+1})$, where $\tilde{p} = \frac{p\lambda}{\lambda-p\beta}$. Moreover,

$$\|\widetilde{I}_\beta(f)\|_{L^{\tilde{p}}(\mathbb{R}^{n+1})} \leq C \|f\|_{\widetilde{M}^{p,\lambda}(\mathbb{R}^{n+1})}^{\frac{\beta p}{\lambda}} \|f\|_{L^p(\mathbb{R}^{n+1})}^{1-\frac{\beta p}{\lambda}} \quad (3.1)$$

$$\|\widetilde{I}_\beta(f)\|_{\widetilde{M}^{\tilde{p},\lambda}(\mathbb{R}^{n+1})} \leq C \|f\|_{\widetilde{M}^{p,\lambda}(\mathbb{R}^{n+1})}. \quad (3.2)$$

(ii) For $0 < \beta < \lambda \leq n+2$, if $f \in L^1(\mathbb{R}^{n+1}) \cap \widetilde{M}^{1,\lambda}(\mathbb{R}^{n+1})$, then $\widetilde{I}_\beta(f) \in L^{\frac{\lambda}{\lambda-\beta},*}(\mathbb{R}^{n+1}) \cap \widetilde{M}_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^{n+1})$. Moreover,

$$\|\widetilde{I}_\beta(f)\|_{L^{\frac{\lambda}{\lambda-\beta},*}(\mathbb{R}^{n+1})} \leq C \|f\|_{\widetilde{M}^{1,\lambda}(\mathbb{R}^{n+1})}^{\frac{\beta}{\lambda}} \|f\|_{L^1(\mathbb{R}^{n+1})}^{1-\frac{\beta}{\lambda}} \quad (3.3)$$

$$\|\widetilde{I}_\beta(f)\|_{\widetilde{M}_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^{n+1})} \leq C \|f\|_{\widetilde{M}^{1,\lambda}(\mathbb{R}^{n+1})}. \quad (3.4)$$

Proof. Assume $f \not\equiv 0$. Then

$$\widetilde{I}_\beta(f)(x, t) = \iint_{P_\epsilon(x, t)} f + \iint_{\mathbb{R}^{n+1} \setminus P_\epsilon(x, t)} f := I_1 + I_2,$$

where $\epsilon > 0$ is to be determined later. For $k \in \mathbb{Z}$, let

$$A_k = \{(y, s) | (y, s) \in P_{2^{k+1}\epsilon}(x, t) \setminus P_{2^k\epsilon}(x, t), \}.$$

Since $\beta \leq n+2$ and $\beta < \frac{\lambda}{p}$, we have

$$\begin{aligned} |I_2| &\leq \sum_{k=1}^{\infty} \iint_{A_k} \delta((x, t), (y, s))^{\beta-(n+2)} |f(y, s)| dy ds \\ &\leq \sum_{k=1}^{\infty} (2^k\epsilon)^{\beta-(n+2)} (2^{k+1}\epsilon)^{n+2-\frac{\lambda}{p}} \mathcal{M}_{\frac{\lambda}{p}} f(x, t) \\ &\leq C \epsilon^{\beta-\frac{\lambda}{p}} \mathcal{M}_{\frac{\lambda}{p}} f(x, t), \end{aligned}$$

where

$$\mathcal{M}_\gamma g(x, t) = \sup_{r>0} r^{\gamma-(n+2)} \iint_{P_r(x, t)} |g(y, s)| dy ds$$

for $0 \leq \gamma \leq n+2$. Notice that $\mathcal{M}_0 g$ is the parabolic Hardy-Littlewood maximal function of g . Similarly, we have

$$\begin{aligned} |I_1| &\leq \sum_{k=1}^{\infty} \iint_{A_{-k}} \delta((x, t), (y, s))^{\beta-(n+2)} |f(y, s)| dy ds \\ &\leq \sum_{k=1}^{\infty} (2^{-k}\epsilon)^{\beta-(n+2)} (2^{-k+1}\epsilon)^{n+2} \mathcal{M}_0(f)(x, t) \\ &\leq C\epsilon^{\beta} \mathcal{M}_0(f)(x, t). \end{aligned}$$

Since

$$\mathcal{M}_{\frac{\lambda}{p}}(f)(x, t) \leq C [\mathcal{M}_{\lambda}(|f|^p)(x, t)]^{\frac{1}{p}} \leq C \|f\|_{\widetilde{M}^{p, \lambda}(\mathbb{R}^{n+1})},$$

we conclude, by choosing $\epsilon = \epsilon(x, t) = \left(\frac{\|f\|_{\widetilde{M}^{p, \lambda}(\mathbb{R}^{n+1})}}{\mathcal{M}_0(f)(x, t)} \right)^{\frac{p}{\lambda}}$, that

$$|\widetilde{I}_{\beta} f(x, t)| \leq C [\|f\|_{\widetilde{M}^{p, \lambda}(\mathbb{R}^{n+1})}]^{\frac{\beta p}{\lambda}} [\mathcal{M}_0 f(x, t)]^{1-\frac{\beta p}{\lambda}}.$$

When $p > 1$, both estimates (3.1) and (3.2) follow by the well-known L^p estimate for $\mathcal{M}_0(f)$ (see [14]).

When $p = 1$, for any $l > 0$, we have

$$\begin{aligned} &\left| \{(x, t) \in \mathbb{R}^{n+1} : |\widetilde{I}_{\beta} f(x, t)| > Cl\} \right| \\ &\leq \left| \{(x, t) \in \mathbb{R}^{n+1} : \mathcal{M}_0(f)(x, t) > \frac{l^{\frac{\lambda}{\lambda-\beta}}}{\|f\|_{\widetilde{M}^{1, \lambda}(\mathbb{R}^{n+1})}}\} \right| \\ &\leq C \frac{\|f\|_{\widetilde{M}^{1, \lambda}(\mathbb{R}^{n+1})}^{\frac{\beta}{\lambda-\beta}}}{l^{\frac{\lambda}{\lambda-\beta}}} \|f\|_{L^1(\mathbb{R}^{n+1})}, \end{aligned}$$

where we have used the weak L^1 -estimate of \mathcal{M}_0 ([14] page 5, Theorem 1). This implies both (3.3) and (3.4). \square

In order to prove Theorem 1.5, we need the following simple estimate on the heat kernel G .

Lemma 3.1 *Let*

$$\begin{aligned} G(x, t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t}\right) && \text{for } t > 0, \\ &= 0 && \text{for } t \leq 0 \end{aligned}$$

be the fundamental solution of the heat equation on \mathbb{R}^n . Then

$$|\nabla G(x, t)| \leq \frac{C}{\delta((x, t), (0, 0))^{n+1}}, \quad \forall (x, t) \in \mathbb{R}^{n+1}. \quad (3.5)$$

Proof. It suffices to show (3.1) for $t > 0$. Direct calculations give

$$\nabla G(x, t) = -\frac{x}{2t} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Set $y = \frac{|x|}{\sqrt{|t|}}$, we have

$$\begin{aligned} |\nabla G(x, t)|(|x| + \sqrt{|t|})^{n+1} &\leq \frac{|x|}{2(\sqrt{|t|})^{n+2}} \exp\left(-\frac{|x|^2}{4t}\right)(|x| + \sqrt{|t|})^{n+1} \\ &\leq y(1+y)^{n+1} \exp\left(-\frac{|y|^2}{4}\right). \end{aligned}$$

Since

$$\lim_{y \rightarrow \infty} y(1+y)^{n+1} \exp\left(-\frac{|y|^2}{4}\right) < +\infty$$

and

$$\delta((x, t), (0, 0)) \leq |x| + \sqrt{|t|} \leq 2\delta((x, t), (0, 0)),$$

the conclusion follows. \square

Proof of Theorem 1.5.

Similar to the proof of Theorem 1.4, we divide it into two claims.

Claim 3. $\nabla u \in M^{2,2-2\alpha}(P_{\frac{r_0}{2}}(x_0))$ for any $\alpha \in (0, 1)$.

To see it, let $P_r(x_1, t_1) \subset P_{r_0}(x_0, t_0)$ and $v : P_r(x_1, t_1) \rightarrow \mathbb{R}^m$ solve

$$v_t - \Delta v = 0 \text{ in } P_r(x_1, t_1), \quad v = u \text{ on } \partial_p P_r(x_1, t_1), \quad (3.6)$$

where $\partial P_r(x_1, t_1) = (B_r(x_1) \times \{t_1 - r^2\}) \cup (\partial B_r(x_1) \times [t_1 - r^2, t_1 + r^2])$ denotes the parabolic boundary of $P_r(x_1, t_1)$.

Multiplying both (1.7) and (3.6) by $u - v$, integrating over $P_r(x_1, t_1)$, we get

$$\int_{P_r(x_1, t_1)} |\nabla(u - v)|^2 \leq C \int_{P_r(x_1, t_1)} |\nabla u|^2 |u - v| \leq C\epsilon_0 \int_{P_r(x_1, t_1)} |\nabla u|^2,$$

where the last inequality follows from the maximum principle of v :

$$\sup_{P_r(x_1, t_1)} |u - v| \leq \text{osc}_{P_r(x_1, t_1)} u \leq \epsilon_0.$$

Hence, for any $\theta \in (0, 1)$,

$$\begin{aligned} (\theta r)^n \int_{P_{\theta r}(x_1, t_1)} |\nabla u|^2 &\leq (\theta r)^{-n} \int_{P_{\theta r}(x_1, t_1)} |\nabla v|^2 + \theta^{-n} r^{-n} \int_{P_r(x_1, t_1)} |\nabla(u - v)|^2 \\ &\leq C(\theta^2 + \epsilon_0 \theta^{-n}) r^{-n} \int_{P_r(x_1, t_1)} |\nabla u|^2, \end{aligned}$$

where we have used the following inequality for v :

$$(\theta r)^{-n} \int_{P_{\theta r}(x_1, t_1)} |\nabla v|^2 \leq C\theta^2 r^{-n} \int_{P_{\theta r}(x_1, t_1)} |\nabla v|^2.$$

For any $\alpha \in (0, 1)$, first choose $\theta = \theta_0(\alpha) \in (0, 1)$ such that $C\theta_0^2 \leq \frac{1}{2}\theta_0^{2\alpha}$ and then choose ϵ_0 such that $C\theta_0^{-n}\epsilon_0 \leq \frac{1}{2}\theta_0^{2\alpha}$. We obtain

$$(\theta_0 r)^{-n} \int_{P_{\theta_0 r}(x_1, t_1)} |\nabla u|^2 \leq \theta_0^{2\alpha} r^{-n} \int_{P_r(x_1, t_1)} |\nabla u|^2.$$

Iterating this inequality finitely many times yields

$$r^{-n} \int_{P_r(x_1, t_1)} |\nabla u|^2 \leq C r^{2\alpha} r_0^{-n} \int_{P_{r_0}(x_0, t_0)} |\nabla u|^2, \quad \forall P_r(x_1, t_1) \subset P_{r_0}(x_0, t_0).$$

This implies Claim 3.

Claim 4. $\nabla u \in L^q(P_{\frac{r_0}{2}}(x_0, t_0))$ for any $1 < q < +\infty$.

For simplicity, assume $(x_0, t_0) = (0, 0)$ and $r_0 = 4$. Let $\eta \in C_0^\infty(P_4)$ be such that $0 \leq \eta(x) \leq 1$, $\eta(x) \equiv 1$ on P_2 , and $|\eta_t| + |\nabla \eta| + |\nabla^2 \eta| \leq 32$. Set $z = (x, t)$ and $v(z) = u(z)\eta(z)$. Then v satisfies in \mathbb{R}^{n+1} :

$$v_t - \Delta v = F, \quad F \equiv \eta Q(u)(\nabla u, \nabla u) - (u - u_4)(\eta_t - \Delta \eta) - 2\nabla u \nabla \eta,$$

where $u_4 = \frac{1}{|P_4|} \int_{P_4} u$ is the average of u over P_4 . Then we have

$$v(z) = \iint_{\mathbb{R}^{n+1}} G(z-w)F(w) dw$$

so that by Lemma 3.1 we have

$$|\nabla v(z)| = \left| \iint_{\mathbb{R}^{n+1}} \nabla G(z-w)F(w) dw \right| \leq C I_1(|F|)(z).$$

Since $F \equiv 0$ outside P_4 , we have $F \in M^{1,2-2\alpha}(\mathbb{R}^{n+1})$ and

$$\begin{aligned} \|F\|_{M^{1,2-2\alpha}(\mathbb{R}^{n+1})} &\leq C \|F\|_{M^{1,2-2\alpha}(P_4)} \\ &\leq C \sup_{P_r(z) \in P_4} \left\{ r^{-2\alpha-n} \int_{P_r(z)} |F(w)| dw \right\} \\ &\leq C [\text{osc}_{P_4} u + \|\nabla u\|_{M^{2,2-2\alpha}(P_4)}] \leq C. \end{aligned}$$

Hence Theorem 3.1 implies that $\nabla v \in L^{\frac{2-2\alpha}{1-2\alpha},*}(\mathbb{R}^{n+1})$. Since $\lim_{\alpha \uparrow \frac{1}{2}} \frac{2-2\alpha}{1-2\alpha} = +\infty$, we conclude that $\nabla u \in L^q(P_2)$ for any $q \in (1, +\infty)$. Now, by $W_q^{2,1}$ -estimate for the heat equation, we conclude that $u \in W_q^{2,1}(P_1)$ for any $q \in (1, +\infty)$. This, combined with the Sobolev's embedding theorem implies that $\nabla u \in C^\alpha(P_1)$ for any $\alpha \in (0, 1)$. Now the Schauder's theory implies that $u \in C^\infty(P_1)$. The proof is now complete. \square

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