On the rigidity of nematic liquid crystal flow on $\mathbb{S}^2$

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Abstract

In this paper we establish the uniformity property of a simplified Ericksen-Leslie system modelling the hydrodynamics of nematic liquid crystals on the two dimensional unit sphere $\mathbb{S}^2$, namely the uniform convergence in L^2 to a steady state exponentially as t tends to infinity. The main assumption, similar to Topping [15], concerns the equation of liquid crystal director d and states that at infinity time, a weak limit d_∞ and any bubble ω_i $(1 \le i \le l)$ share a common orientation. As consequences, the uniformity property holds under various types of small initial data.

1 Introduction

The Ericksen-Leslie system modelling the hydrodynamics of nematic liquid crystals was proposed by Ericksen and Leslie during the period between 1958 and 1968 [2, 4]. It is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity field $u(x,t)$ and the macroscopic description of the microscopic orientation configuration $d(x,t)$ of rod-like liquid crystals (i.e. $d(x,t)$ is a unit vector in $\mathbb{R}^3$). In order to effectively analyze it, Lin [5] proposed a simplified version of the Ericksen-Leslie system, which is called the nematic liquid crystal flow and is given by

$$\begin{cases} u_t + u\cdot\nabla u + \nabla P = \Delta u - \nabla d \Delta d, \\ \text{div } u = 0, \\ d_t + u\cdot\nabla d = \Delta d + |\nabla d|^2 d. \end{cases} \tag{1.1}$$

Roughly speaking, the system (1.1) is a coupling between the non-homogeneous Naiver-Stokes equation and the transported heat flow of harmonic maps to $\mathbb{S}^2$. The system (1.1) has generated great interests and activities among analysts, and there have been many research works on (1.1) recently (see, e.g. [7] [8] [9] [6] [10] [3] [18] and [11]).

In this paper, we are mainly interested in the long time dynamics of the nematic liquid crystal flow (1.1) on $\mathbb{S}^2 \times (0,+\infty)$, where $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is the unit sphere that is equipped with the standard metric g_0. To begin with, we introduce some notations and explain terms in (1.1). First, the fluid velocity field $u(x,t) \in T_x\mathbb{S}^2 \times \{t\} \equiv T_x\mathbb{S}^2$, the liquid crystal molecule director field $d(x,t) \in \mathbb{S}^2$, and the pressure function $P(x,t) \in \mathbb{R}$ for $(x,t) \in \mathbb{S}^2 \times [0,+\infty)$. In the system (1.1), ∇ stands for the gradient operator on $\mathbb{S}^2$, div$(=\text{div}_{g_0})$ and $\Delta(=\Delta_{g_0} = \text{div}_{g_0}\nabla)$

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represent the divergence operator and the Laplace-Beltrami operator on $(\mathbb{S}^2, g_0)$. Hence the second equation $(1.1)_2$ describes the incompressibility of the fluid. The convection term $u \cdot \nabla u$ in the first equation $(1.1)_1$ is the directed differentiation of u with respect to the direction u itself, which is interpreted as the covariant derivative $D_u u$. Here D denotes the covariant derivative operator on $(\mathbb{S}^2, g_0)$. Following the terminology in [14], we know that Δu is also accounted for by the Bochner Laplacian operator $-D^* Du$, where D^* is the adjoint operator of D with respect to the $L^2(\mathbb{S}^2, T\mathbb{S}^2)$ inner product. Write $d = (d^1, d^2, d^3) \in \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. Then in $(1.1)_3$, $\nabla d \Delta d = \sum_{i=1}^{3} \nabla d^i \Delta d^i$, the i-th component of $u \cdot \nabla d$ equals to $g_0(u, \nabla d^i)$, $\Delta d = \big(\Delta_{g_0} d^1, \Delta_{g_0} d^2, \Delta_{g_0} d^3\big)$, and $|\nabla d|^2 = g_0(\nabla d, \nabla d)$.

We will consider the nematic liquid crystal flow (1.1) along with the initial condition:

$$\big(u(x,0), d(x,0)\big) = \big(u_0(x), d_0(x)\big), \ x \in \mathbb{S}^2, \tag{1.2}$$

where $u_0 \in L^2(\mathbb{S}^2, T\mathbb{S}^2)$ is a divergence free tangential vector field on $(\mathbb{S}^2, g_0)$ with zero average:

$$\int_{\mathbb{S}^2} u_0(x)\, dv_{g_0} = 0 \ \text{ and } \ \text{div } u_0 = 0 \text{ on } \mathbb{S}^2, \tag{1.3}$$

and $d_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2) := \Big\{ d \in H^1(\mathbb{S}^2, \mathbb{R}^3) : \ d(x) \in \mathbb{R}^3 \text{ a.e. } x \in \mathbb{S}^2 \Big\}$.

In the recent paper [6], Lin, Lin, and Wang have proved the global existence of weak solutions to the initial and boundary value problem of the nematic liquid crystal flow (1.1) on any bounded domain $\Omega \subset \mathbb{R}^2$, that are smooth away from at most finitely many singular times (see also [3] for discussion in $\mathbb{R}^2$). The uniqueness of the global weak solution constructed by [6] has been given by [10] (see [18] for a different proof). It is an open and challenging problem whether there exists a global weak solution to the problem (1.1)-(1.2) in $\mathbb{R}^3$. On the other hand, concerning the long time behavior of the global solution established in [6], the authors have only shown the subsequential convergence at time infinity. Since the structure of the set of equilibrium to the problem (1.1) is a continuum, whether the limiting point of (1.1) at time infinity is unique is a very interesting problem.

The aim of this paper is to study this problem for (1.1)-(1.2) under certain assumptions on the initial data. The key ingredient is to apply a Łojaciewicz-Simon type approach, which was originally studied by L. Simon in [12] for a large class of nonlinear geometric evolution equations. Due to the nonlinear constraint $|d| = 1$, we cannot establish the related Łojaciewicz-Simon type inequality for the Ginzburg-Landau approximation version of (1.1) as in [16,17]. However, there is a counterpart for our problem if we consider the domain being the unit sphere, namely the heat flow of harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ considered by Topping [15].

We would like to point that by slightly modifying the proof of the regularity Theorem 1.2 in [6], the existence Theorem 1.3 of [6] and the uniqueness theorem of [10] can be extended to (1.1)-(1.2) on $\mathbb{S}^2$ without much difficulty. More precisely, we have

Theorem 1.1. *Suppose $u_0 \in L^2(\mathbb{S}^2, T\mathbb{S}^2)$ satisfies (1.3) and $d_0 \in H^1(\mathbb{S}^2, \mathbb{S}^2)$. Then there exists a unique, global weak solution $(u, d) \in \big(L_t^\infty L_x^2(\mathbb{S}^2 \times [0, +\infty), T\mathbb{S}^2)\big) \times \big(L_t^\infty H_x^1(\mathbb{S}^2 \times [0, +\infty), \mathbb{S}^2)\big)$ of (1.1)-(1.2) such that the following properties hold:*

(i) There exists a non-negative integer L depending only on (u_0, d_0) and $0 < T_1 < \cdots < T_L < +\infty$ such that

$$(u,d) \in C^\infty\Big(\mathbb{S}^2 \times \big((0,+\infty) \setminus \{T_i\}_{i=1}^L\big)\Big).$$

(ii) For $1 \le i \le L$,

$$\limsup_{t\uparrow T_i} \max_{x\in\mathbb{S}^2} \int_{\mathbb{S}^2\cap B_r(x)} (|u|^2 + |\nabla d|^2)(y,t)\, dv_{g_0}(y) \ge 8\pi, \ \forall r > 0. \tag{1.4}$$

(iii) There exist $t_k \uparrow +\infty$, a harmonic map $d_\infty \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, and nontrivial harmonic maps $\{\omega_i\}$ $(1 \le i \le K$ for some non-negative integer $K)$ such that

$$u(t_k) \to 0 \text{ strongly in } H^1(\mathbb{S}^2),\ d(t_k) \to d_\infty \text{ weakly in } H^1(\mathbb{S}^2, \mathbb{S}^2),$$

and

$$\int_{\mathbb{S}^2} |\nabla d(t_k)|^2\, dv_{g_0} \to \int_{\mathbb{S}^2} |\nabla d_\infty|^2\, dv_{g_0} + \sum_{i=1}^K \int_{\mathbb{S}^2} |\nabla \omega_i|^2\, dv_{g_0}. \tag{1.5}$$

(iv) If

$$\int_{\mathbb{S}^2} (|u_0|^2 + |\nabla d_0|^2)\, dv_{g_0} \le 8\pi,$$

then $(u,d) \in C^\infty(\mathbb{S}^2 \times [0,+\infty), T\mathbb{S}^2 \times \mathbb{S}^2)$. Moreover, there exist $t_k \uparrow +\infty$ and a harmonic map $d_\infty \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that

$$\big(u(t_k), d(t_k)\big) \to \big(0, d_\infty\big) \text{ strongly in } H^1(\mathbb{S}^2).$$

The main results in this paper address the issue of unique limit at time infinity of the solution (u,d) to (1.1)-(1.2) given by Theorem 1.1. We find sufficient conditions, similar to that by [15] on the heat flow of harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$, on d_∞ and $\omega_i, 1 \le i \le K$, to guarantee the uniform property of (1.1) at time infinity, i.e., d_∞ is the unique limit of d at $t = +\infty$. It is worth mentioning that in contrast with the heat flow of harmonic maps considered by [13] and [15], because the local energy inequality of (1.1) (see [6] Lemma 4.2) involves L^2-norm of both $|P - c|$ and $|\nabla d|^2$, we cannot show the uniqueness of the location of bubbling positions of the bubbles ω_i $(1 \le i \le K)$.

The paper is written as follows. In §2, we discuss the uniform limit of (1.1) in the space $L^2(\mathbb{S}^2)$ at $t = +\infty$, and prove the first main theorem 2.1 and corollary 2.1. In §3, we discuss the uniform limit of (1.1) in the space $H^k(\mathbb{S}^2)$ $(k \ge 1)$ at $t = +\infty$, and prove the second main theorem 3.1 and corollary 3.1.

2 Uniform limit in $L^2(\mathbb{S}^2)$

Unless explicitly specified, henceforth we will not distinguish the inner product between $T\mathbb{S}^2$ and $\mathbb{R}^3$, nor $d \in \mathbb{S}^2$ and its isometric embedding (d^1, d^2, d^3) into $\mathbb{R}^3$. For the sake of simplicity, $\|\cdot\|_{L^2(\mathbb{S}^2)}$ will be shorthanded by $\|\cdot\|$, and $\|\cdot\|_{L^p(\mathbb{S}^2)}$ will be abbreviated by $\|\cdot\|_{L^p}$ for $p \ne 2$.

We will establish the L^2-convergence of the flow (u,d) to (1.1)-(1.3) to a single steady state solution $(0, d_\infty)$, with $d_\infty \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ a harmonic map, as t tends to infinity. We first recall some notations introduced by Topping [15].

Let us consider a map $d \in H^1(\mathbb{S}^2, \mathbb{S}^2)$. We use $z = x + iy$ as a complex coordinate on the domain $\mathbb{S}^2 \equiv \overline{\mathbb{C}}$, via the stereographic projection. Set $dz = dx + idy, d\bar{z} = dx - idy$, and write the metric g_0 on the domain $\mathbb{S}^2$ as $\sigma(z)^2 dz d\bar{z}$, where

$$\sigma(z) = \frac{2}{1+|z|^2},\ z \in \overline{\mathbb{C}}$$

Let u denote a complex coordinate on the target $\mathbb{S}^2$, and write the metric g_0 on the target $\mathbb{S}^2$ as $\sigma(u)^2 du d\bar{u}$. and write

$$d_z = \frac{1}{2}(d_x - i d_y),\quad d_{\bar{z}} = \frac{1}{2}(d_x + i d_y).$$

The ∂-energy and $\bar{\partial}$-energy of d are given by

$$E_\partial(d) = \frac{i}{2}\int_{\mathbb{C}} \rho^2(d)|d_z|^2\, dz \wedge d\bar{z},\quad E_{\bar{\partial}}(d) = \frac{i}{2}\int_{\mathbb{C}} \rho^2(d)|d_{\bar{z}}|^2\, dz \wedge d\bar{z}.$$

Recall the Dirichlet energy of d is defined by

$$E(d) := \frac{1}{2}\int_{\mathbb{S}^2} |\nabla d|^2\, dv_{g_0}.$$

It is easy to see that

$$E(d) = E_\partial(d) + E_{\bar{\partial}}(d), \tag{2.1}$$

$$4\pi \deg(d) = E_\partial(d) - E_{\bar{\partial}}(d). \tag{2.2}$$

Here $\deg(d)$ denotes the topological degree of $d : \mathbb{S}^2 \to \mathbb{S}^2$, which is well-defined for maps $d \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ (see Brezis-Nirenberg [1]).

Theorem 2.1. *There exist $\epsilon_0 > 0$ and $T_0 \geq 1$ such that if $(u,d) : \mathbb{S}^2 \times (0, +\infty) \to T\mathbb{S}^2 \times \mathbb{S}^2$ is the global solution of the nematic liquid crystal flow (1.1)-(1.3) obtained by Theorem 1.1, satisfying*

$$\frac{1}{2}\big\|u(T_0)\big\|^2 + 2\min\Big\{E_\partial(d(T_0)),\ E_{\bar{\partial}}(d(T_0))\Big\} \leq \epsilon_0. \tag{2.3}$$

then there exist a smooth harmonic map $d_\infty \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, a nonnegative integer k, and $C_1, C_2 > 0$ such that
(i) as $t \to +\infty$, it holds that

$$u(t) \to 0 \text{ strongly in } H^1(\mathbb{S}^2), d(t) \to d_\infty \text{ weakly in } H^1(\mathbb{S}^2) \text{ and strongly in } L^2(\mathbb{S}^2).$$

(ii)

$$\big\|u(t)\big\| + \big\|d(t) - d_\infty\big\| \leq C_1 e^{-C_2 t},\ \forall t \geq T_0. \tag{2.4}$$

(iii)

$$\big|E(d(t)) - E(d_\infty) - 4\pi k\big| \leq C_1 e^{-C_2 t},\ \forall t \geq T_0. \tag{2.5}$$

In order to prove Theorem 2.1, we need a key estimate, originally due to Topping [15], which provides a way to control the ∂-energy of d (or $\bar{\partial}$-energy of d) in terms of its tension field.

Lemma 2.1. *There exist $\epsilon_0 > 0$ and $C_0 > 0$ such that if $d \in H^1(\mathbb{S}^2, \mathbb{S}^2)$ satisfies*

$$\min\Big\{E_\partial(d),\ E_{\bar\partial}(d)\Big\} < \epsilon_0. \tag{2.6}$$

Then

$$\min\Big\{E_\partial(d), E_{\bar\partial}(d)\Big\} \le C_0 \int_{\mathbb{S}^2} \big|\Delta d + |\nabla d|^2 d\big|^2 \, dv_{g_0}. \tag{2.7}$$

We need the energy inequality of the solution of (1.1)-(1.2) obtained by Theorem 1.1.

Lemma 2.2. *Assume that $(u, d) : \mathbb{S}^2 \times [0, +\infty) \to T\mathbb{S}^2 \times \mathbb{S}^2$ is the solution of (1.1)-(1.2) obtained by Theorem 1.1. Then for any $t \in (0, +\infty) \setminus \{T_i\}_{i=1}^L$, the following basic energy law holds:*

$$\frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{S}^2} |u|^2 dv_{g_0} + E(d)\right] = -\int_{\mathbb{S}^2} \big(\,|\nabla u|^2 + \big|\Delta d + |\nabla d|^2 d\big|^2\,\big)\, dv_{g_0} \tag{2.8}$$

Proof. (2.8) follows by multiplying both sides of the equation $(1.1)_1$ by u and both sides of the equation $(1.1)_3$ by $\Delta d + |\nabla d|^2 d$ and integrating the resulting equations over $\mathbb{S}^2$. We refer the interested readers to [6] Lemma 4.1 for the detail. □

We also need the following simple fact on the average of u.

Lemma 2.3. *Assume that $(u, d) : \mathbb{S}^2 \times [0, +\infty) \to T\mathbb{S}^2 \times \mathbb{S}^2$ is the solution of (1.1)-(1.2) obtained by Theorem 1.1. If the condition (1.3) holds, then*

$$\int_{\mathbb{S}^2} u(x, t)\, dv_{g_0} = 0,\ \ \forall t \ge 0. \tag{2.9}$$

Proof. Under the assumption (1.3), (2.9) follows by integrating the equation $(1.1)_1$ over $\mathbb{S}^2$ and the fact that $\Delta d \nabla d = \mathrm{div}\big(\nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2\big)$. Here $(\nabla d \otimes \nabla d)_{ij} = \langle \frac{\partial d}{\partial x_i}, \frac{\partial d}{\partial x_j}\rangle$, for $1 \le i, j \le 3$. □

Now we give the proof of Theorem 2.1.

Proof. Let $\epsilon_0 > 0$ be given by Lemma 2.1 and $T_0 \ge 1$ be sufficiently large so that $(u, d) \in C^\infty(\mathbb{S}^2 \times [T_0, +\infty))$. Without loss of generality, we may assume that

$$E_\partial(d(T_0)) \le E_{\bar\partial}(d(T_0)).$$

It follows from (2.1) and (2.2) that

$$E_\partial(d) = \frac{1}{2}\left[E(d) + 4\pi \mathrm{deg}(d)\right].$$

Since $\mathrm{deg}(d(t))$ is constant for $t \ge T_0$, the basic energy law (2.8) implies that

$$\frac{d}{dt}\Big[\frac{1}{2}\|u\|^2 + 2E_\partial(d)\Big] = \frac{d}{dt}\Big[\frac{1}{2}\|u\|^2 + E(d)\Big] = -\big(\,\|\nabla u\|^2 + \big\|\Delta d + |\nabla d|^2 d\big\|^2\,\big),\ \forall t \ge T_0. \tag{2.10}$$

Therefore, we have that

$$\frac{1}{2}\|u(t)\|^2 + 2E_\partial(d(t)) \le \frac{1}{2}\|u(T_0)\|^2 + 2E_\partial(d(T_0)) \le \epsilon_0,\ \forall t \ge T_0. \tag{2.11}$$

Applying Lemma 2.1 to $d(t)$, and Lemma 2.3 and Poincaré's inequality for $u(t)$, we have

$$\frac{1}{2}\|u(t)\|^2 + 2E_\partial(d(t)) \le C[\|\nabla u(t)\|^2 + \left\|\Delta d(t) + |\nabla d(t)|^2 d(t)\right\|^2], \ \forall t \ge T_0. \tag{2.12}$$

Putting (2.12) together with (2.10), we obtain that for all $t \ge T_0$,

$$\begin{aligned}
-\frac{d}{dt}\Big[\frac{1}{2}\|u\|^2 + 2E_\partial(d)\Big]^{\frac{1}{2}} &= \frac{-\frac{d}{dt}\Big[\frac{1}{2}\|u\|^2 + 2E_\partial(d)\Big]}{2\Big[\frac{1}{2}\|u\|^2 + 2E_\partial(d)\Big]^{\frac{1}{2}}} \\
&\ge C\frac{-\frac{d}{dt}\Big[\frac{1}{2}\|u\|^2 + 2E_\partial(d)\Big]}{\Big[\|\nabla u\|^2 + \|\Delta d + |\nabla d|^2 d\|^2\Big]^{\frac{1}{2}}} \\
&\ge C\left[\|\nabla u\| + \left\|\Delta d + |\nabla d|^2 d\right\|\right] \\
&\ge C\left[\frac{1}{2}\|u\|^2 + 2E_\partial(d)\right]^{\frac{1}{2}}.
\end{aligned} \tag{2.13}$$

Thus by Gronwall's inequality we have

$$\Big[\|u(t)\|^2 + E_\partial(d(t))\Big] \le \Big[\|u(T_0)\|^2 + E_\partial(d(T_0))\Big]e^{-C(t-T_0)} \le C\epsilon_0 e^{-C(t-T_0)}, \quad \forall\, t \ge T_0. \tag{2.14}$$

By integrating (2.13) over $[t, +\infty)$, we obtain that for any $t \ge 2T_0$,

$$\int_t^\infty (\left\|\nabla u\right\| + \left\|\Delta d + |\nabla d|^2 d\right\|)\, d\tau \le C\left[\|u(t)\|^2 + E_\partial(d(t))\right]^{\frac{1}{2}} \le C_1 e^{-C_2 t}. \tag{2.15}$$

Consequently, we infer from $(1.1)_3$, (2.8), (2.13), (2.14) and (2.15) that for any $t_2 \ge t \ge 2T_0$,

$$\begin{aligned}
&\|d(t_2) - d(t)\|_{L^1} \le \int_t^{t_2} \|d_t\|_{L^1} d\tau \\
&\le \int_t^{t_2} \|u\cdot\nabla d\|_{L^1} d\tau + \int_t^{t_2} \left\|\Delta d + |\nabla d|^2 d\right\|_{L^1} d\tau \\
&\le \int_t^{t_2} \|u\|_{L^2}\|\nabla d\|_{L^2} d\tau + 2\sqrt{\pi}\int_t^{t_2} \left\|\Delta d + |\nabla d|^2 d\right\|_{L^2} d\tau \\
&\le C\Big[\int_t^{t_2} \|\nabla u\|_{L^2} d\tau + \int_t^{t_2} \left\|\Delta d + |\nabla d|^2 d\right\| d\tau\Big] \\
&\le C\big[\|u(t)\|^2 + E_\partial(d(t))\big]^{\frac{1}{2}} \le C_1 e^{-C_2 t}.
\end{aligned} \tag{2.16}$$

Thus

$$\|d(t_2) - d(t)\|_{L^2}^2 \le 2\|d(t_2) - d(t)\|_{L^1} \le C_1 e^{-C_2 t}, \tag{2.17}$$

which indicates that as $t \to +\infty$, $d(t)$ converges in $L^2(\mathbb{S}^2)$. It follows from Theorem 1.1 (iii) that there exist a smooth harmonic map $d_\infty \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, nontrivial harmonic maps $\{\omega_i\}_{i=1}^l$ for some nonnegative integer l, and a sequence $t_i \to +\infty$ such that

$$\begin{cases} d(t_i) \to d_\infty \text{ weakly in } H^1(\mathbb{S}^2, \mathbb{S}^2) \text{ and strongly in } L^2(\mathbb{S}^2, \mathbb{S}^2), \\ \displaystyle\int_{\mathbb{S}^2} |\nabla d(t_i)|^2\, dv_{g_0} \to \int_{\mathbb{S}^2} |\nabla d_\infty|^2\, dv_{g_0} + \sum_{i=1}^l \int_{\mathbb{S}^2} |\nabla \omega_i|^2\, dv_{g_0}. \end{cases} \tag{2.18}$$

Therefore, by choosing $t_2 = t_i$ in (2.17) and sending i to ∞, we conclude that

$$\|d(t) - d_\infty\|_{L^2} \le C_1 e^{-C_2 t}, \ \forall t \ge 2T_0. \tag{2.19}$$

In particular, (2.19) implies that $d(t)$ converges to d_∞ in $L^2(\mathbb{S}^2)$ as $t \to +\infty$. To show that $d(t)$ converges to d_∞ weakly in $H^1(\mathbb{S}^2)$. Let $\{t_j\}$ be any sequence tending to $+\infty$. Since $\{d(t_j)\}$ is bounded in $H^1(\mathbb{S}^2)$, there exists a subsequence $t_{j'} \to +\infty$ such that $d(t_{j'})$ weakly converges in $H^1(\mathbb{S}^2)$ and strongly in $L^2(\mathbb{S}^2)$ to a map $d_* \in H^1(\mathbb{S}^2, \mathbb{S}^2)$. Hence $d_* = d_\infty$. This shows that $d(t)$ converges to d_∞ weakly in $H^1(\mathbb{S}^2)$ as $t \to +\infty$. Hence (i) is proven.

To show (iii), integrating (2.10) over $[t, t_2]$ for $t \ge 2T_0$ and applying (2.14) yields

$$\begin{aligned}
&\big[\frac{1}{2}\|u(t)\|^2 + E(d(t))\big] - \big[\frac{1}{2}\|u(t_2)\|^2 + E(d(t_2))\big] \\
&= \big[\frac{1}{2}\|u(t)\|^2 + 2E_\partial(d(t))\big] - \big[\frac{1}{2}\|u(t_2)\|^2 + 2E_\partial(d(t_2))\big] \\
&\le \big[\frac{1}{2}\|u(t)\|^2 + 2E_\partial(d(t))\big] \le C_1 e^{-C_2 t}.
\end{aligned}$$

This implies

$$|E(d(t) - E(d(t_2)| \le \big(\|u(t)\|^2 + \|u(t_2)\|^2\big) + C_1 e^{-C_2 t}, \ \forall t \ge 2T_0. \tag{2.20}$$

Let $t_2 \to +\infty$ be the sequence such that (2.18) holds. Since each harmonic map ω_i, $1 \le i \le l$, is nontrivial and has its energy

$$\int_{\mathbb{S}^2} |\nabla \omega_i|^2 \, dv_{g_0} = 8\pi m_i,$$

for some positive integer m_i, there exists a nonnegative integer k such that

$$\lim_{t_2 \to +\infty} E(d(t_2)) = E(d_\infty) + 8\pi k.$$

Sending t_2 to infinity in (2.20), this implies

$$\Big| E(d(t) - E(d_\infty) - 8\pi k \Big| \le C_1 e^{-C_2 t}, \ \forall t \ge 2T_0. \tag{2.21}$$

The proof is now complete. □

It is well-known that any harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^2$ is either holomorphic or anti-holomorphic. Inspired by [15] Theorem 2, we have

Corollary 2.1. *Suppose $(u, d) : \mathbb{S}^2 \times (0, +\infty) \to T\mathbb{S}^2 \times \mathbb{S}^2$ is the global solution of the nematic liquid crystal flow (1.1)-(1.3) obtained by Theorem 1.1.*
(i) Suppose that the weak limit d_∞ and the bubbles ω_i $(1 \le i \le l)$, associated with a sequence $t_i \uparrow +\infty$, are all holomorphic or all anti-holomorphic. Then there exist a nonnegative integer k, and $C_1, C_2 > 0$ such that

$$u(t) \to 0 \text{ strongly in } H^1(\mathbb{S}^2), d(t) \to d_\infty \text{ weakly in } H^1(\mathbb{S}^2) \text{ and strongly in } H^1(\mathbb{S}^2), \text{ as } t \to +\infty,$$

and

$$\big\|u(t)\big\| + \big\|d(t) - d_\infty\big\| + \big|E(d(t)) - E(d_\infty) - 4\pi k\big| \le C_1 e^{-C_2 t}, \ \forall t \ge T_0. \tag{2.22}$$

(ii) The same conclusions of (i) hold if the initial data (u_0, d_0) satisfies

$$\frac{1}{2}\|u_0\|^2 + 2\min\Big\{E_\partial(d_0), \ E_{\bar\partial}(d_0)\Big\} < 8\pi. \tag{2.23}$$

Proof. For the part (i), it suffices to verify that the condition (2.3) holds. For simplicity, assume that d_∞ and all ω_i's are anti-holomorphic. Thus we have

$$E_\partial(d_\infty) = E_\partial(\omega_1) = \cdots = E_\partial(\omega_l) = 0.$$

From (1.5), we know that

$$\lim_{t_i \uparrow +\infty} \Big[\frac{1}{2}\|u(t_i)\|^2 + E_\partial(d(t_i))\Big] = E_\partial(d_\infty) + \sum_{i=1}^{l} E_\partial(\omega_i) = 0.$$

This clearly implies that there exists a sufficiently large i_0 such that

$$\Big[\frac{1}{2}\|u(t_{i_0})\|^2 + 2E_\partial(d(t_{i_0}))\Big] \le \epsilon_0,$$

which implies (2.3).

For the part (ii), we will show that (2.23) implies the condition in the part (i). For simplicity, assume that $E_\partial(d_0) \le E_{\bar\partial}(d_0)$. Let $0 < T_1 < +\infty$ be the first singular time of the flow (1.1)-(1.3). Since $(u(t), d(t)) \in C^\infty(\mathbb{S}^2 \times (0, T_1))$ and

$$\lim_{t \downarrow 0} \big[\|u(t) - u_0\| + \|\nabla(d(t) - d_0)\|\big] = 0,$$

it is not hard to see

$$\deg(d(t)) = \deg(d_0), \ \forall\, 0 \le t < T_1.$$

Thus the basic energy law (2.8) implies that

$$\frac{d}{dt}\Big[\frac{1}{2}\|u\|^2 + 2E_\partial(d)\Big] = \frac{d}{dt}\Big[\frac{1}{2}\|u\|^2 + E(d)\Big] \le 0. \tag{2.24}$$

Integrating (2.24) from 0 to $0 < t \le T_1$ yields

$$\frac{1}{2}\|u(t)\|^2 + 2E_\partial(d(t)) \le \frac{1}{2}\|u_0\|^2 + 2E_\partial(d_0), \quad 0 \le t \le T_1. \tag{2.25}$$

Let $T_2 \in (T_1, +\infty)$ be the second singular time. Then the same argument yields

$$\frac{1}{2}\|u(t)\|^2 + 2E_\partial(d(t)) \le \frac{1}{2}\|u(T_1)\|^2 + 2E_\partial(d(T_1)), \quad T_1 \le t \le T_2. \tag{2.26}$$

Since there are at most finitely many finite singular times $\{T_i\}_{i=1}^L$ for the flow (1.1)-(1.3), by repeating the argument we would reach that for any $t \ge 0$, it holds that

$$\begin{aligned}\mathcal{E}(t) := \frac{1}{2}\|u(t)\|^2 + 2E_\partial(d(t))) &\le \frac{1}{2}\|u_0\|^2 + 2E_\partial(d_0) \\ &= \frac{1}{2}\|u_0\|^2 + 2\min\Big\{E_\partial(d_0), E_{\bar\partial}(d_0)\Big\} \\ &< 8\pi. \end{aligned} \tag{2.27}$$

By the lower semicontinuity, we have that

$$2E_\partial(d_\infty) \le \lim_{t\to\infty} \mathcal{E}(t) < 8\pi,$$

and

$$2E_\partial(\omega_i) \le \lim_{t\to\infty} \mathcal{E}(t) < 8\pi, \ 1 \le i \le l.$$

This implies that $\omega_1, \cdots, \omega_l$ are all nontrivial anti-holomorphic maps. If d_∞ is not a constant, then d_∞ has to be anti-holomorphic. Therefore d_∞ and all ω_i's are anti-holomorphic.

Thus the conclusions in (i) and (ii) follow from Theorem 2.1. The proof is complete. □

3 Uniform limit in $H^k(\mathbb{S}^2)$ for $k \geq 1$

This subsection is to consider the convergence issues of the nematic liquid crystal flow (1.1)-(1.3) in higher order Sobolev spaces at $t = +\infty$.

Theorem 3.1. *Suppose $(u, d) : \mathbb{S}^2 \times [0, +\infty) \to T\mathbb{S}^2 \times \mathbb{S}^2$ is the global solution of (1.1)-(1.3) obtained by Theorem 1.1. Suppose that there exist a sequence $t_i \uparrow +\infty$ and a smooth harmonic map $d_\infty \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that*

$$\lim_{t_i \uparrow +\infty} \Big[\|u(t_i)\| + \|d(t_i) - d_\infty\| + \|\nabla(d(t_i) - d_\infty)\| \Big] = 0. \tag{3.2}$$

Then for any $k \geq 1$ there exist $C_1, C_2 > 0$ depending only on k such that

$$\Big\|u(t)\Big\|_{H^k(\mathbb{S}^2)} + \Big\|d(t) - d_\infty\Big\|_{H^k(\mathbb{S}^2)} \leq C_1 e^{-C_2 t}. \tag{3.3}$$

In particular, for any $k \geq 1$, $d(t) \to d_\infty$ in $H^k(\mathbb{S}^2)$ as $t \to +\infty$.

Proof. For simplicity, assume that d_∞ is anti-holomorphic, i.e., $\partial_z d_\infty \equiv 0$. Thus we have

$$E_\partial(d(t_i)) = E_\partial(d(t_i) - d_\infty) \leq E(d(t_i) - d_\infty).$$

This, combined with (3.2), implies

$$\lim_{t_i \uparrow +\infty} \Big[\|u(t_i)\| + E_\partial(d(t_i)) \Big] = 0. \tag{3.4}$$

Hence we can apply Theorem 2.1 to conclude that

$$u(t) \to 0 \text{ strongly in } H^1(\mathbb{S}^2), d(t) \to d_\infty \text{ weakly in } H^1(\mathbb{S}^2) \text{ and strongly in } L^2(\mathbb{S}^2), \text{ as } t \to \infty,$$

and

$$\|u(t)\| + \|d(t) - d_\infty\| \leq C_1 e^{-C_2 t}. \tag{3.5}$$

Since it follows from the basic energy law (2.8) and

$$\lim_{t_i \uparrow +\infty} \big(\frac{1}{2} \|u(t_i)\|^2 + E(d(t_i)) \big) = E(d_\infty)$$

that

$$\lim_{t \to +\infty} E(d(t)) = E(d_\infty),$$

we can conclude that

$$\lim_{t \to +\infty} \Big\|d(t) - d_\infty\Big\|_{H^1(\mathbb{S}^2)} = 0. \tag{3.6}$$

For $\epsilon_1 > 0$, let $r_0 = r_0(\epsilon_1) > 0$ be such that

$$\max_{x \in \mathbb{S}^2} \int_{B_{r_0}(x) \cap \mathbb{S}^2} |\nabla d_\infty|^2 \, dv_{g_0} \leq \frac{\epsilon_1}{2}.$$

By (3.6), there exists $T_0 > 0$ such that

$$\sup_{t \geq T} \max_{x \in \mathbb{S}^2} \int_{B_{r_0}(x) \cap \mathbb{S}^2} |\nabla d(t)|^2 \, dv_{g_0} \leq \epsilon_1. \tag{3.7}$$

As in [6], (3.7) then implies the following inequality:

$$\int_{T_0}^{\infty}\int_{\mathbb{S}^2}\left(|\nabla u|^2+|\nabla^2 d|^2\right)\leq C\left(\|u(T_0)\|^2+E(d(T_0))\right). \tag{3.8}$$

With the estimates (3.7) and (3.8), we can apply the regularity Theorem 1.2 of [6] to get that for any $k\geq 0$, there exists $C_k>0$ such that

$$\sup_{t\geq T_0}\Big(\|u(t)\|_{C^k(\mathbb{S}^2)}+\|d(t)\|_{C^{k+1}(\mathbb{S}^2)}\Big)\leq C_k. \tag{3.9}$$

By standard interpolation inequalities, (3.5) and (3.9) imply that (3.3) holds. The proof is now complete. □

It is an interesting question to find sufficient conditions that guarantee the global solution $(u,d):\mathbb{S}^2\times[0,+\infty)\to\mathbb{S}^2$ of the flow (1.1)-(1.3) by Theorem 1.1 has a sequence $t_i\uparrow+\infty$ such that $(u(t_i),d(t_i))\to(0,d_\infty)$ strongly in $L^2(\mathbb{S}^2)\times H^1(\mathbb{S}^2)$.

In this context, we have the following result.

Corollary 3.1. *Suppose $(u,d):\mathbb{S}^2\times[0,+\infty)\to T\mathbb{S}^2\times\mathbb{S}^2$ is the global solution of (1.1)-(1.3) by Theorem 1.1. Then there exists a smooth harmonic map $d_\infty\in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ such that for any $k\geq 1$,*

$$\Big\|u(t)\Big\|_{H^k(\mathbb{S}^2)}+\Big\|d(t)-d_\infty\Big\|_{H^k(\mathbb{S}^2)}\leq C_k e^{-C_k t}, \tag{3.10}$$

provided that one of the following conditions holds:
i) $\frac{1}{2}\|u_0\|^2+E(d_0)\leq 4\pi$.
ii) $d_0(\mathbb{S}^2)$ *is contained in the hemisphere (e.g.* $d_0^3\geq 0$*).*
iii) there exists $C_L>0$ *such that* (u_0,d_0) *satisfies*

$$\exp\big(108C_L^8\big(\|u_0\|^2+\frac{1}{8C_L^4}\big)^2\big)\|\nabla d_0\|^2\leq\frac{1}{8C_L^4}. \tag{3.11}$$

Proof. We will establish that under any one of the three conditions, there exists a sequence $t_i\uparrow+\infty$ such that $u(t_i)\to 0$ in $H^1(\mathbb{S}^2)$ and $d(t_i)$ is strongly convergent in $H^1(\mathbb{S}^2)$.

Let us first consider the condition (i). It has been proved by Theorem 1.1 (iv) that (u,d) has neither finite time singularity nor energy concentration at $t=\infty$. In particular, $(u,d)\in C^\infty(\mathbb{S}^2\times(0,+\infty))$, and there exists $t_i\uparrow+\infty$ and a harmonic map $d_\infty\in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ such that

$$\|u(t_i)\|+\|d(t_i)-d_\infty\|_{H^1(\mathbb{S}^2)}\to 0.$$

Moreover, it has been shown in [6] that d_∞ is constant, unless $\big(\frac{1}{2}\|u_0\|^2+E(d_0)\big)=4\pi$ which would imply that $u\equiv u_0\equiv 0$, and $d\equiv d_0\equiv d_\infty$ is a harmonic map of degree one.

Now let us consider the condition (ii). Since $d_0^3\geq 0$, it follows from the maximum principle on the equation $(1.1)_3$ that $d^3(t)\geq 0$ for all $t\geq 0$. Since there doesn't exist non-constant harmonic maps from $\mathbb{S}^2$ to the hemisphere $\mathbb{S}^2_+$, we can apply [6] Theorem 1.3 to conclude that there is neither finite time singularity nor any energy concentration at $t=\infty$ for (u,d) (see also [11] for a different proof). In particular, there exist a point $p\in\mathbb{S}^2_+$ and $t_i\uparrow+\infty$ such that

$$\|u(t_i)\|+\|d(t_i)-p\|_{H^1(\mathbb{S}^2)}\to 0.$$

Finally let us consider the condition (iii). Recall that under the condition (iii), it has been proven by Xu-Zhang [18] that (u, d) is smooth when the domain is $\mathbb{R}^2$. Here we indicate how to extend the argument by [18] to $\mathbb{S}^2$.

Multiplying both sides of the equation $(1.1)_3$ by $-(\Delta d+|\nabla d|^2 d)$ and integrating the resulting equation over $\mathbb{S}^2$ yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla d\|^2 + \int_{\mathbb{S}^2} \left|\Delta d + |\nabla d|^2 d\right|^2 \, dv_{g_0} = \int_{\mathbb{S}^2} u \cdot \nabla d \Delta d \, dv_{g_0}. \tag{3.12}$$

To estimate the right side of (3.12), we apply the Ricci identity on $(\mathbb{S}^2, g_0)$ and the Poincaré inequality for u and ∇d to obtain

$$\int_{\mathbb{S}^2} |\Delta d|^2 \, dv_{g_0} = \int_{\mathbb{S}^2} (|\nabla^2 d|^2 + |\nabla d|^2) \, dv_{g_0} \geq \int_{\mathbb{S}^2} |\nabla^2 d|^2 \, dv_{g_0}, \tag{3.13}$$

$$\|u\| \leq C_{\mathrm{p}}\|\nabla u\| \quad \text{and} \quad \|\nabla d\| \leq C_{\mathrm{p}}\|\nabla^2 d\| \leq C_{\mathrm{p}}\|\Delta d\|, \tag{3.14}$$

where $C_{\mathrm{p}} > 0$ is the constant in the Poincaré inequality. Also recall the Ladyzhenskaya inequality on $\mathbb{S}^2$:

$$\|f\|_{L^4} \leq C_1\|f\|^{\frac{1}{2}}\|\nabla f\|^{\frac{1}{2}} + C_2\|f\|.$$

Combining these inequalities with Young's inequality and (2.8), we have

$$\begin{aligned}
&\Big|\int_{\mathbb{S}^2} u \cdot \nabla d \Delta d \, dv_g\Big| \leq \|\Delta d\|\|\nabla d\|_{L^4}\|u\|_{L^4} \\
&\leq \|\Delta d\|\big(C_1\|\nabla d\|^{\frac{1}{2}}\|\Delta d\|^{\frac{1}{2}} + C_2\|\nabla d\|\big)\big(C_1\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}} + C_2\|u\|\big) \\
&\leq \|\Delta d\|\Big(C_1\|\nabla d\|^{\frac{1}{2}}\|\Delta d\|^{\frac{1}{2}} + C_2 C_{\mathrm{p}}^{\frac{1}{2}}\|\nabla d\|^{\frac{1}{2}}\|\Delta d\|^{\frac{1}{2}}\Big)\Big(C_1\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}} + C_2 C_{\mathrm{p}}^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\Big) \\
&\leq C_L^2\|\Delta d\|^{\frac{3}{2}}\|\nabla d\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}} \\
&\leq \frac{\|\Delta d\|^2}{8} + 54C_L^8\|u\|^2\|\nabla u\|^2\|\nabla d\|^2 \\
&\leq \frac{\|\Delta d\|^2}{8} + 54C_L^8\left(\|u_0\|^2 + \|\nabla d_0\|^2\right)\|\nabla u\|^2\|\nabla d\|^2.
\end{aligned} \tag{3.15}$$

Here

$$C_L \doteq \big(C_1 + C_2 C_{\mathrm{p}}^{\frac{1}{2}}\big). \tag{3.16}$$

On the other hand, since

$$\left|\Delta d + |\nabla d|^2 d\right|^2 = |\Delta d|^2 - |\nabla d|^4,$$

we have

$$\int_{\mathbb{S}^2} \left||\nabla d|^2 d + \Delta d\right|^2 \, dv_g = \|\Delta d\|^2 - \|\nabla d\|_{L^4}^4 \geq \|\Delta d\|^2 - C_L^4\|\nabla d\|^2\|\Delta d\|^2. \tag{3.17}$$

If d_0 satisfies $\|\nabla d_0\|^2 < \frac{1}{8C_L^4}$, then there exists $T_1 > 0$ such that for any $t \in [0, T_1]$,

$$\|\nabla d(t)\|^2 \leq \frac{1}{8C_L^4}. \tag{3.18}$$

Let T_1^* denote the maximal time such that (3.18) holds on $[0, T_1^*]$. Then, by (3.12)-(3.17) we have that for any $t \in [0, T_1^*]$,

$$\frac{d}{dt}\|\nabla d\|^2 + \frac{1}{4}\|\Delta d\|^2 \leq 108C_L^8\Big(\|u_0\|^2 + \frac{1}{8C_L^4}\Big)\|\nabla u\|^2\|\nabla d\|^2. \tag{3.19}$$

Using Gronwall's inequality and (2.8), we deduce from (3.19) that for any $0 \leq t \leq T_1^*$,

$$
\begin{aligned}
\|\nabla d(t)\|^2 + \frac{1}{4}\int_0^t \|\Delta d(\tau)\|^2 d\tau &\leq \exp\left(108C_L^8(\|u_0\|^2 + \frac{1}{8C_L^4})\int_0^{T_1^*}\|\nabla u(\tau)\|^2 d\tau\right)\|\nabla d_0\|^2 \\
&\leq \exp\left(108C_L^8(\|u_0\|^2 + \frac{1}{8C_L^4})^2\right)\|\nabla d_0\|^2, \qquad (3.20)
\end{aligned}
$$

which implies that $T_1^* = T$ and

$$\|\nabla d(t)\|^2 + \frac{1}{4}\int_0^t \|\Delta d(\tau)\|^2 d\tau \leq \frac{1}{8C_L^4}, \qquad (3.21)$$

holds for all $0 \leq t \leq T$, provided that (u_0, d_0) satisfies

$$\exp\left(108C_L^8(\|u_0\|^2 + \frac{1}{8C_L^4})^2\right)\|\nabla d_0\|^2 \leq \frac{1}{8C_L^4}. \qquad (3.22)$$

Let T^* be the maximal existence time for the solution (u, d). Then (2.8) and (3.21) ensure that $T^* = +\infty$ by the continuity argument . Moreover, $(u, d) \in C^\infty(\mathbb{S}^2 \times (0, +\infty))$ by Theorem 1.2 in [6]. Note that (3.21) and (3.13) imply that

$$\int_0^\infty \|\nabla^2 d(\tau)\|^2 d\tau \leq \frac{1}{2C_L^4}. \qquad (3.23)$$

It follows from (2.8), (3.23), and (3.14) that there is a sequence $t_i \uparrow +\infty$ such that

$$\lim_{t_i \uparrow +\infty}\left(\|u(t_i)\| + \|\nabla d(t_i)\| + \|\nabla^2 d(t_i)\|\right) = 0.$$

In particular, there exists $p \in \mathbb{S}^2$ such that $d(t_i) \to p$ strongly in $H^1(\mathbb{S}^2)$ as $t_i \uparrow +\infty$.

We have verified the condition of Theorem 3.1 holds under all the three conditions. Thus the conclusion follows from Theorem 3.1. □

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