Energy identity of approximate biharmonic maps to Riemannian manifolds and its application

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Abstract

We consider in dimension four weakly convergent sequences of approximate biharmonic maps to a Riemannian manifold with bi-tension fields bounded in L^p for $p > \frac{4}{3}$. We prove an energy identity that accounts for the loss of hessian energies by the sum of hessian energies over finitely many nontrivial biharmonic maps on $\mathbb{R}^4$. As a corollary, we obtain an energy identity for the heat flow of biharmonic maps at time infinity.

1 Introduction

This is a continuation of our previous work [26] on the blow-up analysis of approximate biharmonic maps in dimension 4. In [26], we obtained in dimension four an energy identity for approximate biharmonic maps into sphere with bounded L^p bi-tension field for $p > 1$, and an energy identity of the heat flow of biharmonic map into sphere at time infinity. The aim of this paper is to extend the main theorems of [26] to any compact Riemannian manifold without boundary, under the additional assumption that the bi-tension fields are bounded in L^p for $p > \frac{4}{3}$. The main results of this paper was announced in [26].

Let $\Omega \subset \mathbb{R}^4$ be a bounded smooth domain, and (N, h) be a compact n-dimensional Riemannian manifold without boundary, embedded into K-dimensional Euclidean space $\mathbb{R}^K$. Recall the Sobolev space $W^{l,p}(\Omega, N)$, $1 \le l < +\infty$ and $1 \le p < +\infty$, is defined by

$$W^{l,p}(\Omega, N) = \Big\{ v \in W^{l,p}(\Omega, \mathbb{R}^K) : \ v(x) \in N \ \text{ a.e. } x \in \Omega \Big\}.$$

In this paper we will discuss the limiting behavior of weakly convergent sequences of approximate (extrinsic) biharmonic maps $\{u_k\} \subset W^{2,2}(\Omega, N)$ in dimension $n = 4$, especially an energy identity during the process of convergence. First we recall the notion of approximate (extrinsic) biharmonic maps.

Definition 1.1 *A map $u \in W^{2,2}(\Omega, N)$ is called an approximate biharmonic map if there exists a bi-tension field $h \in L^1_{\text{loc}}(\Omega, \mathbb{R}^K)$ such that*

$$\Delta^2 u = \Delta(\mathbb{B}(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla(\mathbb{P}(u))\rangle - \langle \Delta(\mathbb{P}(u)), \Delta u\rangle + h \tag{1.1}$$

in the distribution sense, where $\mathbb{P}(y) : \mathbb{R}^K \to T_yN$ is the orthogonal projection from $\mathbb{R}^K$ to the tangent space of N at $y \in N$, and $\mathbb{B}(y)(X, Y) = -\nabla_X \mathbb{P}(y)(Y)$, $\forall X, Y \in T_yN$, is the second fundamental form of $N \subset \mathbb{R}^K$. In particular, if $h = 0$ then u is called a biharmonic map to N.

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Note that biharmonic maps to a Riemannian manifold N are critical points of the hessian energy functional

$$E_2(u) = \int_\Omega |\nabla^2 u|^2 \, dx$$

over $W^{2,2}(\Omega, N)$. Biharmonic maps are higher order extensions of harmonic maps. The study of regularity of biharmonic maps has generated considerable interests after the initial work by Chang-Wang-Yang [3], the readers can refer to Wang [22, 23, 24], Strzelecki [19], Lamm-Rivière [12], Struwe [21], Scheven [17, 18] (see also Ku [10] and Gong-Lamm-Wang [6] for the boundary regularity). In particular, the interior regularity theorem asserts that the smoothness of $W^{2,2}$-biharmonic maps holds in dimension $n = 4$, and the partial regularity of stationary $W^{2,2}$-biharmonic maps holds in dimensions $n \geq 5$.

It is an important observation that biharmonic maps are invariant under dilations in $\mathbb{R}^n$ for $n = 4$. Such a property leads to non-compactness of biharmonic maps in dimension 4, which prompts recent studies by Wang [22] and Hornung-Moser [8] concerning the failure of strong convergence for weakly convergent biharmonic maps. Roughly speaking, the results in [22] and [8] assert that the failure of strong convergence occurs at finitely many concentration points of hessian energy, where finitely many bubbles (i.e. nontrivial biharmonic maps on $\mathbb{R}^4$) are generated, and the total hessian energies from these bubbles account for the total loss of hessian energies during the process of convergence.

Our first result is to extend the results from [22] and [8] to the context of suitable approximate biharmonic maps to a compact Riemannian manifold N. More precisely, we have

Theorem 1.2 *For $n = 4$, suppose $\{u_k\} \subset W^{2,2}(\Omega, N)$ is a sequence of approximate biharmonic maps, which are bounded in $W^{2,2}(\Omega, N)$ and have their bi-tension fields h_k bounded in L^p for $p > \frac{4}{3}$, i.e.*

$$M := \sup_k \left(\|u_k\|_{W^{2,2}} + \|h_k\|_{L^p} \right) < +\infty. \tag{1.2}$$

Assume $u_k \rightharpoonup u$ in $W^{2,2}$ and $h_k \rightharpoonup h$ in L^p. Then
(i) u is an approximate biharmonic map to N with h as its bi-tension field.
(ii) there exist a nonnegative integer L depending on M and L points $\{x_1, \cdots, x_L\} \subset \Omega$ such that $u_k \to u$ strongly in $W^{2,2}_{\text{loc}} \cap C^0_{\text{loc}}(\Omega \setminus \{x_1, \cdots, x_L\}, N)$.
(iii) For $1 \leq i \leq L$, there exist a positive integer L_i depending on M and L_i nontrivial smooth biharmonic map ω_{ij} from $\mathbb{R}^4$ to N with finite hessian energy, $1 \leq j \leq L_i$, such that

$$\lim_{k\to\infty} \int_{B_{r_i}(x_i)} |\nabla^2 u_k|^2 = \int_{B_{r_i}(x_i)} |\nabla^2 u|^2 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla^2 \omega_{ij}|^2, \tag{1.3}$$

and

$$\lim_{k\to\infty} \int_{B_{r_i}(x_i)} |\nabla u_k|^4 = \int_{B_{r_i}(x_i)} |\nabla u|^4 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla \omega_{ij}|^4, \tag{1.4}$$

where $r_i = \dfrac{1}{2} \min\limits_{1\leq j\leq L,\ j\neq i} \{|x_i - x_j|,\ \text{dist}(x_i, \partial\Omega)\}$.

As an application of Theorem 1.2, we study asymptotic behavior at time infinity for the heat flow of biharmonic maps in dimension 4.

Let's review the studies on the heat flow of biharmonic maps undertaken by Lamm [11], Gastel [5], Wang [25], and Moser [16]. The equation of heat flow of (extrinsic) biharmonic maps into N is to seek $u : \Omega \times [0, +\infty) \to N$ that solves:

$$u_t + \Delta^2 u = \Delta(\mathbb{B}(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla(\mathbb{P}(u))\rangle - \langle \Delta(\mathbb{P}(u)), \Delta u\rangle, \ \Omega \times (0, +\infty) \quad (1.5)$$

$$u = u_0, \ \Omega \times \{0\} \quad (1.6)$$

$$(u, \frac{\partial u}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu}), \ \partial\Omega \times (0, +\infty), \quad (1.7)$$

where $u_0 \in W^{2,2}(\Omega, N)$ is a given map. Note that any time independent solution $u : \Omega \to N$ of (1.5) is a biharmonic map to N.

In dimension $n = 4$, Lamm [11] established the existence of global smooth solutions to (1.5)-(1.7) for $u_0 \in W^{2,2}(\Omega, N)$ with small $W^{2,2}$-norm, and Gastel [5] and Wang [25] independently showed that there exists a unique global weak solution to (1.5))-(1.7) for any initial data $u_0 \in W^{2,2}(\Omega, N)$ that has at most finitely many singular times. Moreover, such a solution enjoys the energy inequality:

$$2\int_0^T \int_\Omega |u_t|^2 + \int_\Omega |\Delta u|^2(T) \le \int_\Omega |\Delta u_0|^2, \ \forall \ 0 < T < +\infty. \quad (1.8)$$

Recently, Moser [16] showed the existence of a global weak solution to (1.5)-(1.7) for any target manifold N in dimensions $n \le 8$.

It follows from (1.8) that there exists a sequence $t_k \uparrow \infty$ such that $u_k := u(\cdot, t_k) \in W^{2,2}(\Omega, N)$ satisfies
(i) $\tau_2(u_k) := u_t(t_k)$ satisfies $\|\tau_2(u_k)\|_{L^2} \to 0$; and
(ii) u_k satisfies in the distribution sense

$$-\Delta^2 u_k + \Delta(\mathbb{B}(u_k)(\nabla u_k, \nabla u_k)) + 2\nabla \cdot \langle \Delta u_k, \nabla(\mathbb{P}(u_k))\rangle - \langle \Delta(\mathbb{P}(u_k)), \Delta u_k\rangle = \tau_2(u_k). \quad (1.9)$$

By Definition 1.1 $\{u_k\}$ is a sequence of approximate biharmonic maps to N, which are bounded in $W^{2,2}$ and have their bi-tension fields bounded in L^2. Hence, as an immediate corollary, we obtain

Theorem 1.3 *For $n = 4$ and $u_0 \in W^{2,2}(\Omega, N)$, let $u : \Omega \times \mathbb{R}_+ \to N$, with $u \in L^\infty(\mathbb{R}_+, W^{2,2}(\Omega))$ and $u_t \in L^2(\mathbb{R}_+, L^2(\Omega))$, be a global weak solution of (1.5)-(1.7) that satisfies the energy inequality (1.8). Then there exist $t_k \uparrow +\infty$, a biharmonic map $u_\infty \in C^\infty \cap W^{2,2}(\Omega, N)$ with $u_\infty = u_0$ on $\partial\Omega$, and a nonnegative integer L and L points $\{x_1, \cdots, x_L\} \subset \Omega$ such that*
(i) $u_k := u(\cdot, t_k) \rightharpoonup u_\infty$ in $W^{2,2}(\Omega, N)$.
(ii) $u_k \to u_\infty$ in $C^0_{\rm loc} \cap W^{2,2}_{\rm loc}(\Omega \setminus \{x_1, \cdots, x_L\}, N)$.
(iii) for $1 \le i \le L$, there exist a positive integer L_i and L_i nontrivial biharmonic maps $\{\omega_{ij}\}_{j=1}^{L_i}$ on $\mathbb{R}^4$ with finite hessian energies such that

$$\lim_{k\to\infty} \int_{B_{r_i}(x_i)} |\nabla^2 u_k|^2 = \int_{B_{r_i}(x_i)} |\nabla^2 u_\infty|^2 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla^2 \omega_{ij}|^2, \quad (1.10)$$

and

$$\lim_{k\to\infty} \int_{B_{r_i}(x_i)} |\nabla u_k|^4 = \int_{B_{r_i}(x_i)} |\nabla u_\infty|^4 + \sum_{j=1}^{L_i} \int_{\mathbb{R}^4} |\nabla \omega_{ij}|^4, \quad (1.11)$$

where $r_i = \frac{1}{2} \min_{1\le j\le L, \ j\ne i} \{|x_i - x_j|, \ \mathrm{dist}(x_i, \partial\Omega)\}$.

The main ideas to prove Theorem 1.2 can be outlined as follows. First, we adapt the arguments from [23, 24] to establish an ε_0−regularity theorem for any approximate biharmonic map u with bi-tension field $h \in L^p$ for $p > 1$, which asserts that $u \in C^\alpha$ for any $\alpha \in (0, \min\{1, 4(1-\frac{1}{p})\})$ and $\nabla^4 u \in L^p$. Second, we prove that for $p > \frac{4}{3}$ there is no concentration of angular hessian energy in the neck region by comparing the approximate biharmonic maps with radial biharmonic functions over annulus. This is a well known technique in harmonic maps in dimension two (see [4]). For biharmonic maps in dimension 4, it was derived by Hornung-Moser [8]. Third, we use a Pohozaev type argument to control the radial hessian energy by the angular hessian energy and L^p-norm of bi-tension fields in the neck region. The assumption $p \geq \frac{4}{3}$ seems to be necessary to validate the Pohozaev type argument, since we need $\Delta^2 u_k \cdot (x \cdot \nabla u_k) \in L^1$ and $h_k \cdot (x \cdot \nabla u_k) \in L^1$. It remains to be an open question whether Theorem 1.2 holds for $1 < p \leq \frac{4}{3}$.

The paper is organized as follows. In §2, we establish the Hölder continuity and $W^{4,p}$-regularity for any approximate biharmonic map with its bi-tension field in L^p for $p > 1$. In §3, we show the strong convergence under the smallness condition of hessian energy and set up the bubbling process. In §4, we prove Theorem 1.2 by establishing (i) there is no concentration of angular hessian energy in the neck region; and (ii) control the radial hessian energy in the neck region by angular hessian energy and L^p-norm of bi-tension field through a Pohozaev type argument.

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Added Note. After we completed this paper, we noticed from a closely related preprint posted in arxiv.org at December 23, 2011 by Laurain and Rivière (arXiv:1112.5393v1), in which they claimed Theorem 1.2 holds for all $p > 1$. Since the main ideas of our proof are different from theirs, we believe that our work here shall have its own interest.

2 A priori estimates of approximate biharmonic maps

In this section, we will establish both Hölder continuity and $W^{4,p}$-regularity for any approximate biharmonic map with bi-tension field $h \in L^p$ for $p > 1$, under the smallness condition of hessian energy. The proof of Hölder continuity is based on suitable modifications and extensions of that by Wang [23] Theorem A on the regularity of biharmonic maps to the context of approximate biharmonic maps in dimension 4. The proof of a bootstrap type argument, which may have its own interest. It is needed for the ε_0-compactness lemma and Pohozaev argument for approximate biharmonic maps.

Denote by $B_r(x) \subset \mathbb{R}^4$ the ball with center x and radius r, and $B_r = B_r(0)$. First, we have

Lemma 2.1 *For any $\alpha \in \Big(0, \min\{1, 4(1-\frac{1}{p})\}\Big)$, there exist $\varepsilon_0 > 0$ such that if $u \in W^{2,2}(B_2, N)$ is an approximate biharmonic map with its bi-tension field $h \in L^p(B_1)$ for $p > 1$, and satisfies*

$$\int_{B_2} \Big(|\nabla^2 u|^2 + |\nabla u|^4\Big) \leq \varepsilon_0^2. \tag{2.1}$$

then $u \in C^\alpha(B_{\frac{1}{2}}, N)$ and

$$\Big[u\Big]_{C^\alpha(B_{\frac{1}{2}})} \leq C\Big(\varepsilon_0 + \|h\|_{L^p(B_1)}\Big). \tag{2.2}$$

Proof. We follow [23] §2, §3 and §4 closely and only sketch the main steps of proof here. First, by choosing ε_0 sufficiently small, Proposition 3.2 and Theorem 3.3 of [23] imply that there exists an adopted frame $\{e_\alpha\}_{\alpha=1}^m$ ($m = \dim N$) along with u^*TN on B_1 such that its connection form $A = (\langle De_\alpha, e_\beta\rangle)$ satisfies:

$$\begin{cases} d^*A = 0 \text{ in } B_1; \quad x\cdot A = 0 \text{ on } \partial B_1 \\ \|A\|_{L^4(B_1)} + \|\nabla A\|_{L^2(B_1)} \le C\|\nabla u\|^2_{L^4(B_1)} \\ \|\nabla A\|_{L^{2,1}(B_{\frac12})} + \|A\|_{L^{4,1}(B_{\frac12})} \le C(\|\nabla^2 u\|_{L^2(B_1)} + \|\nabla u\|_{L^4(B_1)}). \end{cases} \tag{2.3}$$

Here $L^{r,s}$ denotes Lorentz spaces for $1 \le r < +\infty, 1 \le s \le \infty$, see [23] §2 or [7] for its definition and basic properties.

Utilizing $\{e_\alpha\}_{\alpha=1}^m$, we can rewrite (1.1) into the following form (see [23] Lemma 4.1): for $1 \le \alpha \le m$,

$$\Delta\nabla\cdot\langle\nabla u, e_\alpha\rangle = G(u, e_\alpha) + \langle h, e_\alpha\rangle, \tag{2.4}$$

where

$$G(u, e_\alpha) = \begin{cases} \Delta\langle\nabla u, \nabla e_\alpha\rangle + \nabla\cdot\langle\Delta u, \nabla e_\alpha\rangle \\ +\sum_\beta \Big(\nabla\cdot(\langle\Delta u, e_\beta\rangle\langle\nabla e_\alpha, e_\beta\rangle) - \langle(\Delta u)^T, e_\beta\rangle\langle\nabla e_\alpha, e_\beta\rangle\Big) \\ -\sum_\beta\langle\Delta u, e_\beta\rangle\langle\mathbb{B}(u)(e_\alpha, \nabla u), e_\beta\rangle + \Big\langle\nabla(\mathbb{B}(u)(\nabla u, \nabla u)), \nabla e_\alpha\Big\rangle. \end{cases} \tag{2.5}$$

As in [23] Lemma 4.3, we let $\tilde u \in W^{2,2}(\mathbb{R}^4, \mathbb{R}^K)$, $\{\tilde e_\alpha\}_{\alpha=1}^n$, $\tilde A$ and $\tilde h$ be the extensions of u, $\{e_\alpha\}_{\alpha=1}^n$, A and h from $B_{\frac12}$ to $\mathbb{R}^4$ such that $|\tilde e_\alpha| \le 1$, $\tilde A^{\alpha\beta} = \langle\nabla\tilde e_\alpha, \tilde e_\beta\rangle$, and

$$\begin{cases} \|\nabla^2\widetilde{u}\|_{L^2(\mathbb{R}^4)} \le C\|\nabla^2 u\|_{L^2(B_{\frac12})}, \quad \|\nabla\widetilde{u}\|_{L^4(\mathbb{R}^4)} \le C\|\nabla u\|_{L^4(B_{\frac12})}; \\ \|\nabla\widetilde{u}\|_{L^{4,\infty}(\mathbb{R}^4)} \le C\|\nabla u\|_{L^{4,\infty}(B_{\frac12})}, \quad \|\nabla^2\widetilde{u}\|_{L^{2,\infty}(\mathbb{R}^4)} \le C\|\nabla^2 u\|_{L^{2,\infty}(B_{\frac12})}; \\ \|\tilde A\|_{L^{4,1}(\mathbb{R}^4)} + \|\nabla\tilde A\|_{L^{2,1}(\mathbb{R}^4)} \le C(\|\nabla u\|_{L^4(B_1)} + \|\nabla^2 u\|_{L^2(B_1)}); \\ \|\tilde h\|_{L^q(\mathbb{R}^4)} \le C\|h\|_{L^q(B_1)}, \quad \forall 1 \le q \le p. \end{cases} \tag{2.6}$$

Let $\Gamma(x-y) = c_4\ln|x-y|$ be the fundamental solution of Δ^2 in $\mathbb{R}^4$. Set

$$\begin{aligned} W_\alpha(x) &= \int_{\mathbb{R}^4}\Gamma(x-y)G(\tilde u, \tilde e_\alpha)(y)dy + \int_{\mathbb{R}^4}\Gamma(x-y)\langle\tilde h, \tilde e_\alpha\rangle(y)dy \\ &= J_1^\alpha(x) + J_2^\alpha(x) \quad x \in \mathbb{R}^4. \end{aligned} \tag{2.7}$$

Then we have

$$\Delta^2 W_\alpha = G(\tilde u, \tilde e_\alpha) + \langle\tilde h, \tilde e_\alpha\rangle, \text{ in } \mathbb{R}^4. \tag{2.8}$$

For J_2^α, observe that

$$\nabla^4 J_2^\alpha(x) = \int_{\mathbb{R}^4}\nabla_y^4\Gamma(x-y)\langle\widetilde{h}, \widetilde{e}_\alpha\rangle(y)\,dy.$$

Hence by Calderon-Zygmund's $W^{4,p}$-theorey, we have $J_2^\alpha \in W^{4,p}(\mathbb{R}^4)$ and

$$\Big\|\nabla^4 J_2^\alpha\Big\|_{L^p(\mathbb{R}^4)} \le C\Big\|\widetilde{h}\Big\|_{L^p(\mathbb{R}^4)} \le C\Big\|h\Big\|_{L^p(B_1)}. \tag{2.9}$$

By Sobolev embedding theorem, we have that $\nabla^2 J_1^\alpha \in L^{\bar p}(B_1)$, with $\bar p = \frac{2p}{2-p}$ for $1 < p < 2$ or $\bar p$ is any finite number for $p \ge 2$, and

$$\Big\|\nabla^2 J_2^\alpha\Big\|_{L^{\bar p}(B_1)} \le C\Big\|h\Big\|_{L^p(B_1)}. \tag{2.10}$$

By Hölder inequality, (2.9) and (2.10) imply that for any $\theta \in (0, \frac{1}{2})$, it holds

$$\left\|\nabla J_2^\alpha\right\|_{L^{4,\infty}(B_\theta)} + \left\|\nabla^2 J_2^\alpha\right\|_{L^{2,\infty}(B_\theta)} \leq C\theta^{4(1-\frac{1}{p})}\left\|\nabla^2 J_2^\alpha\right\|_{L^{\bar{p}}(B_\theta)} \leq C\theta^{4(1-\frac{1}{p})}\left\|h\right\|_{L^p(B_1)}. \tag{2.11}$$

For J_1^α, we follow exactly [23] Lemma 4.4, Lemma 4.5, and Lemma 4.6 to obtain that $\nabla^2 J_1^\alpha \in L^{2,\infty}(\mathbb{R}^4)$ and

$$\left\|\nabla^2 J_1^\alpha\right\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C\varepsilon_0\Big(\left\|\nabla u\right\|_{L^{4,\infty}(B_{\frac{1}{2}})} + \left\|\nabla^2 u\right\|_{L^{2,\infty}(B_{\frac{1}{2}})}\Big). \tag{2.12}$$

By Sobolev embedding theorem in Lorentz spaces, (2.12) yields

$$\left\|\nabla J_1^\alpha\right\|_{L^{4,\infty}(\mathbb{R}^4)} \leq C\left\|\nabla^2 J_1^\alpha\right\|_{L^{2,\infty}(\mathbb{R}^4)}. \tag{2.13}$$

Combining (2.12) with (2.13) yields

$$\left\|\nabla J_1^\alpha\right\|_{L^{4,\infty}(\mathbb{R}^4)} + \left\|\nabla^2 J_1^\alpha\right\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C\varepsilon_0\Big(\left\|\nabla u\right\|_{L^{4,\infty}(B_{\frac{1}{2}})} + \left\|\nabla^2 u\right\|_{L^{2,\infty}(B_{\frac{1}{2}})}\Big). \tag{2.14}$$

Now, as in [23] Lemma 4.6, we consider the Hodge decomposition of the 1-form $\langle d\tilde{u}, \tilde{e}_\alpha\rangle$. It is well-known [9] that there exist $F_\alpha \in W^{1,4}(\mathbb{R}^4)$ and $H_\alpha \in W^{1,4}(\mathbb{R}^4, \wedge^2\mathbb{R}^4)$ such that

$$\langle d\tilde{u}, \tilde{e}_\alpha\rangle = dF_\alpha + d^*H_\alpha, \quad dH_\alpha = 0 \quad \text{in} \quad \mathbb{R}^4, \tag{2.15}$$

$$\left\|\nabla F_\alpha\right\|_{L^4(\mathbb{R}^4)} + \left\|\nabla H_\alpha\right\|_{L^4(\mathbb{R}^4)} \leq C\left\|\nabla\tilde{u}\right\|_{L^4(\mathbb{R}^4)} \leq C\left\|\nabla u\right\|_{L^4(B_{\frac{1}{2}})}. \tag{2.16}$$

It is easy to see that H_α satisfies

$$\Delta H_\alpha = d\widetilde{u} \wedge d\widetilde{e}_\alpha \qquad \text{in} \quad \mathbb{R}^4, \tag{2.17}$$

and

$$\Delta^2 F_\alpha = \Delta\nabla\cdot\langle\nabla u, e_\alpha\rangle = \Delta^2 W_\alpha \quad \text{in} \quad B_{\frac{1}{2}}. \tag{2.18}$$

By Calderon-Zygmund's $L^{r,s}$-theory and Sobolev embedding theorem, we have that $\nabla^2 H_\alpha \in L^{2,\infty}(\mathbb{R}^4)$ and

$$\left\|\nabla H_\alpha\right\|_{L^{4,\infty}(\mathbb{R}^4)} + \left\|\nabla^2 H_\alpha\right\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C\left\|d\widetilde{u} \wedge d\widetilde{e}_\alpha\right\|_{L^{2,\infty}(\mathbb{R}^4)} \leq C\varepsilon_0\left\|\nabla u\right\|_{L^{4,\infty}(B_{\frac{1}{2}})}. \tag{2.19}$$

By (2.18), we have that $F_\alpha - W_\alpha$ is a biharmonic function on $B_{\frac{1}{2}}$. By the standard estimate of biharmonic functions, we have (see [23] Lemma 4.7) that for any $\theta \in (0, \frac{1}{2})$, it holds

$$\begin{aligned} &\left\|\nabla(F_\alpha - W_\alpha)\right\|_{L^{4,\infty}(B_\theta)} + \left\|\nabla^2(F_\alpha - W_\alpha)\right\|_{L^{2,\infty}(B_\theta)} \\ \leq\ & C\theta\Big(\left\|\nabla(F_\alpha - W_\alpha)\right\|_{L^{4,\infty}(B_1)} + \left\|\nabla^2(F_\alpha - W_\alpha)\right\|_{L^{2,\infty}(B_1)}\Big). \end{aligned} \tag{2.20}$$

Putting (2.11), (2.14), (2.19), and (2.20) together, we can argue, similar to [23] page 84, to reach that for any $\theta \in (0, \frac{1}{2})$, it holds

$$\begin{aligned} \left\|\nabla u\right\|_{L^{4,\infty}(B_\theta)} + \left\|\nabla^2 u\right\|_{L^{2,\infty}(B_\theta)} \ \leq\ & C(\varepsilon + \theta)\Big(\left\|\nabla u\right\|_{L^{4,\infty}(B_1)} + \left\|\nabla^2 u\right\|_{L^{2,\infty}(B_1)}\Big) \\ & + C\theta^{4(1-\frac{1}{p})}\left\|h\right\|_{L^p(B_1)}. \end{aligned} \tag{2.21}$$

It is readily seen that for any $\alpha \in (0, \min\{1, 4(1-\frac{1}{p})\})$, we can choose both $\theta \in (0, \frac{1}{2})$ and $\varepsilon_0 \in (0,1)$ sufficiently small so that

$$\Big\|\nabla u\Big\|_{L^{4,\infty}(B_\theta)} + \Big\|\nabla^2 u\Big\|_{L^{2,\infty}(B_\theta)} \le \theta^\alpha\Big(\Big\|\nabla u\Big\|_{L^{4,\infty}(B_1)} + \Big\|\nabla^2 u\Big\|_{L^{2,\infty}(B_1)} + \Big\|h\Big\|_{L^p(B_1)}\Big). \tag{2.22}$$

In fact, by iterating (2.22) finitely many times on $B_{\frac{1}{2}}(x)$ for $x \in B_{\frac{1}{2}}$, we would have that for $0 < r \le \frac{1}{4}$,

$$\Big\|\nabla u\Big\|_{L^{4,\infty}(B_r(x))} + \Big\|\nabla^2 u\Big\|_{L^{2,\infty}(B_r(x))} \le Cr^\alpha\Big(\Big\|\nabla u\Big\|_{L^{4,\infty}(B_1)} + \Big\|\nabla^2 u\Big\|_{L^{2,\infty}(B_1)} + \Big\|h\Big\|_{L^p(B_1)}\Big). \tag{2.23}$$

Since $L^q(B_r(x)) \subset L^{s,\infty}(B_r(x))$ for any $1 \le q < s$, (2.23) implies that for any $1 < q < 2$,

$$r^{2q-4}\int_{B_r(x)} (|\nabla u|^{2q} + |\nabla^2 u|^q) \le Cr^{2\alpha}\Big(\Big\|\nabla u\Big\|^4_{L^4(B_1)} + \Big\|\nabla^2 u\Big\|^2_{L^2(B_1)} + \Big\|h\Big\|^p_{L^p(B_1)}\Big). \tag{2.24}$$

This, with the help of Morrey's decay lemma, immediately implies that $u \in C^\alpha(B_{\frac{1}{2}})$ and (2.2) holds. This completes the proof of Lemma 2.1. □

In order to show ε_0-compactness and Pohozaev argument for approximate biharmonic maps, we will establish the higher order Sobolev type regularity for approximate biharmonic maps.

The proof utilizes Adams' Reisz potential estimate between Morry spaces, we briefly recall Morrey spaces and Adams' estimates (see [1] and [24] for more details). For an open set $U \subset \mathbb{R}^4$, $1 \le p < +\infty$, $0 < \lambda \le 4$, the Morrey space $M^{p,\lambda}(U)$ is defined by

$$M^{p,\lambda}(U) = \Big\{f \in L^p(U) : \|f\|^p_{M^{p,\lambda}} = \sup_{B_r \subset U} r^{\lambda-4}\int_{B_r} |f|^p < +\infty\Big\}. \tag{2.25}$$

The weak Morrey space $M^{p,\lambda}_*(U)$ is the set of functions $f \in L^{p,\infty}(U)$ satisfying

$$\|f\|^p_{M^{p,\lambda}_*(U)} \equiv \sup_{B_r \subset U}\Big\{\rho^{\lambda-n}\|f\|^p_{L^{p,\infty}(B_r)}\Big\} < \infty. \tag{2.26}$$

For $0 < \beta < 4$, let $I_\beta(f)$ be the Riesz potential of order defined by

$$I_\beta(f)(x) \equiv \int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^{4-\beta}}\,dy, \qquad x \in \mathbb{R}^4. \tag{2.27}$$

Recall Adams' estimate [1] in dimension 4:

Lemma 2.2 *(1) For any $\beta > 0, 0 < \lambda \le 4, 1 < p < \frac{\lambda}{\beta}$, if $f \in M^{p,\lambda}(\mathbb{R}^4)$, then $I_\beta(f) \in M^{\tilde{p},\lambda}(\mathbb{R}^4)$, where $\tilde{p} = \frac{\lambda p}{\lambda - p\beta}$. Moreover,*

$$\|I_\beta(f)\|_{M^{\tilde{p},\lambda}(\mathbb{R}^4)} \le C\|f\|_{M^{p,\lambda}(\mathbb{R}^4)}. \tag{2.28}$$

(2) For $0 < \beta < \lambda \le 4$, if $f \in M^{1,\lambda}(\mathbb{R}^4)$, then $I_\beta(f) \in M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^4)$. Moreover,

$$\|I_\beta(f)\|_{M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^4)} \le C\|f\|_{M^{1,\lambda}(\mathbb{R}^4)}. \tag{2.29}$$

Now we are ready to prove

Lemma 2.3 *There exists $\varepsilon_0 > 0$ such that if $u \in W^{2,2}(B_1, N)$ is an approximate biharmonic map with its bi-tension field $h \in L^p(B_1)$ for $p > 1$, and satisfies*

$$\int_{B_1} |\nabla u|^4 + |\nabla^2 u|^2 \le \varepsilon_0^2. \tag{2.30}$$

Then $u \in W^{4,p}(B_{\frac{1}{8}}, N)$ and

$$\Big\|\nabla^4 u\Big\|_{L^p(B_{\frac{1}{8}})} \le C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)} + \Big\|\nabla^2 u\Big\|_{L^2(B_1)} + \Big\|h\Big\|_{L^p(B_1)}\Big). \tag{2.31}$$

Proof. By (2.2) of Lemma 2.1, we have that for any $\alpha \in (0, \min\{1, 4(1-\frac{1}{p})\})$,

$$\mathrm{osc}_{B_r(x)} u \in Cr^{\alpha},\ \forall B_r(x) \subset B_{\frac{1}{2}}. \tag{2.32}$$

We now divide the proof into three steps.
Step1. There exists $\alpha_0 \in (0, \alpha]$ such that $\nabla u \in M^{4,4-2\alpha_0}(B_{\frac{1}{2}})$, $\nabla^2 u \in M^{2,4-2\alpha_0}(B_{\frac{1}{2}})$, and

$$\Big\|\nabla u\Big\|_{M^{4,4-2\alpha_0}(B_{\frac{1}{2}})} + \Big\|\nabla^2 u\Big\|_{M^{2,4-2\alpha_0}(B_{\frac{1}{2}})} \le C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)} + \Big\|\nabla^2 u\Big\|_{L^2(B_1)} + \Big\|h\Big\|_{L^p(B_1)}\Big). \tag{2.33}$$

Observe that (2.33) is a refined version of (2.23) and (2.24). It is obtained by the hole filling argument as follows. For any $B_r(x) \subset B_{\frac{1}{2}}$, let $\phi \in C_0^\infty(B_r(x))$ such that $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_{\frac{r}{2}}(x)$. Multiplying (1.1) by $\phi(u - u_{x,r})$, where $u_{x,r}$ is the average of u on $B_r(x)$, and integrating over $B_r(x)$, we would have

$$\begin{aligned}
&\int_{B_r(x)} |\Delta(\phi(u - u_{x,r}))|^2 \\
\le\ & \int_{B_r(x)} \Delta((1-\phi)(u - u_{x,r})) \cdot \Delta(\phi(u - u_{x,r})) + C\int_{B_r(x)} |\nabla u|^2 |\Delta(\phi(u - u_{x,r}))| \\
+\ & C\int_{B_r(x)} |\nabla^2 u||\nabla u||\nabla(\phi(u - u_{x,r})| + C\Big(\int_{B_r(x)} |\Delta u|^2 + |h|\Big)\mathrm{osc}_{B_r(x)} u.
\end{aligned} \tag{2.34}$$

It is not hard to see that by Hölder inequality, Sobolev's inequality and (2.32), (2.34) implies

$$\int_{B_{\frac{r}{2}}(x)} (|\nabla u|^4 + |\nabla^2 u|^2) \le \theta \int_{B_r(x)} (|\nabla u|^4 + |\nabla^2 u|^2) + Cr^{\alpha}, \tag{2.35}$$

where $\theta = \frac{C}{C+1} < 1$. Now we can iterate (2.35) finitely many times and achieve that there exists $\alpha_0 \in (0, \alpha)$ such that

$$\sup_{B_r(x) \subset B_{\frac{1}{2}}} r^{-2\alpha_0} \int_{B_r(x)} (|\nabla u|^4 + |\nabla^2 u|^2) \le C\Big(\|\nabla u\|^4_{L^4(B_1)} + \|\nabla^2 u\|^2_{L^2(B_1)} + \|h\|^p_{L^p(B_1)}\Big). \tag{2.36}$$

This yields (2.33).
Step 2. Set $\bar{p} > 2$ by

$$\bar{p} = \begin{cases} \frac{2p}{2-p} & \text{if } 1 < p < 2 \\ \text{any } 2 < q < +\infty & \text{if } p \ge 2. \end{cases}$$

Then $u \in W^{2,\bar{p}}(B_{\frac{1}{4}})$ and

$$\Big\|\nabla u\Big\|_{L^{2\bar{p}}(B_{\frac{1}{4}})} + \Big\|\nabla^2 u\Big\|_{L^{\bar{p}}(B_{\frac{1}{4}})} \le C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)} + \Big\|\nabla^2 u\Big\|_{L^2(B_1)} + \Big\|h\Big\|_{L^p(B_1)}\Big). \tag{2.37}$$

To show (2.37), let $\widetilde{u}, \widetilde{h} : \mathbb{R}^4 \to \mathbb{R}^K$ be extensions of u and h on $B_{\frac{1}{2}}$ such that

$$\|\nabla^2 \widetilde{u}\|_{M^{2,4-2\alpha_0}(\mathbb{R}^4)} \le C\|\nabla^2 u\|_{M^{2,4-2\alpha_0}(B_{\frac{1}{2}})}, \quad \|\nabla \widetilde{u}\|_{M^{4,4-2\alpha_0}(\mathbb{R}^4)} \le C\|\nabla u\|_{M^{4,4-2\alpha_0}(B_{\frac{1}{2}})}, \tag{2.38}$$

and

$$\|\widetilde{h}\|_{L^p(\mathbb{R}^4)} \le C\|h\|_{L^p(B_1)}. \tag{2.39}$$

Define $w : \mathbb{R}^4 \to \mathbb{R}^K$ by

$$\begin{aligned} w(x) &= \int_{\mathbb{R}^4} \Gamma(x-y)\widetilde{h}(y)\,dy + \int_{\mathbb{R}^4} \Delta_y \Gamma(x-y)(\mathbb{B}(\tilde{u})(\nabla\tilde{u}, \nabla\tilde{u}))(y)\,dy \\ &- 2\int_{\mathbb{R}^4} \nabla_y \Gamma(x-y)\langle \Delta\tilde{u}, \nabla(\mathbb{P}(\tilde{u}))\rangle(y)\,dy - \int_{\mathbb{R}^4} \Gamma(x-y)\langle \Delta(\mathbb{P}(\tilde{u})), \Delta\tilde{u}\rangle(y)\,dy \\ &= w_1(x) + w_2(x) + w_3(x) + w_4(x), \ x \in \mathbb{R}^4. \end{aligned} \tag{2.40}$$

Then it is readily seen that

$$\Delta^2(u - w) = 0 \quad \text{on} \quad B_{\frac{1}{2}}, \tag{2.41}$$

or $u - w$ is a biharmonic function on $B_{\frac{1}{2}}$.

Now we estimate w_i, $1 \le i \le 4$, as follows. For w_1, by Calderon-Zygmund's $W^{4,p}$-theory we have that $w_1 \in W^{4,p}(\mathbb{R}^4)$ so that $\nabla^2 w_1 \in L^{\bar{p}}(B_{\frac{1}{2}})$ and

$$\Big\|\nabla^2 w_1\Big\|_{L^{\bar{p}}(B_{\frac{1}{2}})} \le C\|h\|_{L^p(B_1)}. \tag{2.42}$$

For w_3, since $|\nabla^2\widetilde{u}||\nabla\widetilde{u}| \in M^{\frac{4}{3},4-2\alpha_0}(\mathbb{R}^4)$, $|\nabla w_3| \le CI_2(|\nabla^2\widetilde{u}||\nabla\widetilde{u}|)$ and $|\nabla^2 w_3| \le CI_1(|\nabla^2\widetilde{u}||\nabla\widetilde{u}|)$, Lemma 2.2 implies that $\nabla w_3 \in M^{\frac{4(2-\alpha_0)}{2-3\alpha_0},4-2\alpha_0}(\mathbb{R}^4)$, $\nabla^2 w_3 \in M^{\frac{2(4-2\alpha_0)}{4-3\alpha_0},4-2\alpha_0}(\mathbb{R}^4)$, and

$$\begin{aligned} &\Big\|\nabla w_3\Big\|_{M^{\frac{4(2-\alpha_0)}{2-3\alpha_0},4-2\alpha_0}(\mathbb{R}^4)} + \Big\|\nabla^2 w_3\Big\|_{M^{\frac{2(4-2\alpha_0)}{4-3\alpha_0},4-2\alpha_0}(\mathbb{R}^4)} \\ \le\ & C\Big\||\nabla^2\widetilde{u}||\nabla\widetilde{u}|\Big\|_{M^{\frac{4}{3},4-2\alpha_0}(\mathbb{R}^4)} \le C\Big\|\nabla^2\widetilde{u}\Big\|_{M^{2,4-2\alpha_0}(\mathbb{R}^4)}\Big\|\nabla\widetilde{u}\Big\|_{M^{4,4-2\alpha_0}(\mathbb{R}^4)} \\ \le\ & C\Big\|\nabla^2 u\Big\|_{M^{2,4-2\alpha_0}(B_{\frac{1}{2}})}\Big\|\nabla u\Big\|_{M^{4,4-2\alpha_0}(B_{\frac{1}{2}})} \\ \le\ & C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)} + \Big\|\nabla^2 u\Big\|_{L^2(B_1)} + \Big\|h\Big\|_{L^p(B_1)}\Big). \end{aligned} \tag{2.43}$$

For w_4, it is easy to see that $|\nabla w_4| \le CI_3(|\nabla^2\widetilde{u}|^2 + |\nabla\widetilde{u}|^4)$ and $|\nabla^2 w_4| \le CI_2(|\nabla^2\widetilde{u}|^2 + |\nabla\widetilde{u}|^4)$. Since $(|\nabla^2\widetilde{u}|^2 + |\nabla\widetilde{u}|^4) \in M^{1,4-2\alpha_0}(\mathbb{R}^4)$, Lemma 2.2 implies that $|\nabla w_4| \in M_*^{\frac{4-2\alpha_0}{1-2\alpha_0},4-2\alpha_0}(\mathbb{R}^4)$ and $|\nabla^2 w_4| \in M_*^{\frac{2-\alpha_0}{1-\alpha_0},4-2\alpha_0}(\mathbb{R}^4)$ and

$$\begin{aligned} &\Big\|\nabla w_4\Big\|_{M_*^{\frac{4-2\alpha_0}{1-2\alpha_0},4-2\alpha_0}(\mathbb{R}^4)} + \Big\|\nabla^2 w_4\Big\|_{M_*^{\frac{2-\alpha_0}{1-\alpha_0},4-2\alpha_0}(\mathbb{R}^4)} \\ \le\ & C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)} + \Big\|\nabla^2 u\Big\|_{L^2(B_1)} + \Big\|h\Big\|_{L^p(B_1)}\Big). \end{aligned} \tag{2.44}$$

For w_2, since $|\nabla w_2| \leq CI_1(|\nabla\widetilde{u}|^2)$, $|\nabla\widetilde{u}|^2 \in M^{2,4-2\alpha_0}(\mathbb{R}^4)$, Lemma 2.2 implies that $|\nabla w_2| \in M^{\frac{4-2\alpha_0}{1-\alpha_0},4-2\alpha_0}(\mathbb{R}^4)$ and

$$\begin{aligned}\Big\|\nabla w_2\Big\|_{M^{\frac{4-2\alpha_0}{1-\alpha_0},4-2\alpha_0}(\mathbb{R}^4)} &\leq C\Big\||\nabla\widetilde{u}|^2\Big\|_{M^{2,4-2\alpha_0}(\mathbb{R}^4)}\\ &\leq C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)}+\Big\|\nabla^2 u\Big\|_{L^2(B_1)}+\Big\|h\Big\|_{L^p(B_1)}\Big).\end{aligned} \tag{2.45}$$

It is not hard to see from (2.41) and the standard estimate on biharmonic function, and the estimates (2.42), (2.43), (2.44), and (2.45) that there exist $1<q<\min\Big\{\frac{p}{4-3p},\frac{4-2\alpha_0}{4(1-\alpha_0)},\frac{2-\alpha_0}{2-3\alpha_0}\Big\}$ and $0<\alpha_1\leq\min\Big\{\alpha_0,\frac{(4-3p)q}{2p}\Big\}$ such that $\nabla u\in M^{4q,4-2\alpha_1}(B_{\frac{3}{8}})$ and $\nabla^2 u\in M^{2q,4-2\alpha_1}(B_{\frac{3}{8}})$, and

$$\Big\|\nabla u\Big\|_{M^{4q,4-2\alpha_1}(B_{\frac{3}{8}})}+\Big\|\nabla^2 u\Big\|_{M^{2q,4-2\alpha_1}(B_{\frac{3}{8}})}\leq C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)}+\Big\|\nabla^2 u\Big\|_{L^2(B_1)}+\Big\|h\Big\|_{L^p(B_1)}\Big). \tag{2.46}$$

With (2.46), we can repeat the same argument to bootstrap the integrablity of $\nabla^2 u$ and finally get that $\nabla^2 u\in L^{\bar{p}}(B_{\frac{1}{4}})$ and (2.37) holds.

Step 3. $\nabla^4 u\in L^p(B_{\frac{1}{8}})$ and

$$\Big\|\nabla^4 u\Big\|_{L^p(B_{\frac{1}{8}})}\leq C\Big(\Big\|\nabla u\Big\|_{L^4(B_1)}+\Big\|\nabla^2 u\Big\|_{L^2(B_1)}+\Big\|h\Big\|_{L^p(B_1)}\Big). \tag{2.47}$$

To prove (2.47), first observe that the equation (1.1) can be written as

$$\Delta^2 u=\mathrm{div}(E(u))+G(u)+h,\quad \text{in}\quad B_{\frac{1}{4}}, \tag{2.48}$$

where $E(u)=\nabla(B(u)(\nabla u,\nabla u))+2\langle\Delta u,\nabla(\mathbb{P}(u))\rangle$ and $G(u)=-\langle\Delta(\mathbb{P}(u)),\Delta u\rangle$ so that

$$|E(u)|\leq C(|\nabla u|^3+|\nabla^2 u||\nabla u|),\quad |G(u)|\leq C(|\nabla^2 u|^2+|\nabla u|^4).$$

By (2.37) and Sobolev's inequality, we have $\nabla u\in L^{\frac{4p}{4-3p}}(B_{\frac{1}{4}})$ so that $G(u)\in L^{\frac{p}{2-p}}(B_{\frac{1}{4}})$ and $E(u)\in L^{\frac{4p}{8-5p}}(B_{\frac{1}{4}})$. Note that we can write $u=u_1+u_2+u_3+u_4$ in $B_{\frac{1}{4}}$, where

$$\Delta^2 u_1=G(u)\text{ in }B_{\frac{1}{4}};\ (u_1,\nabla u_1)=(0,0)\text{ on }\partial B_{\frac{1}{4}}, \tag{2.49}$$

$$\Delta^2 u_2=h\text{ in }B_{\frac{1}{4}};\ (u_2,\nabla u_2)=(0,0)\text{ on }\partial B_{\frac{1}{4}}, \tag{2.50}$$

$$\Delta^2 u_3=\mathrm{div}(E(u))\ B_{\frac{1}{4}};\ (u_3,\nabla u_3)=(0,0)\text{ on }\partial B_{\frac{1}{4}}, \tag{2.51}$$

and

$$\Delta^2 u_4=0\ B_{\frac{1}{4}};\ (u_3,\nabla u_3)=(u,\nabla u)\text{ on }\partial B_{\frac{1}{4}}. \tag{2.52}$$

By Calderon-Zygmund's L^q-theory, we have

$$\Big\|u_1\Big\|_{W^{4,\frac{p}{2-p}}(B_{\frac{1}{4}})}+\Big\|u_2\Big\|_{W^{4,p}(B_{\frac{1}{4}})}\leq C(\Big(\Big\|\nabla u\Big\|_{L^4(B_1)}+\Big\|\nabla^2 u\Big\|_{L^2(B_1)}+\Big\|h\Big\|_{L^p(B_1)}\Big), \tag{2.53}$$

and

$$\Big\|u_3\Big\|_{W^{3,\frac{4p}{4-p}}(B_{\frac{1}{4}})}+\Big\|u_4\Big\|_{W^{4,p}(B_{\frac{1}{5}})}\leq C(\Big(\Big\|\nabla u\Big\|_{L^4(B_1)}+\Big\|\nabla^2 u\Big\|_{L^2(B_1)}+\Big\|h\Big\|_{L^p(B_1)}\Big), \tag{2.54}$$

In particular, we can conclude that $u \in W^{3,\frac{4p}{4-p}}(B_{\frac{1}{5}})$ and

$$\left\|u\right\|_{W^{3,\frac{4p}{4-p}}(B_{\frac{1}{5}})} \leq C(\Big(\left\|\nabla u\right\|_{L^4(B_1)} + \left\|\nabla^2 u\right\|_{L^2(B_1)} + \left\|h\right\|_{L^p(B_1)}\Big). \tag{2.55}$$

By Hölder inequality, (2.55) then implies

$$|\mathrm{div}(E(u))| \leq C(|\nabla^3 u||\nabla u| + |\nabla^2 u|^2 + |\nabla u|^3 + |\nabla u|^4) \in L^{\frac{p}{2-p}}(B_{\frac{1}{5}}).$$

Hence applying $W^{4,q}$-estimate of (2.51) yields that $u_3 \in W^{4,\frac{p}{2-p}}(B_{\frac{1}{8}})$ and

$$\left\|u_3\right\|_{W^{4,\frac{p}{2-p}}(B_{\frac{1}{8}})} \leq C(\Big(\left\|\nabla u\right\|_{L^4(B_1)} + \left\|\nabla^2 u\right\|_{L^2(B_1)} + \left\|h\right\|_{L^p(B_1)}\Big). \tag{2.56}$$

Since $\frac{p}{2-p} > p$, by combining (2.56) with (2.53) and (2.54), we finally obtain that $u \in W^{4,p}(B_{\frac{1}{8}})$ and (2.47) holds. This completes the proof of Lemma 2.3. $\square$

3 Blow up analysis and energy inequality

This section is devoted to ϵ_0-compactness lemma and preliminary steps on the blow up analysis of approximate biharmonic maps with bi-tension fields bounded in L^p for $p > 1$.

First we have

Lemma 3.1 *For $n = 4$, there exists an $\epsilon_0 > 0$ such that if $\{u_k\} \subset W^{2,2}(B_1, N)$ is a sequence of approximate biharmonic maps satisfying*

$$\sup_k \Big(\|\nabla u_k\|^4_{L^4(B_1)} + \left\|\nabla^2 u_k\right\|^2_{L^2(B_1)}\Big) \leq \epsilon_0, \tag{3.1}$$

and $u_k \rightharpoonup u$ in $W^{2,2}(B_1)$ and $h_k \rightharpoonup h$ in $L^p(B_1)$ for some $p > 1$. Then $u \in C^0 \cap W^{4,p}(\Omega, N)$ is an approximate biharmonic map with bi-tension field h, and

$$\lim_{k\to\infty} \left\|u_k - u\right\|_{W^{2,2}(B_{\frac{1}{2}})} = 0. \tag{3.2}$$

Proof. The first assertion follows easily from (1.1) and (3.2). To show (3.2), it suffices to show that $\{u_k\}$ is a Cauchy sequence in $W^{2,2}(B_{\frac{1}{2}})$. By (3.1) and Lemma 2.1, there exist $\alpha \in (0, 1)$ and $q > 2$ such that

$$\sup_k \Big[\left\|u_k\right\|_{C^\alpha(B_{\frac{3}{4}})} + \left\|\nabla^2 u_k\right\|_{L^q(B_{\frac{3}{4}})}\Big] \leq C.$$

Hence we may assume that

$$\lim_{k,l\to\infty} \left\|u_k - u_l\right\|_{L^\infty(B_{\frac{3}{4}})} = 0.$$

For $\eta \in C_0^\infty(B_{\frac{3}{4}})$ be a cut-off function of $B_{\frac{1}{2}}$, multiplying the equations of u_k and u_l by $(u_k - u_l)\phi^2$

and integrating over B_1, we obtain

$$\begin{aligned}
&\int_{B_1} |\Delta(u_k-u_l)|^2\phi^2 \\
\le\ & \int_{B_1} |\Delta(u_k-u_l)|(2|\nabla(u_k-u_l)||\nabla\phi^2|+|u_k-u_l||\Delta\phi^2|)+\int_{B_1}|h_k-h_l||u_k-u_l|\phi^2 \\
&+3\int_{B_1}(|\Delta u_l|^2+|\Delta u_k|^2)|u_k-u_l|\phi^2 \\
&+4\int_{B_1}|\nabla^2 u_k||\nabla u_k||\nabla(u_k(u_k-u_l)\phi^2)| \\
&+4\int_{B_1}|\nabla^2 u_l||\nabla u_l||\nabla(u_l(u_k-u_l)\phi^2)| \\
=\ & I+II+III+IV+V.
\end{aligned}$$

It is easy to see

$$|I|\le C(\|\nabla(u_k-u_l)\|_{L^2(B_{\frac34})}+\|u_k-u_l\|_{L^\infty(B_{\frac34})})\to 0,$$

$$|II|\le C\|h_k-h_l\|_{L^1(B_{\frac34})}\|u_k-u_l\|_{L^\infty(B_{\frac34})}\to 0,$$

$$|III|\le C(\|\nabla^2u_k\|^2_{L^2(B_{\frac34})}+\|\nabla^2u_l\|^2_{L^2(B_{\frac34})})\|u_k-u_l\|_{L^\infty(B_{\frac34})}\to 0.$$

For IV, observe that for $1<r<4$ with $\frac14+\frac1q+\frac1r=1$, we have

$$\begin{aligned}
|IV| \ \le\ & C\Big(\|\nabla^2u_k\|_{L^2(B_{\frac34})}\|\nabla u\|^2_{L^4(B_{\frac34})}\|u_k-u_l\|_{L^\infty(B_{\frac34})} \\
&+\|\nabla^2u_k\|_{L^q(B_{\frac34})}\|\nabla u_k\|_{L^4(B_{\frac34})}\|\nabla(u_k-u_l)\|_{L^r(B_{\frac34})}\Big)\to 0,
\end{aligned}$$

since $\|\nabla(u_k-u_l)\|_{L^r(B_{\frac34})}\to 0$. Similarly, we can show

$$|V|\to 0.$$

Hence $\{u_k\}$ is a Cauchy sequence in $W^{2,2}(B_{\frac12})$. This completes the proof. □

Lemma 3.2 *Under the same assumptions as Theorem 1.2, there exists a finite subset $\Sigma\subset\Omega$ such that $u_k\to u$ in $W^{2,2}_{\rm loc}\cap C^0_{\rm loc}(\Omega\backslash\Sigma,N)$. Moreover, $u\in W^{4,p}\cap C^0(\Omega,N)$ is an approximate biharmonic map with bi-tension field h.*

Proof. Let $\epsilon_0>0$ be given by Lemma 2.1, and define

$$\Sigma:=\bigcap_{r>0}\Big\{x\in\Omega:\liminf_{k\to\infty}\int_{B_r(x)}(|\nabla^2u_k|^2+|\nabla u_k|^4)>\epsilon_0^2\Big\}. \tag{3.3}$$

Then by a simple covering argument we have that Σ is a finite set and

$$H^0(\Sigma)\le\frac{1}{\epsilon_0^2}\sup_k\int_\Omega(|\nabla^2u_k|^2+|\nabla u_k|^4)<+\infty.$$

For any $x_0\in\Omega\setminus\Sigma$, there exists $r_0>0$ such that

$$\liminf_{k\to\infty}\int_{B_{r_0}(x_0)}(|\nabla^2u_k|^2+|\nabla u_k|^4)\le\epsilon_0^2.$$

Hence Lemma 2.1 and Lemma 3.1 imply that there exists $\alpha \in (0,1)$ such that

$$\Big\|u_k\Big\|_{C^\alpha(B_{\frac{r_0}{2}}(x_0))} \leq C,$$

so that $u_k \to u$ in $C^0 \cap W^{2,2}(B_{\frac{r_0}{2}}(x_0))$. This proves that $u_k \to u$ in $W^{2,2}_{\rm loc} \cap C^0_{\rm loc}(\Omega \setminus \Sigma)$. It is clear that $u \in W^{2,2}(\Omega)$ is an approximate biharmonic map with bi-tension field $h \in L^p(\Omega)$. Applying Lemma 2.1 and Lemma 2.3 again, we conclude that $u \in C^0(\Omega, N) \cap W^{4,p}(\Omega, N)$. □

Proof of Theorem 1.2:

The proof of (1.10) with "=" replaced by"≥" is similar to [22] Lemma 3.3. Here we sketch it. For any $x_0 \in \Sigma$, there exist $r_0 > 0$, $x_k \to x_0$ and $r_k \downarrow 0$ such that

$$\max_{x \in B_{r_0}(x_0)} \Big\{ \int_{B_{r_k}(x)} (|\nabla^2 u_k|^2 + |\nabla u_k|^4) \Big\} = \frac{\epsilon_0^2}{2} = \int_{B_{r_k}(x_k)} (|\nabla^2 u_k|^2 + |\nabla u_k|^4).$$

Define $v_k(x) = u_k(x_k + r_k x) : r_k^{-1}\Big(B_{r_0}(x_0) \setminus \{x_k\}\Big) \to N$. Then v_k is an approximate biharmonic map, with bi-tension field $\widetilde{h}_k(x) = r_k^4 h(x_k + r_k x)$, that satisfies

$$\int_{B_1(x)} (|\nabla^2 v_k|^2 + |\nabla v_k|^4) \leq \frac{\epsilon_0^2}{2},\ \forall x \in r_k^{-1}\Big(B_{r_0}(x_0) \setminus \{x_k\}\Big),\ \text{and}\ \int_{B_1(0)} (|\nabla^2 v_k|^2 + |\nabla v_k|^4) = \frac{\epsilon_0^2}{2},$$

and

$$\Big\|\widetilde{h}_k\Big\|_{L^p\left(r_k^{-1}(B_{r_0}(x_0) \setminus \{x_k\}\right)} \leq r_k^{4(1-\frac{1}{p})} \Big\|h_k\Big\|_{L^p(\Omega)} \to 0.$$

Thus Lemma 2.3 and Lemma 3.1 imply that, after taking possible subsequences, there exists a nontrivial biharmonic map $\omega : \mathbb{R}^4 \to N$ with

$$\frac{\epsilon_0^2}{2} \leq \int_{\mathbb{R}^4} (|\nabla^2 \omega|^2 + |\nabla \omega|^4) < +\infty$$

such that $v_k \to \omega$ in $W^{2,2}_{\rm loc} \cap C^0_{\rm loc}(\mathbb{R}^4)$. Performing such a blow-up argument at all $x_i \in \Sigma$, $1 \leq i \leq L$, we can find all possible nontrivial biharmonic maps $\{\omega_{ij}\} \in W^{2,2}(\mathbb{R}^4)$ for $1 \leq j \leq L_i$, with $L_i \leq CM\epsilon_0^{-2}$. Moreover, by the lower semicontinuity, we have that the part "≥" of (1.10) holds.

The other half, "≤", of (1.10) will be proved in the next section.

4 No hessian energy concentration in the neck region

In order to show the part "≤" of (1.10), we need to show that there is no concentration of hessian energy in the neck region. This will be done in two steps. The first step is to show that there is no angular hessian energy concentration in the neck region by comparing with radial biharmonic functions over dydaic annulus. The second step is to use the type of almost hessian energy monotonicity inequality, which is obtained by the Pohozaev type argument, to control the radial component of hessian energy by the angular component of hessian energy. We require $p > \frac{4}{3}$ in both steps.

Suppose that $\{u_k\} \subset W^{2,2}(B_1, N)$ is a sequence of approximate biharmonic maps satisfying for some $p > \frac{4}{3}$,

$$\int_{B_1} (|\nabla^2 u_k|^2 + |\nabla u_k|^4 + |h_k|^p) \leq C,\ \forall k \geq 1. \tag{4.1}$$

Without loss of generality, we assume that $u_k \rightharpoonup u$ in $W^{2,2}$, $h_k \rightharpoonup h$ in L^p, and $u_k \to u$ in $W^{2,2}_{\rm loc}(B_1\backslash\{0\}, N)$ but not in $W^{2,2}(B_1, N)$. Furthermore, as in Ding-Tian [4], we may assume that the total number of bubbles generated at 0 is $L = 1$. Then for any $\varepsilon > 0$ there exist $r_k \downarrow 0$, $R \geq 1$ sufficiently large, and $0 < \delta \leq \varepsilon^{\frac{p}{4(p-1)}}$ so that for k sufficiently large, the following property holds

$$\int_{B_{2\rho}\backslash B_\rho} (|\nabla^2 u_k|^2 + |\nabla u_k|^4) \leq \epsilon^2, \qquad \forall\ \frac{1}{16} Rr_k \leq \rho \leq 16\delta. \tag{4.2}$$

Step 1. *Angular hessian energy estimate in the neck region*:

Since $p > \frac{4}{3}$, it follows from (4.2), Lemma 2.1, Lemma 2.3, and Sobolev embedding theorem that for any $\alpha \in \Big(0, 4(1-\frac{1}{p})\Big)$, $u_k \in C^\alpha \cap W^{4,p}(B_{2\rho} \setminus B_\rho)$, $\nabla u_k \in C^0(B_{2\rho} \setminus B_\rho)$, and

$$\Big[u\Big]_{C^\alpha(B_{2\rho}\backslash B_\rho)} + \Big\|\nabla u_k\Big\|_{L^\infty(B_{2\rho}\backslash B_\rho)} \leq C\Big(\varepsilon + \rho^{4(1-\frac{1}{p})}\Big) \leq C\varepsilon,\ \forall\ \frac{1}{8} Rr_k \leq \rho \leq 8\delta. \tag{4.3}$$

It follows from Lemma 2.3 that

$$\Big\|\nabla^3 u_k\Big\|_{L^{\frac{4}{3}}(B_{2\rho}\backslash B_\rho)} \leq C\varepsilon,\ \forall\ \frac{1}{8} Rr_k \leq \rho \leq 8\delta. \tag{4.4}$$

To handle the contributions of various boundary terms during the argument, by Fubini's theorem we assume that $R > 0$ and $\delta > 0$ are chosen so that for k sufficiently large, the following property

$$r\int_{\partial B_r} (|\nabla u_k|^4 + |\nabla^2 u_k|^2 + |\nabla^3 u_k|^{\frac{4}{3}}) \leq 8 \sup_k \int_{B_{2r}\backslash B_{\frac{r}{2}}} (|\nabla u_k|^4 + |\nabla^2 u_k|^2 + |\nabla^3 u_k|^{\frac{4}{3}}) \leq C\varepsilon^2, \tag{4.5}$$

holds for $r = \frac{1}{2}Rr_k, Rr_k, \delta, 2\delta, 4\delta$. For simplicity, we assume (4.5) holds for all $k \geq 1$. Here we indicate (4.5) for $r = Rr_k$: set $\widetilde{u}_k = u_k(r_k x) : B_{\delta r_k^{-1}} \to N$. Then by Fatou's lemma we have

$$\int_{\frac{1}{2}R}^{2R} \liminf_k \int_{\partial B_r} (|\nabla \widetilde{u}_k|^4 + |\nabla^2 \widetilde{u}_k|^2 + |\nabla^3 \widetilde{u}_k|^{\frac{4}{3}}) \leq \liminf_k \int_{B_{2R}\backslash B_{\frac{1}{2}R}} (|\nabla \widetilde{u}_k|^4 + |\nabla^2 \widetilde{u}_k|^2 + |\nabla^3 \widetilde{u}_k|^{\frac{4}{3}}).$$

By Fubini's theorem and scalings, this easily imply (4.5) for $r \approx Rr_k$ (for simplicity, we can assume $r = Rr_k$).

Let $N_k \in \mathbb{N}$ be such that $2^{N_k} = [\frac{2\delta}{Rr_k}]$. Set

$$\mathcal{A}_k^i := B_{2^{i+1}Rr_k} \setminus B_{2^i Rr_k} \ \text{ and }\ \mathcal{B}_k^i := B_{2^{i+2}Rr_k} \setminus B_{2^{i-1}Rr_k},\ 1 \leq i \leq N_k - 1. \tag{4.6}$$

Now we define a radial biharmonic function v_k on the annulus $B_{2\delta}\backslash B_{Rr_k}$ as follows. For simplicity, we may assume $\frac{2\delta}{Rr_k}$ is a positive integer so that $B_{2\delta}\backslash B_{Rr_k} = \bigcup_{i=0}^{N_k-1} \mathcal{A}_k^i$. For $0 \leq i \leq N_k - 1$, $v_k(x) = v_k(|x|)$ satisfies

$$\begin{cases} \Delta^2 v_k = 0 & \text{in } \mathcal{A}_k^i, \\ v_k(r) = ⨍_{\partial B_{2^{i+1}Rr_k}} u_k, \quad v_k'(r) = ⨍_{\partial B_{2^{i+1}Rr_k}} \frac{\partial u_k}{\partial r}, & \text{if } r = 2^{i+1}Rr_k, \\ v_k(r) = ⨍_{\partial B_{2^i Rr_k}} u_k, \quad v_k'(r) = ⨍_{\partial B_{2^i Rr_k}} \frac{\partial u_k}{\partial r}, & \text{if } r = 2^i Rr_k. \end{cases} \tag{4.7}$$

Here $\fint$ denotes the average integral. By the standard estimate of biharmonic functions (see, e.g., [8] Lemma 5.1) and (4.3), we have that $v_k \in W^{4,p}(\mathcal{A}_k^i)$ for $0 \le i \le N_k - 1$ and

$$\Big[v_k\Big]_{C^\alpha(\mathcal{A}_k^i)} \le C\Big(\Big[u_k\Big]_{C^\alpha(\mathcal{A}_k^i)} + \Big[\nabla u_k\Big]_{L^\infty(\mathcal{A}_k^i)}\Big) \le C\varepsilon.$$

In particular, we have

$$\sup_{0\le i\le N_k-1} \operatorname{osc}_{\mathcal{A}_k^i}(u_k - v_k) \le C\varepsilon. \tag{4.8}$$

We now perform the estimate, similar to the arguments by [20] or [4] on harmonic maps and [8] on biharmonic maps. First, since $u_k - v_k \in W^{4,p}(\mathcal{A}_k^i)$, we can apply the Green's identity to get that for $0 \le i \le N_k - 1$,

$$\begin{aligned}\int_{\mathcal{A}_k^i} \Delta^2(u_k - v_k)(u_k - v_k) &= \int_{\mathcal{A}_k^i} |\Delta(u_k - v_k)|^2 + \int_{\partial\mathcal{A}_k^i} \frac{\partial(\Delta(u_k - v_k))}{\partial\nu}(u_k - v_k)\\ &- \int_{\partial\mathcal{A}_k^i} \frac{\partial(u_k - v_k)}{\partial\nu}\Delta(u_k - v_k).\end{aligned} \tag{4.9}$$

Summing over $0 \le i \le N_k - 1$, we obtain

$$\begin{aligned}\int_{B_{2\delta}\setminus B_{Rr_k}} |\Delta(u_k - v_k)|^2 &= \sum_{i=0}^{N_k-1} \int_{\mathcal{A}_k^i} \Delta^2(u_k - v_k)(u_k - v_k)\\ &+\Big(\int_{\partial B_{2\delta}} - \int_{\partial B_{Rr_k}}\Big)\frac{\partial(u_k - v_k)}{\partial\nu}\Delta(u_k - v_k)\\ &-\Big(\int_{\partial B_{2\delta}} - \int_{\partial B_{Rr_k}}\Big)\frac{\partial(\Delta(u_k - v_k))}{\partial\nu}(u_k - v_k)\\ &= \sum_{i=0}^{N_k-1} \int_{\mathcal{A}_k^i} \Delta^2 u_k(u_k - v_k) + \Big(\int_{\partial B_{2\delta}} - \int_{\partial B_{Rr_k}}\Big)\frac{\partial(u_k - v_k)}{\partial\nu}\Delta u_k\\ &-\Big(\int_{\partial B_{2\delta}} - \int_{\partial B_{Rr_k}}\Big)\frac{\partial\Delta u_k}{\partial\nu}(u_k - v_k).\end{aligned} \tag{4.10}$$

Here we haved use that $\Delta^2 v_k = 0$ in $\mathcal{A}_k^i$, and $\int_{\partial B_\rho} \frac{\partial(u_k - v_k)}{\partial\nu}\Delta v_k = \int_{\partial B_\rho} \frac{\partial\Delta v_k}{\partial\nu}(u_k - v_k) = 0$ for $\rho = 2\delta$ and Rr_k due to the radial form of v_k and the choices of boundary conditions of v_k.

We can check that the last two terms in the right hand side of (4.10) converge to zero as $k \to \infty$. In fact, by (4.5), Hölder inequality, and (4.2) we have that

$$\begin{aligned}\Big|\int_{\partial B_{2\delta}} \frac{\partial(u_k - v_k)}{\partial\nu}\Delta u_k\Big| &\le \int_{\partial B_{2\delta}} |\nabla u_k||\Delta u_k| + (\fint_{\partial B_{2\delta}} |\nabla u_k|)\int_{\partial B_{2\delta}} |\Delta u_k|\\ &\le C\Big(\delta\int_{\partial B_{2\delta}} |\nabla u_k|^4\Big)^{\frac14}\Big(\delta\int_{\partial B_{2\delta}} |\nabla u_k|^2\Big)^{\frac12} \le C\varepsilon^{\frac32}.\end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned}\Big|\int_{\partial B_{Rr_k}} \frac{\partial(u_k - v_k)}{\partial\nu}\Delta u_k\Big| &\le \int_{\partial B_{Rr_k}} |\nabla u_k||\Delta u_k| + (\fint_{\partial B_{Rr_k}} |\nabla u_k|)\int_{\partial B_{Rr_k}} |\Delta u_k|\\ &\le C\Big(Rr_k\int_{\partial B_{Rr_k}} |\nabla u_k|^4\Big)^{\frac14}\Big(Rr_k\int_{\partial B_{Rr_k}} |\nabla u_k|^2\Big)^{\frac12} \le C\varepsilon^{\frac32}.\end{aligned} \tag{4.12}$$

For the last term, by Fubini's theorem and (4.8) we have

$$\begin{aligned}\Big|\int_{\partial B_{2\delta}}\frac{\partial\Delta u_k}{\partial\nu}(u_k-v_k)\Big| &\le C\Big(\delta\int_{\partial B_{2\delta}}|\nabla^3 u_k|^{\frac{4}{3}}\Big)^{\frac{3}{4}}\max_{\partial B_{2\delta}}|u_k-v_k| \\ &\le C\epsilon\Big(\int_{B_{4\delta}\setminus B_\delta}|\nabla^3 u_k|^{\frac{4}{3}}\Big)^{\frac{3}{4}}\le C\varepsilon. \qquad (4.13)\end{aligned}$$

and, similarly,

$$\begin{aligned}\Big|\int_{\partial B_{Rr_k}}\frac{\partial\Delta u_k}{\partial\nu}(u_k-v_k)\Big| &\le C\Big(Rr_k\int_{\partial B_{Rr_k}}|\nabla^3 u_k|^{\frac{4}{3}}\Big)^{\frac{3}{4}}\max_{\partial B_{Rr_k}}|u_k-v_k| \\ &\le C\varepsilon\Big(\int_{B_{2Rr_k}\setminus B_{\frac{1}{2}Rr_k}}|\nabla^3 u_k|^{\frac{4}{3}}\Big)^{\frac{3}{4}}\le C\varepsilon. \qquad (4.14)\end{aligned}$$

Therefore we conclude that the last two terms in the right hand side of (4.10) converge to zero as $k\to\infty$.

For the first term in the right hand side of (4.10), we proceed as follows. First, we can rewrite the equation for u_k as

$$\Delta^2 u_k=\mathrm{div}(E(u_k))+G(u_k)+h_k, \qquad (4.15)$$

where

$$|E(u_k)|\le C\Big(|\nabla^2 u_k||\nabla u_k|+|\nabla u_k|^3\Big),\quad |G(u_k)|\le C\Big(|\nabla^2 u_k|^2+|\nabla u_k|^4\Big). \qquad (4.16)$$

Hence

$$\begin{aligned}\int_{\mathcal{A}_k^i}\Delta^2 u_k(u_k-v_k) &= \int_{\mathcal{A}_k^i}\mathrm{div}(E(u_k))(u_k-v_k)+\int_{\mathcal{A}_k^i}G(u_k)(u_k-v_k)+\int_{\mathcal{A}_k^i}h_k(u_k-v_k) \\ &= I_k^i+II_k^i+III_k^i. \qquad (4.17)\end{aligned}$$

By (4.8) we have

$$|II_k^i|\le C\Big(\int_{\mathcal{A}_k^i}|\nabla^2 u_k|^2+|\nabla u_k|^4\Big)\mathrm{osc}_{\mathcal{A}_k^i}u_k\le C\varepsilon\int_{\mathcal{A}_k^i}|\nabla^2 u_k|^2+|\nabla u_k|^4 \qquad (4.18)$$

and

$$|III_k^i|\le C\mathrm{osc}_{\mathcal{A}_k^i}u_k\int_{\mathcal{A}_k^i}|h_k|\le C\varepsilon\int_{\mathcal{A}_k^i}|h_k|. \qquad (4.19)$$

For I_k^i, by integration by parts we obtain

$$I_k^i=\int_{\mathcal{A}_k^i}E(u_k)\cdot\nabla(u_k-v_k)+\int_{\partial\mathcal{A}_k^i}E(u_k)(u_k-v_k)\cdot\nu, \qquad (4.20)$$

so that after summing over $0\le i\le N_k-1$ we have

$$\begin{aligned}\sum_{i=0}^{N_k-1}I_k^i &= \sum_{i=0}^{N_k-1}\int_{\mathcal{A}_k^i}E(u_k)\cdot\nabla(u_k-v_k)+\Big(\int_{\partial B_{2\delta}}-\int_{\partial B_{Rr_k}}\Big)E(u_k)(u_k-v_k)\cdot\nu \\ &\le C\sum_{i=0}^{N_k-1}\int_{\mathcal{A}_k^i}\Big(|\nabla^2 u_k||\nabla u_k|+|\nabla u_k|^3\Big)|\nabla(u_k-v_k)| \\ &+ C\Big(\int_{\partial B_{2\delta}}+\int_{\partial B_{Rr_k}}\Big)\Big(|\nabla^2 u_k||\nabla u_k|+|\nabla u_k|^3\Big)|u_k-v_k| \\ &\le IV_k+V_k. \qquad (4.21)\end{aligned}$$

By (4.5) and Hölder inequality, we see that

$$V_k \le C\varepsilon.$$

For IV_k, we proceed the estimate as follows. By Nirenberg's interpolation inequality and (4.8), we have

$$\begin{aligned}\Big(\int_{\mathcal{A}_k^i}|\nabla u_k|^4\Big)^{\frac14} &\le C\Big[u_k\Big]^{\frac12}_{\mathrm{BMO}(\mathcal{B}_k^i)}\Big(\int_{\mathcal{B}_k^i}(|\nabla^2 u_k|^2+\frac{|\nabla u_k|^2}{(2^iRr_k)^2})\Big)^{\frac14}\\ &\le C\varepsilon^{\frac12}\Big(\int_{\mathcal{B}_k^i}(|\nabla^2 u_k|^2+\frac{|\nabla u_k|^2}{(2^iRr_k)^2})\Big)^{\frac14}. \qquad (4.22)\end{aligned}$$

Since v_k is a biharmonic function on $\mathcal{A}_k^i$, we also have

$$\begin{aligned}\Big(\int_{\mathcal{A}_k^i}|\nabla(u_k-v_k)|^4\Big)^{\frac14} &\le C\Big(\int_{\mathcal{A}_k^i}|\nabla u_k|^4\Big)^{\frac14}\\ &\le C\Big[u_k\Big]^{\frac12}_{\mathrm{BMO}(\mathcal{B}_k^i)}\Big(\int_{\mathcal{B}_k^i}|\nabla^2 u_k|^2+\frac{|\nabla u_k|^2}{(2^iRr_k)^2}\Big)^{\frac14}\\ &\le C\varepsilon^{\frac12}\Big(\int_{\mathcal{B}_k^i}(|\nabla^2 u_k|^2+\frac{|\nabla u_k|^2}{(2^iRr_k)^2})\Big)^{\frac14}. \qquad (4.23)\end{aligned}$$

Therefore we have

$$\begin{aligned}IV_k &= \sum_{i=0}^{N_k-1}\int_{\mathcal{A}_k^i}|\nabla^2 u_k||\nabla u_k||\nabla(u_k-v_k)|\\ &\le C\sum_{i=0}^{N_k-1}\Big(\int_{\mathcal{A}_k^i}|\nabla^2 u_k|^2\Big)^{\frac12}\Big(\int_{\mathcal{A}_k^i}|\nabla u_k|^4\Big)^{\frac14}\Big(\int_{\mathcal{A}_k^i}|\nabla(u_k-v_k)|^4\Big)^{\frac14}\\ &\le C\varepsilon\Big(\sum_{i=0}^{N_k-1}\int_{\mathcal{A}_k^i}|\nabla^2 u_k|^2\Big)^{\frac12}\Big(\sum_{i=0}^{N_k-1}\int_{\mathcal{B}_k^i}|\nabla^2 u_k|^2+\frac{|\nabla u_k|^2}{(2^iRr_k)^2}\Big)^{\frac12}\\ &\le C\varepsilon\Big(\int_{B_{2\delta}\setminus B_{Rr_k}}|\nabla^2 u_k|^2\Big)^{\frac12}\Big(\int_{B_{4\delta}\setminus B_{\frac12 Rr_k}}|\nabla^2 u_k|^2+\frac{|\nabla u_k|^2}{|x|^2}\Big)^{\frac12}. \qquad (4.24)\end{aligned}$$

Applying Lemma 5.2 in [8], we have the following Hardy inequality:

$$\int_{B_{4\delta}\setminus B_{\frac12 Rr_k}}\frac{|\nabla u_k|^2}{|x|^2}\le\int_{B_{4\delta}\setminus B_{\frac12 Rr_k}}|\nabla^2 u_k|^2+\Big(\int_{\partial B_{4\delta}}-\int_{\partial B_{\frac12 Rr_k}}\Big)(\frac{1}{|x|}|\nabla u_k|^2). \qquad (4.25)$$

By (4.5) and Hölder inequality, we have that

$$\frac1\delta\int_{\partial B_{4\delta}}|\nabla u_k|^2\le C\Big(\delta\int_{\partial B_{4\delta}}|\nabla u_k|^4\Big)^{\frac12}\le C\Big(\int_{B_{8\delta}\setminus B_{2\delta}}|\nabla u_k|^4\Big)^{\frac12}\le C\varepsilon, \qquad (4.26)$$

and, similarly,

$$\frac{1}{Rr_k}\int_{\partial B_{\frac12 Rr_k}}|\nabla u_k|^2\le C\Big(Rr_k\int_{\partial B_{\frac12 Rr_k}}|\nabla u_k|^4\Big)^{\frac12}\le C\Big(\int_{B_{Rr_k}\setminus B_{\frac14 Rr_k}}|\nabla u_k|^4\Big)^{\frac12}\le C\varepsilon. \qquad (4.27)$$

Substituting (4.25), (4.26) and (4.27) into (4.24), we obtain

$$IV_k \leq C\varepsilon\Big(\int_{B_{4\delta}\setminus B_{\frac{1}{2}Rr_k}} |\nabla^2 u_k|^2\Big) + C\varepsilon. \tag{4.28}$$

Using the same argument to estimate the second term of IV_k, we have

$$IV_k \leq C\varepsilon\Big(\int_{B_{2\delta}\setminus B_{Rr_k}} |\nabla^2 u_k|^2\Big) + C\varepsilon. \tag{4.29}$$

Combining the estimates together yields

$$\int_{B_{2\delta}\setminus B_{Rr_k}} |\Delta(u_k - v_k)|^2 \leq C\varepsilon\Big(\int_{B_{4\delta}\setminus B_{\frac{1}{2}Rr_k}} |\nabla^2 u_k|^2 + |\nabla u_k|^4 + |h_k|\Big) + C\varepsilon. \tag{4.30}$$

This, combined with Calderon-Zygmund's $W^{2,2}$ estimate, yields

$$\int_{B_{\delta}\setminus B_{2Rr_k}} |\nabla^2(u_k - v_k)|^2 \leq C\varepsilon\Big(\int_{B_{4\delta}\setminus B_{\frac{1}{2}Rr_k}} |\nabla^2 u_k|^2 + |\nabla u_k|^4 + |h_k|\Big) + C\varepsilon. \tag{4.31}$$

Step 2. *Control of radial component of hessian energy in the neck region*:

Since v_k is radial, it is easy to see that (4.31) yields

$$\begin{aligned}\int_{B_{\delta}\setminus B_{2Rr_k}} |\nabla_T \nabla u_k|^2 \ &\leq \ C\varepsilon\Big(\int_{B_{4\delta}\setminus B_{\frac{1}{2}Rr_k}} |\nabla^2 u_k|^2 + |\nabla u_k|^4 + |h_k|\Big) + C\varepsilon \\ &\leq \ C\varepsilon. \end{aligned}\tag{4.32}$$

Here $\nabla_T \nabla u_k = \nabla^2 u_k - \frac{\partial}{\partial r}(\nabla u_k)$ denotes the tangential component of ∇u_k.

Next, we want to apply the Pohozaev type argument to $W^{4,p}$-approximate biharmonic maps u_k with L^p bi-tension field h_k for $p \geq \frac{4}{3}$ to control $\int_{B_{\delta}\setminus B_{2Rr_k}} \Big|\frac{\partial^2 u_k}{\partial r^2}\Big|^2$ by $\int_{B_{\delta}\setminus B_{2Rr_k}} |\nabla_T \nabla u_k|^2$ and $\|h_k\|_{L^p(B_{2\delta})}$. This type of argument is well-known in the blow up analysis of harmonic or approximate harmonic maps on Riemann surfaces (see [20], [14], [15], and [13]). In the context of biharmonic maps, it was first derived by Hornung-Moser [8].

By (4.2) and Lemma 2.3, we see that $u_k \in W^{4,p}(B_{2\delta} \setminus B_{\frac{1}{2}Rr_k})$. While in $B_{\frac{1}{2}Rr_k}$, since $\widetilde{u}_k(x) = u_k(r_k x) : B_R \to N$ is an approximate biharmonic map that converges to the bubble ω_1, we conclude that $\|\widetilde{u}_k\|_{W^{4,p}(B_R)} < +\infty$ and hence $u_k \in W^{4,p}(B_{\frac{1}{2}Rr_k})$ by scaling. From this, we then see $u_k \in W^{4,p}(B_{2\delta})$. Since $x \cdot \nabla u_k \in L^4(B_{2\delta})$ and $p \geq \frac{4}{3}$, we see that $\Delta^2 u_k \cdot (x \cdot \nabla u_k) \in L^1(B_{2\delta})$ and $h_k \cdot (x \cdot \nabla u_k) \in L^1(B_{2\delta})$. Since the equation (1.1) implies that $(\Delta^2 u_k - h_k)(x) \perp T_{u_k(x)}N$ a.e. $x \in B_{2\delta}$. Note also that $x \cdot \nabla u_k(x) \in T_{u_k(x)}N$ for a.e. $x \in B_{2\delta}$. Multiplying the equation (1.1) by $x \cdot \nabla u_k$ and integrating over B_r for any $0 < r \leq 2\delta$, we have

$$\int_{B_r} \Delta^2 u_k \cdot (x \cdot \nabla u_k) = \int_{B_r} h_k \cdot (x \cdot \nabla u_k). \tag{4.33}$$

Applying Green's identity, we have

$$\begin{aligned}&\int_{B_r} \Delta^2 u_k \cdot (x \cdot \nabla u_k) \\ = \ &\int_{B_r} \Delta u_k \Delta(x \cdot \nabla u_k) + r\int_{\partial B_r} \frac{\partial}{\partial r}(\Delta u_k)\frac{\partial u_k}{\partial r} - \int_{\partial B_r} \Delta u_k\Big(\frac{\partial u_k}{\partial r} + r\frac{\partial^2 u_k}{\partial r^2}\Big). \end{aligned}\tag{4.34}$$

Direct calculations yield

$$\begin{aligned}\int_{B_r}\Delta u_k\Delta(x\cdot\nabla u_k) &= \int_{B_r} x\cdot\nabla\Big(\frac{|\Delta u_k|^2}{2}\Big)+2|\Delta u_k|^2\\ &= \int_{B_r}\operatorname{div}\Big(\frac{|\Delta u_k|^2}{2}x\Big)=r\int_{\partial B_r}\frac{|\Delta u_k|^2}{2}. \end{aligned}\tag{4.35}$$

Putting (4.35), (4.34), and (4.33) together yields

$$r\int_{\partial B_r}\frac{|\Delta u_k|^2}{2}+r\int_{\partial B_r}\frac{\partial}{\partial r}(\Delta u_k)\frac{\partial u_k}{\partial r}-\int_{\partial B_r}\Delta u_k\big(\frac{\partial u_k}{\partial r}+r\frac{\partial^2 u_k}{\partial r^2}\big)=\int_{B_r}h_k\cdot(x\cdot\nabla u_k). \tag{4.36}$$

Applying integration by parts multi-times to (4.36) in the same way as [3] or Angelsberg [2], we can obtain that for a.e. $0<r\le 2\delta$,

$$\begin{aligned}\int_{\partial B_r}|\Delta u_k|^2 &= 4\int_{\partial B_r}\Big(\frac{|u_{k,\alpha}+x^\beta u_{k,\alpha\beta}|^2}{r^2}+2\frac{|x\cdot\nabla u_k|^2}{r^4}\Big)\\ &+ 2\frac{d}{dr}\int_{\partial B_r}\Big(-\frac{x^\alpha u_{k,\beta}u_{k,\alpha\beta}}{r}+2\frac{|x\cdot\nabla u_k|^2}{r^3}-2\frac{|\nabla u_k|^2}{r}\Big)\\ &+ \frac{1}{r}\int_{B_r}h_k\cdot(x\cdot\nabla u_k). \end{aligned}\tag{4.37}$$

Recall that in the spherical coordinate, we have

$$\Delta u_k=u_{k,rr}+\frac{3}{r}u_{k,r}+\frac{1}{r^2}\Delta_{S^3}u_k,$$

where Δ_{S^3} denotes the Laplace operator on the standard three sphere S^3. Hence we have

$$\begin{aligned}\int_{\partial B_r}|\Delta u_k|^2 &= \int_{\partial B_r}\Big[|u_{k,rr}|^2+\frac{9}{r^2}|u_{k,r}|^2+\frac{6}{r}u_{k,r}u_{k,rr}\Big]\\ &+ \int_{\partial B_r}\Big[\frac{1}{r^4}|\Delta_{S^3}u_k|^2+(u_{k,rr}+\frac{3}{r}u_{k,r})\cdot(\frac{2}{r^2}\Delta_{S^3}u_k)\Big]. \end{aligned}\tag{4.38}$$

On the other hand, we have

$$\begin{aligned}&4\int_{\partial B_r}\Big(\frac{|u_{k,\alpha}+x^\beta u_{k,\alpha\beta}|^2}{r^2}+2\frac{|x\cdot\nabla u_k|^2}{r^4}\Big)\\ =& \int_{\partial B_r}\Big[\frac{12}{r^2}|u_{k,r}|^2+4|u_{k,rr}|^2+\frac{4}{r^2}|\nabla_{S^3}u_{k,r}|^2+\frac{8}{r}u_{k,r}u_{k,rr}\Big]. \end{aligned}\tag{4.39}$$

Substituting (4.38) and (4.39) into (4.37) and integrating over $r\in[Rr_k,\delta]$, we obtain

$$\begin{aligned}&\int_{B_\delta\setminus B_{Rr_k}}\Big[3|u_{k,rr}|^2+\frac{3}{r^2}|u_{k,r}|^2+\frac{2}{r}u_{k,r}u_{k,rr}\Big]\\ \le& \int_{B_\delta\setminus B_{Rr_k}}\Big[\frac{1}{r^4}|\Delta_{S^3}u_k|^2+(u_{k,rr}+\frac{3}{r}u_{k,r})\cdot(\frac{2}{r^2}\Delta_{S^3}u_k)\Big]+\int_{Rr_k}^{\delta}\int_{B_r}|h_k||u_{k,r}|\\ +& 2\Big(\int_{\partial B_\delta}-\int_{\partial B_{Rr_k}}\Big)\Big(\frac{x^\alpha u_{k,\beta}u_{k,\alpha\beta}}{r}-2\frac{|x\cdot\nabla u_k|^2}{r^3}+2\frac{|\nabla u_k|^2}{r}\Big)\\ =& I_k+II_k+III_k. \end{aligned}\tag{4.40}$$

By Hölder inequality, (4.32), (4.25), (4.26), and (4.27), we have

$$|I_k| \leq \int_{B_\delta \setminus B_{Rr_k}} |\nabla_T \nabla u_k|^2 + \Big(\int_{B_\delta \setminus B_{Rr_k}} |\nabla^2 u_k|^2\Big)^{\frac{1}{2}} \Big(\int_{B_\delta \setminus B_{Rr_k}} |\nabla_T \nabla u_k|^2\Big)^{\frac{1}{2}} + C\varepsilon \leq C\varepsilon.$$

For II_k we have

$$|II_k| \leq \delta \|h_k\|_{L^p(B_\delta)} \|\nabla u_k\|_{L^{\frac{p}{p-1}}(B_\delta)} \leq C\delta \|\nabla u_k\|_{L^4(B_\delta)} \leq C\delta,$$

where we have used the fact $p \geq \frac{4}{3}$ so that $\frac{p}{p-1} \leq 4$.

We use (4.5) to estimate III_k as follows. First we have

$$\begin{aligned} & \Big|\int_{\partial B_\delta} \Big(\frac{x^\alpha u_{k,\beta} u_{k,\alpha\beta}}{r} - 2\frac{|x \cdot \nabla u_k|^2}{r^3} + 2\frac{|\nabla u_k|^2}{r}\Big)\Big| \\ \leq\ & C\Big[\int_{\partial B_\delta} |\nabla u_k||\nabla^2 u_k| + \delta^{-1} \int_{\partial B_\delta} |\nabla u_k|^2\Big] \\ \leq\ & C\Big(\delta \int_{\partial B_\delta} |\nabla u_k|^4\Big)^{\frac{1}{4}} \Big(\delta \int_{\partial B_\delta} |\nabla^2 u_k|^2\Big)^{\frac{1}{2}} + C\Big(\delta \int_{\partial B_\delta} |\nabla u_k|^4\Big)^{\frac{1}{2}} \\ \leq\ & C\Big[\|\nabla u_k\|_{L^4(B_{2\delta} \setminus B_{\frac{\delta}{2}})} \|\nabla^2 u_k\|_{L^2(B_{2\delta} \setminus B_{\frac{\delta}{2}})} + \|\nabla u_k\|^2_{L^4(B_{2\delta} \setminus B_{\frac{\delta}{2}})}\Big] \leq C\varepsilon. \end{aligned}$$

Similarly, by (4.5) we have

$$\begin{aligned} & \Big|\int_{\partial B_{Rr_k}} \Big(\frac{x^\alpha u_{k,\beta} u_{k,\alpha\beta}}{r} - 2\frac{|x \cdot \nabla u_k|^2}{r^3} + 2\frac{|\nabla u_k|^2}{r}\Big)\Big| \\ \leq\ & C\Big[\int_{\partial B_{Rr_k}} |\nabla u_k||\nabla^2 u_k| + (Rr_k)^{-1} \int_{\partial B_{Rr_k}} |\nabla u_k|^2\Big] \\ \leq\ & C\Big(Rr_k \int_{\partial B_{Rr_k}} |\nabla u_k|^4\Big)^{\frac{1}{4}} \Big(Rr_k \int_{\partial B_{Rr_k}} |\nabla^2 u_k|^2\Big)^{\frac{1}{2}} + C\Big(Rr_k \int_{\partial B_{Rr_k}} |\nabla u_k|^4\Big)^{\frac{1}{2}} \\ \leq\ & C\Big[\|\nabla u_k\|_{L^4(B_{2Rr_k} \setminus B_{\frac{1}{2}Rr_k})} \|\nabla^2 u_k\|_{L^2(B_{2Rr_k} \setminus B_{\frac{1}{2}Rr_k})} + \|\nabla u_k\|^2_{L^4(B_{2Rr_k} \setminus B_{\frac{1}{2}Rr_k})}\Big] \leq C\varepsilon. \end{aligned}$$

Therefore, by putting these estimates together, we have

$$\int_{B_\delta \setminus B_{Rr_k}} \Big[3|u_{k,rr}|^2 + \frac{3}{r^2}|u_{k,r}|^2 + \frac{2}{r} u_{k,r} u_{k,rr}\Big] \leq C(\varepsilon + \delta). \tag{4.41}$$

Competition of Proof of Theorem 1.2:

Since $\dfrac{2}{r} u_{k,r} u_{k,rr} \geq -(|u_{k,rr}|^2 + \dfrac{1}{r^2}|u_{k,r}|^2)$, (4.41) implies

$$\int_{B_\delta \setminus B_{Rr_k}} |u_{k,rr}|^2 \leq C(\varepsilon + \delta),$$

this, combined with (4.32), implies

$$\int_{B_\delta \setminus B_{Rr_k}} |\nabla^2 u_k|^2 \leq C(\varepsilon + \delta).$$

Thus there is no concentration of hessian energy in the neck region. It is well known that this yields the energy identity (1.10). To show (1.11), observe that Nirenberg's interpolation inequality and (1.10) imply

$$\begin{aligned}\|\nabla u_k\|^2_{L^4(B_\delta\setminus B_{Rr_k})} &\le C\|\nabla u_k\|_{L^\infty(B_{2\delta})}\Big(\|\nabla u_k\|_{L^2(B_{2\delta})}+\|\nabla^2 u_k\|_{L^2(B_{2\delta}\setminus B_{\frac12 Rr_k})}\Big)\\ &\le C(\epsilon+\delta+o(1))\end{aligned}$$

where we have used that $\|\nabla u_k\|_{L^2(B_{2\delta})}=\|\nabla u\|_{L^2(B_{2\delta})}+o(1)=o(1)$. Thus (1.11) also holds. □

Proof of Corollary 1.3:

It follows from the energy inequality (1.8) that there exists $t_k\uparrow+\infty$ such that $u_k(\cdot)=u(\cdot,t_k)$ is an approximate biharmonic map into N with bi-tension field $h_k=u_t(\cdot,t_k)\in L^2(\Omega)$ satisfying

$$\Big\|h_k\Big\|_{L^2(\Omega)}=\Big\|u_t(\cdot,t_k)\Big\|_{L^2(\Omega)}\to 0.$$

Moreover,

$$\Big\|u_k\Big\|_{W^{2,2}(\Omega)}\le C\Big\|u_0\Big\|_{W^{2,2}(\Omega)}.$$

Therefore we may assume that after taking another subsequence, $u_k\rightharpoonup u_\infty$ in $W^{2,2}(\Omega,N)$. It is easy to see that u_∞ is a biharmonic map so that $u_\infty\in C^\infty(\Omega,N)$ (see [23]). All other conclusions follow directly from Theorem 1.2. □

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