Blow up criterion for nematic liquid crystal flows

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Abstract

In this paper, we establish a blow up criterion for the short time classical solution of the nematic liquid crystal flow, a simplified version of Ericksen-Leslie system modeling the hydrodynamic evolution of nematic liquid crystals, in dimensions two and three. More precisely, $0 < T_* < +\infty$ is the maximal time interval iff (i) for $n = 3$, $|\omega|+|\nabla d|^2 \notin L^1_t L^\infty_x(\mathbb{R}^3 \times [0, T_*])$; and (ii) for $n = 2$, $|\nabla d|^2 \notin L^1_t L^\infty_x(\mathbb{R}^2 \times [0, T_*])$.

1 Introduction

In this paper, we consider the Cauchy problem to the following hydrodynamic flow of nematic liquid crystals in $\mathbb{R}^n$ ($n = 2$ or 3):

$$u_t + u \cdot \nabla u - \nu \triangle u + \nabla p = -\Delta d \cdot \nabla d, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$d_t + u \cdot \nabla d = \triangle d + |\nabla d|^2 d, \tag{1.3}$$

$$(u, d)\big|_{t=0} = (u_0, d_0), \tag{1.4}$$

where $u : \mathbb{R}^n \to \mathbb{R}^n$ represents the velocity field of the incompressible viscous fluid, $\nu > 0$ is the Kinematic viscosity, $p : \mathbb{R}^3 \to \mathbb{R}$ represents the pressure function, $d : \mathbb{R}^n \to \mathbb{S}^2$ represents the macroscopic average of the nematic liquid crystal orientation field, $\nabla\cdot$ and Δ denotes the divergence operator and the Laplace operator respectively, $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a given initial velocity field with $\nabla \cdot u_0 = 0$, and $d_0 : \mathbb{R}^n \to S^2$ is a given initial liquid crystal orientation field.

The system (1.1)-(1.3) is a simplified version of the Ericksen-Leslie system modeling the hydrodynamics of nematic liquid crystals developed during the period of 1958 through 1968 ([3] [2] [11]). It is a macroscopic continuum description of the time evolution of the material under the influence of both the flow field $u(x, t)$, and the macroscopic description of the microscopic orientation configurations $d(x, t)$ of rod-like liquid crystals. Recall that the Ericksen-Leslie theory reduces to the Ossen-Frank theory in the static case, see Hardt-Lin-Kinderlehrer [4] and references therein. The system (1.1)-(1.3) was first introduced by Lin and Liu in their important works [6, 7] during the 1990's. Roughly speaking, (1.1)-(1.3) is a system that couples between the non-homogeneous Navier-Stokes equation and the transported heat flow of harmonic maps into S^2. For dimension $n = 2$, Lin-Lin-Wang [9] have proved the global existence of Leray-Hopf type weak solutions to (1.1)-(1.3) on bounded domains in $\mathbb{R}^2$ under the initial and boundary value conditions (see [5] for the

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case $\Omega = \mathbb{R}^2$), and Lin-Wang [8] have further established the uniqueness for such weak solutions. It is an interesting and challenging problem to study the nematic liquid crystal flow equation (1.1)-(1.3) in dimension three, such as the global existence of weak solutions and the partial regularity of suitable weak solutions.

In this paper, we will consider the short time classical solution to (1.1) -(1.4) and address some criterion that characterizes the first finite singular time. It is well-known that if the initial velocity $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, S^2)$ for $s \geq n$, then there is $T_0 > 0$ depending only on $\|u_0\|_{H^s}$ and $\|d_0\|_{H^{s+1}}$ such that (1.1)-(1.4) has a unique, classical solution (u, d) in $\mathbb{R}^n \times [0, T_0)$ satisfying

$$\begin{aligned} &u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^n)) \text{ and} \\ &d \in C([0,T], H^{s+1}(\mathbb{R}^n, S^2)) \cap C^1([0,T], H^s(\mathbb{R}^n, S^2)), \end{aligned} \tag{1.5}$$

for any $0 < T < T_0$. Assume $T_* > 0$ is the maximum value such that (1.5) holds with $T_0 = T_*$. We would like to characterize such a T_*.

Recall that when d is a constant vector, (1.1)-(1.4) becomes the Navier-Stokes equation. In their famous work [1], Beale-Kato-Majda proved that for $n = 3$, if $T_* > 0$ is the first finite singular time, then the vorticity $\omega = \nabla \times u$ doesn't belong to $L^1_t L^\infty_x(\mathbb{R}^3 \times [0, T_*))$. On the other hand, when $u = 0$, (1.1)-(1.4) becomes the heat flow of harmonic maps into S^2, Wang proved in [12] that for $n \geq 2$, if $\nabla d \in L^\infty_t L^n_x(\mathbb{R}^n \times [0,T])$, then $d \in C^\infty(\mathbb{R}^n \times (0,T])$. Our main result on (1.1)-(1.4) is a natural extension of [1] and [12].

Theorem 1.1 *For $n = 3$, $s \geq 3$, $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, S^2)$, let $T_* > 0$ be the maximum value such that (1.1)-(1.4) has a unique solution (u, d) satisfying (1.5) with T_0 replaced by T_*. If $T_* < +\infty$, then*

$$\int_0^{T_*} \left(\|\omega(t)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla d(t)\|^2_{L^\infty(\mathbb{R}^3)} \right) dt = \infty, \tag{1.6}$$

where $\omega = \nabla \times u$ is the vorticity. In particular,

$$\limsup_{t \nearrow T_*} \left(\|\omega(t)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla d(t)\|^2_{L^\infty(\mathbb{R}^3)} \right) = \infty. \tag{1.7}$$

As a byproduct of the proof of theorem 1.1 and the regularity theorem by [9], we obtain a corresponding criterion in dimension $n = 2$. More precisely, we have

Corollary 1.2 *For $n = 2$, $s \geq 2$, $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, S^2)$, let $T_* > 0$ be the maximum value such that (1.1)-(1.4) has a unique solution (u, d) satisfying (1.5) with T_0 replaced by T_*. If $T_* < +\infty$, then*

$$\int_0^{T_*} \|\nabla d(t)\|^2_{L^\infty(\mathbb{R}^2)} \, dt = \infty. \tag{1.8}$$

In particular,

$$\limsup_{t \nearrow T_*} \|\nabla d(t)\|_{L^\infty(\mathbb{R}^2)} = \infty. \tag{1.9}$$

2 Proof of Theorem 1.1

For simplicity, we assume $\nu = 1$. We need the following lemma to prove theorem 1.1.

Lemma 2.1 *For $n = 2$ or 3, $s \geq n$, $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, S^2)$, $M > 0$, and $T_0 > 0$, let (u, d) be a solution to (1.1)-(1.4) satisfying (1.5) and*

$$\begin{cases} \int_0^{T_0} \left(\|\omega(t)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d(t)\|^2_{L^\infty(\mathbb{R}^n)} \right) dt \leq M & \text{for } n = 3, \\ \text{or} \qquad \int_0^{T_0} \|\nabla d(t)\|^2_{L^\infty(\mathbb{R}^n)} dt \leq M & \text{for } n = 2. \end{cases} \tag{2.1}$$

Then

$$\sup_{0 \leq t \leq T_0} \left(\|\omega(t)\|_{L^2(\mathbb{R}^n)} + \|\nabla^2 d(t)\|_{L^2(\mathbb{R}^n)} \right) \leq C, \tag{2.2}$$

where $C > 0$ depends only on u_0, d_0 and M.

Proof. Taking $\nabla \times$ on (1.1), we obtain

$$\omega_t - \triangle \omega + u \cdot \nabla \omega = \begin{cases} \omega \cdot \nabla u - \nabla \times (\triangle d \cdot \nabla d) & \text{if } n = 3, \\ -\nabla \times (\triangle d \cdot \nabla d) & \text{if } n = 2 \end{cases} \tag{2.3}$$

Multiplying ω and integrating over $\mathbb{R}^n$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\omega|^2 \, dx + \int_{\mathbb{R}^n} |\nabla \omega|^2 \, dx = \begin{cases} \int_{\mathbb{R}^n} [(\omega \cdot \nabla) u \cdot \omega + (\triangle d \cdot \nabla d) \cdot (\nabla \times) \omega] \, dx & \text{for } n = 3, \\ \int_{\mathbb{R}^n} (\triangle d \cdot \nabla d) \cdot (\nabla \times \omega) \, dx & \text{for } n = 2. \end{cases} \tag{2.4}$$

where we have used the fact

$$\int_{\mathbb{R}^n} (u \cdot \nabla) \omega \cdot \omega \, dx = \frac{1}{2} \int_{\mathbb{R}^n} (u \cdot \nabla) |\omega|^2 \, dx = 0.$$

Since

$$\nabla u = (-\triangle)^{-1} \nabla (\nabla \times \omega),$$

we have $\|\nabla u\|_{L^2} \leq C \|\omega\|_{L^2}$ and

$$\left| \int_{\mathbb{R}^n} (\omega \cdot \nabla) u \cdot \omega \, dx \right| \leq C \|\omega\|_{L^\infty} \|\omega\|^2_{L^2}. \tag{2.5}$$

By Young's inequality, we obtain

$$\left| \int_{\mathbb{R}^n} (\triangle d \cdot \nabla d) \cdot (\nabla \times \omega) \, dx \right| \leq C \int_{\mathbb{R}^n} |\triangle d|^2 |\nabla d|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \omega|^2 \, dx. \tag{2.6}$$

Combining (2.4), (2.5), with (2.6) , we have

$$\frac{d}{dt} \|\omega\|^2_{L^2} + \|\nabla \omega\|^2_{L^2} \leq \begin{cases} C \|\omega\|_{L^\infty} \|\omega\|^2_{L^2} + C \int_{\mathbb{R}^n} |\triangle d|^2 |\nabla d|^2 \, dx & \text{for } n = 3, \\ C \int_{\mathbb{R}^n} |\triangle d|^2 |\nabla d|^2 \, dx & \text{for } n = 2. \end{cases} \tag{2.7}$$

Taking $\triangle$ on (1.3), multiplying $\triangle d$ and integrating over $\mathbb{R}^n$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\triangle d\|^2_{L^2} + \|\nabla \triangle d\|^2_{L^2} = \int_{\mathbb{R}^n} \triangle (|\nabla d|^2 d) \cdot \triangle d \, dx - \int_{\mathbb{R}^n} \triangle (u \cdot \nabla d) \cdot \triangle d \, dx. \tag{2.8}$$

Since

$$\int_{\mathbb{R}^n} (u \cdot \nabla \triangle d) \cdot \triangle d \, dx = \frac{1}{2} \int_{\mathbb{R}^n} (u \cdot \nabla)(|\triangle d|^2) \, dx = 0,$$

and

$$\nabla \times \omega = \nabla \times (\nabla \times u) = \nabla(\nabla \cdot u) - \triangle u = -\triangle u,$$

we obtain,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \triangle(u \cdot \nabla d) \cdot \triangle d \, dx \right| &\leq \int_{\mathbb{R}^n} |\triangle u||\nabla d||\triangle d| \, dx + 2 \int_{\mathbb{R}^n} |\nabla u||\nabla^2 d||\triangle d| \, dx \\ &\leq \int_{\mathbb{R}^n} |\nabla \omega||\nabla d||\triangle d| \, dx + 2 \int_{\mathbb{R}^n} |\nabla u||\nabla^2 d||\triangle d| \, dx \\ &= I_1 + I_2. \end{aligned} \tag{2.9}$$

$$I_1 \leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\triangle d\|_{L^2}^2. \tag{2.10}$$

$$\begin{aligned} I_2 &\leq C \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^4}^2 \\ &\leq C \|\omega\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla^3 d\|_{L^2} \\ &\leq C \|\omega\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla \triangle d\|_{L^2} \\ &\leq \frac{1}{4} \|\nabla \triangle d\|_{L^2}^2 + C \|\omega\|_{L^2}^2 \|\nabla d\|_{L^\infty}^2, \end{aligned} \tag{2.11}$$

where we have used Nirenberg's interpolation inequality: for nonegative integers k and l with $k \leq l - 1$,

$$\|\nabla^k f\|_{L^{\frac{2l}{k}}}^{\frac{2l}{k}} \leq C \|f\|_{L^\infty}^{\frac{2l}{k}-2} \|\nabla^l f\|_{L^2}^2.$$

Combining (2.9), (2.10), with (2.11), we have

$$\left| \int_{\mathbb{R}^n} \triangle(u \cdot \nabla d) \cdot \triangle d \, dx \right| \leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + \frac{1}{4} \|\nabla \triangle d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 (\|\triangle d\|_{L^2}^2 + \|\omega\|_{L^2}^2). \tag{2.12}$$

Now we need to estimate the first term in the right hand side of (2.8).

$$\begin{aligned} &\int_{\mathbb{R}^n} \triangle(|\nabla d|^2 d) \cdot \triangle d \, dx \\ =&\int_{\mathbb{R}^n} \triangle(|\nabla d|^2) d \cdot \triangle d \, dx + \int_{\mathbb{R}^n} |\nabla d|^2 |\triangle d|^2 \, dx + \int_{\mathbb{R}^n} 2\nabla |\nabla d|^2 \cdot \nabla d \triangle d \, dx \\ =&I_3 + I_4 + I_5. \end{aligned} \tag{2.13}$$

By integration by parts, we obtain

$$\begin{aligned} |I_3| &= \left| - \int_{\mathbb{R}^n} (\nabla(|\nabla d|^2) \nabla d \cdot \triangle d + \nabla(|\nabla d|^2) d \cdot \nabla \triangle d) \, dx \right| \\ &\leq 2 \int_{\mathbb{R}^n} (|\nabla d|^2 |\nabla^2 d||\triangle d| + |\nabla d||\nabla^2 d||\nabla \triangle d|) \, dx \\ &\leq C \|\nabla d\|_{L^\infty}^2 \|\triangle d\|_{L^2}^2 + \frac{1}{4} \|\nabla \triangle d\|_{L^2}^2. \end{aligned} \tag{2.14}$$

$$|I_4| \leq \|\nabla d\|_{L^\infty}^2 \|\triangle d\|_{L^2}^2. \tag{2.15}$$

$$
\begin{aligned}
|I_5| &\le 4\int_{\mathbb{R}^n} |\nabla d|^2|\nabla^2 d||\triangle d|\,dx \\
&\le C\|\nabla d\|_{L^\infty}^2\|\nabla^2 d\|_{L^2}\|\triangle d\|_{L^2} \\
&\le C\|\nabla d\|_{L^\infty}^2\|\triangle d\|_{L^2}^2.
\end{aligned}
\tag{2.16}
$$

Combining (2.13), (2.14), (2.15), with (2.16), we have

$$
\int_{\mathbb{R}^3} \triangle(|\nabla d|^2 d)\cdot\triangle d\,dx \le C\|\nabla d\|_{L^\infty}^2\|\triangle d\|_{L^2}^2 + \frac{1}{4}\|\nabla\triangle d\|_{L^2}^2. \tag{2.17}
$$

Combining (2.8), (2.12) and (2.17), we obtain

$$
\frac{d}{dt}\|\triangle d\|_{L^2}^2 + \|\nabla\triangle d\|_{L^2}^2 \le \frac{1}{2}\|\nabla\omega\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2(\|\triangle d\|_{L^2}^2 + \|\omega\|_{L^2}^2). \tag{2.18}
$$

Adding (2.7) and (2.18) together, we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\triangle d\|_{L^2}^2) + \frac{1}{2}\|\nabla\omega\|_{L^2}^2 + \|\nabla\triangle d\|_{L^2}^2 \\
&\le \begin{cases} C\left(\|\omega\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right)\left(\|\triangle d\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) & \text{for } n=3 \\ C\|\nabla d\|_{L^\infty}^2\left(\|\triangle d\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) & \text{for } n=2 \end{cases}
\end{aligned}
\tag{2.19}
$$

Then by Gronwall's inequality,

$$
\begin{aligned}
&\|\omega(T_0)\|_{L^2}^2 + \|\triangle d(T_0)\|_{L^2}^2 \\
&\le \begin{cases} \left(\|\triangle d_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2\right)\exp\left(C\int_0^{T_0}\left(\|\omega(t)\|_{L^\infty} + \|\nabla d(t)\|_{L^\infty}^2\right)dt\right) & \text{for } n=3 \\ \left(\|\triangle d_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2\right)\exp\left(C\int_0^{T_0}\|\nabla d(t)\|_{L^\infty}^2\,dt\right) & \text{for } n=2. \end{cases}
\end{aligned}
\tag{2.20}
$$

Since

$$
\int_{\mathbb{R}^n} |\Delta d|^2\,dx = \int_{\mathbb{R}^n} |\nabla^2 d|^2\,dx,
$$

this yields the conclusion and hence completes the proof of lemma 2.1. □

Proof of Theorem 1.1: We prove the theorem by contradiction. Assume that (1.6) were not true. Then there is $0 < M < \infty$ such that

$$
\int_0^{T_*}\left(\|\omega(t)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla d(t)\|_{L^\infty(\mathbb{R}^3)}^2\right)dt \le M. \tag{2.21}
$$

Then by lemma 2.1, we have

$$
\sup_{0\le t\le T_*}\left(\|\omega(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla^2 d(t)\|_{L^2(\mathbb{R}^3)}\right) < C, \tag{2.22}
$$

where $C > 0$ depends on u_0, d_0 and M.

If we could control $\|u(t)\|_{H^3} + \|\nabla d(t)\|_{H^3}$ for any $0 \le t \le T_*$ in terms of u_0, d_0 and M, we would reach a contradiction. To do this, we need higher order energy estimates, which can be done as follows.

For any multi-index s with $|s|=3$, taking D^s on (1.1), multiplying $D^s u$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\frac{|D^s u|^2}{2}\,dx+\int_{\mathbb{R}^3}|D^{s+1}u|^2\,dx\\
=&-\int_{R^3}[D^s(u\cdot\nabla u)-u\cdot\nabla D^s u]\cdot D^s u\,dx\\
&-\int_{\mathbb{R}^3}D^s(\triangle d\cdot\nabla d)\cdot D^s u\,dx\\
=&J_1+J_2.
\end{aligned}\tag{2.23}$$

For J_1, we need to use the following estimate (see [10])

$$\|D^s(fg)-fD^s g\|_{L^2}\le C(\|f\|_{H^3}\|g\|_{L^\infty}+\|\nabla f\|_{L^\infty}\|g\|_{H^2}).$$

Setting $f=u$ and $g=\nabla u$, we have

$$\begin{aligned}
|J_1|&\le C\|D^s(u\cdot\nabla u)-u\cdot\nabla D^s u\|_{L^2}\|D^s u\|_{L^2}^2\\
&\le C\|\nabla u\|_{L^\infty}\|u\|_{H^3}^2.
\end{aligned}\tag{2.24}$$

Applying the Leibniz's rule and Nirenberg's interpolation inequality, we have

$$\begin{aligned}
J_2&=\int_{\mathbb{R}^3}D^{s-1}(\triangle d\cdot\nabla d)\cdot D^{s+1}u\,dx\\
&\le\frac{1}{2}\|D^{s+1}u\|_{L^2}^2+C\int_{\mathbb{R}^3}|D^{s-1}(\triangle d\cdot\nabla d)|^2\,dx\\
&\le\frac{1}{2}\|D^{s+1}u\|_{L^2}^2+C\int_{\mathbb{R}^3}(|\nabla^4 d|^2|\nabla d|^2+|\nabla^2 d|^2|\nabla^3 d|^2)\,dx\\
&\le\frac{1}{2}\|D^{s+1}u\|_{L^2}^2+C\|\nabla d\|_{L^\infty}^2\|\nabla d\|_{H^3}^2+C\|\nabla^2 d\|_{L^6}^2\|\nabla^3 d\|_{L^3}^2\\
&\le\frac{1}{2}\|D^{s+1}u\|_{L^2}^2+C\|\nabla d\|_{L^\infty}^2\|\nabla d\|_{H^3}^2.
\end{aligned}\tag{2.25}$$

Combining (2.23), (2.24), with (2.25), we have

$$\frac{d}{dt}\|D^s u\|_{L^2}^2+\|D^{s+1}u\|_{L^2}^2\le C\left(\|\nabla u\|_{L^\infty}\|u\|_{H^3}^2+\|\nabla d\|_{L^\infty}^2\|\nabla d\|_{H^3}^2\right).\tag{2.26}$$

Taking D^{s+1} on (1.3), multiplying $D^{s+1}d$ and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\frac{|D^{s+1}d|^2}{2}\,dx+\int_{\mathbb{R}^3}|D^{s+2}d|^2\,dx\\
=&-\int_{R^3}\left(D^{s+1}(u\cdot\nabla d)-u\cdot\nabla D^{s+1}d\right)D^{s+1}d\,dx\\
&+\int_{\mathbb{R}^3}D^{s+1}(|\nabla d|^2 d)\cdot D^{s+1}d\,dx\\
=&J_3+J_4.
\end{aligned}\tag{2.27}$$

For J_3, similar as the proof of (2.24), we have

$$\begin{aligned}
J_3 &\leq C\|D^{s+1}(u\cdot\nabla d) - u\cdot\nabla D^{s+1}d\|_{L^2}\|D^{s+1}d\|_{L^2}\\
&\leq C\|\nabla d\|_{L^\infty}\|u\|_{H^4}\|D^{s+1}d\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla d\|_{H^3}\|D^{s+1}d\|_{L^2}\\
&\leq \frac{\epsilon}{2}\|u\|_{H^4}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d\|_{H^3}^2 + \|\nabla u\|_{L^\infty}\|\nabla d\|_{H^3}^2,
\end{aligned} \tag{2.28}$$

where ϵ will be chosen below. Since $|\nabla d|^2 + d\cdot\triangle d = 0$, we have

$$\begin{aligned}
J_4 =& \int_{\mathbb{R}^3}\left(-D^s(|\nabla d|^2)d\cdot D^{s+2}d - D^s(|\nabla d|^2)Dd\cdot D^{s+1}d\right)\,dx\\
&+\int_{\mathbb{R}^3}\left(D^s(|\nabla d|^2)Dd\cdot D^{s+1}d + D^{s-1}(|\nabla d|^2)D^2d\cdot D^{s+1}d\right)\,dx\\
&+\int_{\mathbb{R}^3}\left(D(|\nabla d|^2)D^sd\cdot D^{s+1}d + |\nabla d|^2|D^{s+1}d|^2\right)\,dx\\
\leq& C\int_{\mathbb{R}^3}\left(|\nabla^4 d||\nabla d| + |\nabla^2 d||\nabla^3 d|\right)|D^{s+2}d|\,dx\\
&+C\int_{\mathbb{R}^3}\left(|\nabla^2 d|^3|\nabla^4 d| + |\nabla d||\nabla^2 d||\nabla^3 d||\nabla^4 d| + |\nabla d|^2|D^{s+1}d|^2\right)\,dx\\
\leq& \frac{1}{2}\|D^{s+2}d\|_{L^2}^2 + C\int_{\mathbb{R}^3}(|\nabla^4 d|^2|\nabla d|^2 + |\nabla^2 d|^2|\nabla^3 d|^2)\,dx + \|\nabla^2 d\|_{L^6}^3\|\nabla^4 d\|_{L^2}\\
\leq& \frac{1}{2}\|D^{s+2}d\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d\|_{H^3}^2.
\end{aligned} \tag{2.29}$$

Combining (2.27), (2.28), with (2.29), we have

$$\frac{d}{dt}\|D^{s+1}d\|_{L^2}^2 + \|D^{s+2}d\|_{L^2}^2 \leq \epsilon\|u\|_{H^4}^2 + C\left(\|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right)\left(\|u\|_{H^3}^2 + \|\nabla d\|_{H^3}^2\right). \tag{2.30}$$

Combining (2.26) with (2.30), we have

$$\begin{aligned}
&\frac{d}{dt}(\|D^{s+1}d\|_{L^2}^2 + \|D^s u\|_{L^2}^2) + \|D^{s+2}d\|_{L^2}^2 + \|D^{s+1}u\|_{L^2}^2\\
\leq&\epsilon\|u\|_{H^4}^2 + C\left(\|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right)\left(\|u\|_{H^3}^2 + \|\nabla d\|_{H^3}^2\right).
\end{aligned}$$

We can prove similar inequalities for all $|s| < 3$. Summing over all s with $|s| \leq 3$, and taking ϵ small enough, we have

$$\frac{d}{dt}(\|\nabla d\|_{H^3}^2 + \|u\|_{H^3}^2) \leq C\left(\|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right)\left(\|u\|_{H^3}^2 + \|\nabla d\|_{H^3}^2\right). \tag{2.31}$$

We now end our argument as follows. Set

$$m(t) = e + \|u\|_{H^3} + \|\nabla d\|_{H^3}.$$

Then by (2.31), we have

$$\frac{dm(t)}{dt} \leq C\left(\|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right)m(t).$$

By Gronwall's inequality, we obtain,

$$m(t) \leq m(0)\exp\left(C\int_0^t(\|\nabla u(t)\|_{L^\infty} + \|\nabla d(t)\|_{L^\infty}^2)dt\right). \tag{2.32}$$

By combining the following critical Sobolev embedding inequality (see [1] for the detail)

$$\|\nabla u\|_{L^\infty} \le C\left(1+\|\omega\|_{L^2}+\|\omega\|_{L^\infty}\ln(e+\|u\|_{H^3})\right),$$

with (2.22), we have

$$\|\nabla u\|_{L^\infty} \le C\left(1+\|\omega\|_{L^\infty}\ln(e+\|u\|_{H^3})\right).$$

Combining this inequality with (2.32) and the ineqaulity $\ln(m(t)) \ge 1$, we have,

$$\begin{aligned}\frac{d}{dt}\ln(m(t)) &\le C(1+\|\nabla d\|_{L^\infty}^2)+C\|\omega\|_{L^\infty}\ln(m(t))\\ &\le C(1+\|\nabla d\|_{L^\infty}^2+\|\omega\|_{L^\infty})\ln(m(t)).\end{aligned}$$

By Gronwall's equality, we have

$$\ln(m(t)) \le \ln(m(0))\exp\left(C\int_0^t(\|\nabla d(t)\|_{L^\infty}^2+\|\omega(t)\|_{L^\infty})dt\right),$$

Or equivalently,

$$m(t) \le \exp\left(\ln(m(0))\exp\left(C\int_0^t(\|\nabla d(t)\|_{L^\infty}^2+\|\omega(t)\|_{L^\infty})dt\right)\right),$$

for any $0 \le t \le T_*$. This completes the proof. □

Proof of Corollary 1.2: Assume that (1.8) were not true. Then there is $0 < M_1 < \infty$ such that

$$\int_0^{T_*}\|\nabla d(t)\|_{L^\infty(\mathbb{R}^2)}^2dt \le M_1. \tag{2.33}$$

Then lemma 2.1 implies

$$\sup_{0\le t\le T_*}\left(\|\omega(t)\|_{L^2(\mathbb{R}^2)}+\|\nabla^2 d(t)\|_{L^2(\mathbb{R}^2)}\right) \le C_1, \tag{2.34}$$

where $C_1 > 0$ depends only on u_0, d_0 and M_1. In particular, we have

$$(\nabla u, \nabla^2 d) \in L_t^2L_x^2(\mathbb{R}^2\times[0,T_*]).$$

On the other hand, since (u, d) satisfies (1.5), the following energy inequality holds (cf. [9]):

$$\begin{aligned}&\int_{\mathbb{R}^2}(|u(t)|^2+|\nabla d(t)|^2)\,dx+2\int_0^t\int_{\mathbb{R}^2}(|\nabla u|^2+|\Delta d+|\nabla d|^2d|^2)\,dxdt\\ &=\int_{\mathbb{R}^2}(|u_0|^2+|\nabla d_0|^2)\,dx\end{aligned}\tag{2.35}$$

for any $0 < t \le T_*$. Therefore, we have

$$(u, \nabla d) \in L_t^\infty L_x^2(\mathbb{R}^2\times[0,T_*])\cap L_t^2H_x^1(\mathbb{R}^2\times[0,T_*]).$$

Applying the regularity theorem 1.2 of [9], we conclude $(u, d) \in C^\infty(\mathbb{R}^2\times(0,T_*])$. This contradicts the assumption that $0 < T_* < \infty$ is the first singular time. The proof is complete. □

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