Blow up criterion for compressible nematic liquid crystal flows in dimension three

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April 29, 2011

Abstract

In this paper, we consider the short time strong solution to a simplified hydrodynamic flow modeling the compressible, nematic liquid crystal materials in dimension three. We establish a criterion for possible breakdown of such solutions at finite time in terms of the temporal integral of both the maximum norm of the deformation tensor of velocity gradient and the square of maximum norm of gradient of liquid crystal director field.

1 Introduction

Nematic liquid crystals are aggregates of molecules which possess same orientational order and are made of elongated, rod-like molecules. The continuum theory of liquid crystals was developed by Ericksen [9] and Leslie [28] during the period of 1958 through 1968, see also the book by de Gennes [11]. Since then there have been remarkable research developments in liquid crystals from both theoretical and applied aspects. When the fluid containing nematic liquid crystal materials is at rest, we have the well-known Ossen-Frank theory for static nematic liquid crystals. The readers can refer to the poineering work by Hardt-Lin-Kinderlehrer [12] on the analysis of energy minimal configurations of namatic liquid crystals. In general, the motion of fluid always takes place. The so-called Ericksen-Leslie system is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity field u and the macroscopic description of the microscopic orientation configurations d of rod-like liquid crystals.

When the fluid is an incompressible, viscous fluid, Lin [18] first derived a simplified Ericksen-Leslie equation modeling liquid crystal flows in 1989. Subsequently, Lin and Liu [19, 20] made some important analytic studies, such as the existence of weak and strong solutions and the partial regularity of suitable solutions, of the simplified Ericksen-Leslie system, under the assumption that the liquid crystal director field is of varying length by Leslie's terminology or variable degree of orientation by Ericksen's terminology.

When the fluid is allowed to be compressible, the Ericksen-Leslie system becomes more complicate and there seems very few analytic works available yet. We would like to mention that very recently, there have been both modeling study, see Morro [29], and numerical study, see Zakharov-Vakulenko [36], on the hydrodynamics of compressible nematic liquid crystals under the influence of temperature gradient or electromagnetic forces.

The main aim of this paper, and the companion paper [17] as well, is an attemption to initiate some analytic study for the flow of compressible nematic liquid crystals. We will mainly address

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several issues on the strong solutions. More precisely, we will focus on the blow-up criterion on strong solutions in this paper.

Now we start to describe the problem. Let $\Omega \subset \mathbb{R}^3$ be either a bounded smooth domain or the entire $\mathbb{R}^3$, we will consider a simplified version of Ericksen-Leslie equation that models the hydrodynamic flow of compressible, nematic liquid crystals in Ω:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.1}$$

$$\rho u_t + \rho u \cdot \nabla u + \nabla(P(\rho)) = \mathcal{L}u - \Delta d \cdot \nabla d, \tag{1.2}$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \tag{1.3}$$

where $\rho : \Omega \to \mathbb{R}_+$ is the density of the fluid, $u : \Omega \to \mathbb{R}^3$ is the fluid velocity field, $P(\rho) : \Omega \to \mathbb{R}_+$ denotes the pressure of the fluid, $d : \Omega \to S^2$ represents the macroscopic average of the nematic liquid crystal orientation field, $\nabla \cdot (= \mathrm{div})$ denotes the divergence operator on $\mathbb{R}^3$, and $\mathcal{L}$ denotes the Lamé operator defined by

$$\mathcal{L}u = \mu \Delta u + (\mu + \lambda)\nabla(\nabla \cdot u), \tag{1.4}$$

where μ and λ are the shear viscosity and the bulk viscosity coefficients of the fluid repsectively, which are assumed to satisfy the following physical condition:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0. \tag{1.5}$$

The pressure $P(\rho)$, as a given continuous function of ρ, is usually determined by the equation of states. Through this paper, we assume that

$$P : [0, +\infty) \to \mathbb{R} \text{ is a locally Lipschitz continuous function.} \tag{1.6}$$

Notice that (1.1) is the equation for conservation of mass, (1.2) is the linear momemtum equation, and (1.3) is the angular momentum equation. We would like to point out that the system (1.1)-(1.3) includes several important equations as special cases:

(i) When ρ is constant, the equation (1.1) reduces to the incompressibility condition of the fluid ($\nabla \cdot u = 0$), and the system (1.1)-(1.3) becomes the equation of incompressible flow of namatic liquid crystals provided that P is a unknown pressure function. This was previously proposed by Lin [18] as a simplified Ericksen-Leslie equation modeling incompressible liquid crystal flows.

(ii) When d is a constant vector field, the system (1.1)-(1.2) becomes a compressible Navier-Stokes equation, which is an extremely important equation to describe motion of compressible fluids. It has attracted great interests among many analysts and there have been many important developments (see, for example, Lions [26], Feireisl [10] and references therein).

(iii) When both ρ and d are constants, the system (1.1)-(1.2) becomes the incompressible Naiver-Stokes equation provided that P is a unknown pressure function, the fundamental equation to describe Newtonian fluids (see, Lions [25] and Temam [30] for survey of important developments).

(iv) When ρ is constant and $u = 0$, the system (1.1)-(1.3) reduces to the equation for heat flow of harmonic maps into S^2. There have been extensive studies on the heat flow of harmonic maps in the past few decades (see, for example, the monograph by Lin-Wang [23] and references therein).

From the viewpoint of partial differential equations, the system (1.1)-(1.3) is a highly nonlinear system coupling between hyperbolic equations and parabolic equations. It is very challenging to understand and analyze such a system, especially when the density function ρ may vanish or the fluid takes vacuum states.

In this paper, the system (1.1)-(1.3) will be studied along with the initial condition:

$$(\rho, u, d)\Big|_{t=0} = (\rho_0, u_0, d_0), \tag{1.7}$$

and one of the following three types of boundary conditions:
(i) Cauchy problem:

$$\Omega = \mathbb{R}^3,\ \rho, u \text{ vanish and } d \text{ is constant at infinity (in some weak sense).} \tag{1.8}$$

(ii) Dirichlet and Neumann boundary condition for (u, d): $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, and

$$(u,\ \frac{\partial d}{\partial \nu})\Big|_{\partial\Omega} = 0, \tag{1.9}$$

where ν is the unit outward normal vector of $\partial\Omega$.
(iii) Navier-slip and Neumann boundary condition for (u, d): $\Omega \subset \mathbb{R}^3$ is a bounded, simply connected, smooth domain, and

$$(u \cdot \nu,\ \operatorname{curl} u \times \nu,\ \frac{\partial d}{\partial \nu})\Big|_{\partial\Omega} = 0, \tag{1.10}$$

where $\operatorname{curl} u = \nabla \times u$ denotes the vorticity field of the fluid.

In order to state the definition of strong solutions to the initial and boundary value problem (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10), we introduce some notations.

We denote

$$\int f\,dx = \int_\Omega f\,dx.$$

For $1 \le r \le \infty$, denote the L^r spaces and the standard Sobolev spaces as follows:

$$L^r = L^r(\Omega),\ D^{k,r} = \left\{u \in L^1_{\text{loc}}(\Omega) : \|\nabla^k u\|_{L^r} < \infty\right\},$$

$$W^{k,r} = L^r \cap D^{k,r},\ H^k = W^{k,2},\ D^k = D^{k,2},$$

$$D^1_0 = \left\{u \in L^6 :\ \|\nabla u\|_{L^2} < \infty, \text{ and satisfies (1.8) or (1.9) or (1.10) for the part of } u\right\},$$

$$H^1_0 = L^2 \cap D^1_0,\ \|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r}.$$

Denote

$$Q_T = \Omega \times [0, T]\ (T > 0),$$

and let

$$\mathcal{D}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right)$$

denote the deformation tensor, which is the symmetric part of the velocity gradient.

Definition 1.1 *For $T > 0$, (ρ, u, d) is called a strong solution to the compressible nematic liquid crystal flow (1.1)-(1.3) in $\Omega \times (0, T]$, if for some $q \in (3, 6]$,*

$$\begin{gathered}
0 \le \rho \in C([0,T]; W^{1,q} \textstyle\bigcap H^1),\ \rho_t \in C([0,T]; L^2 \textstyle\bigcap L^q);\\
u \in C([0,T]; D^2 \textstyle\bigcap D^1_0) \textstyle\bigcap L^2(0,T; D^{2,q}),\ u_t \in L^2(0,T; D^1_0),\ \sqrt{\rho} u_t \in L^\infty(0,T; L^2);\\
\nabla d \in C([0,T]; H^2) \textstyle\bigcap L^2(0,T; H^3),\ d_t \in C([0,T]; H^1) \textstyle\bigcap L^2(0,T; H^2),\ |d| = 1 \text{ in } \overline{Q}_T;
\end{gathered}$$

and (ρ, u, d) satisfies (1.1)-(1.3) a.e. in $\Omega \times (0, T]$.

For the existence of local strong solutions associated with the three types of boundary conditions, we have obtained the following theorem in the paper [17].

Theorem 1.2 *Assume that P satisfies (1.6), $\rho_0 \geq 0$, $\rho_0 \in W^{1,q} \bigcap H^1 \bigcap L^1$ for some $q \in (3,6]$, $u_0 \in D^2 \bigcap D_0^1$, $\nabla d_0 \in H^2$ and $|d_0| = 1$ in $\overline{\Omega}$. If, in additions, the following compatibility condition*

$$\mathcal{L}u_0 - \nabla(P(\rho_0)) - \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0} g \text{ for some } g \in L^2(\Omega, \mathbb{R}^3) \tag{1.11}$$

holds, then there exist a positive time $T_0 > 0$ and a unique strong solution (ρ, u, d) of (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10) in $\Omega \times (0, T_0]$.

We would like to point out that an analogous existence theorem of local strong solutions to the isentropic compressible Naiver-Stokes equation, under the first two boundary conditions (1.8) and (1.9), has been previously established by Choe-Kim [3] and Cho-Choe-Kim [4]. A byproduct of our theorem 1.2 also yields the existence of local strong solutions to the isentropic compressible Navier-Stokes equation under the Navier-slip boundary condition (1.10).

In dimension one, Ding-Lin-Wang-Wen [7] have proven that the local strong solution to (1.1)-(1.3) under (1.7) and (1.9) is global. For dimensions at least two, it is reasonable to believe that the local strong solution to (1.1)-(1.3) may cease to exist globally. In fact, there exist finite time singularities of the (transported) heat flow of harmonic maps (1.3) in dimensions two or higher (we refer the interested readers to [23] for the exact references). An important question to ask would be what is the main mechanism of possible break down of local strong (or smooth) solutions.

Such a question has been studied for the incompressible Euler equation or the Navier-Stokes equation by Beale-Kato-Majda in their poineering work [1], which showed that the L^∞-bound of vorticity $\nabla \times u$ must blow up. Later, Ponce [27] rephrased the BKM-criterion in terms of the deformation tensor $\mathcal{D}(u)$.

When dealing with the *isentropic*[1] compressible Navier-Stokes equation, there have recently been several very interesting works on the blow up criterion. For example, if $0 < T_* < +\infty$ is the maximum time for strong solution, then (i) Huang-Li-Xin [14] established a Serrin type criterion: $\lim_{T\uparrow T_*} \left(\|\text{div} u\|_{L^1(0,T;L^\infty)} + \|\rho^{\frac{1}{2}} u\|_{L^s(0,T;L^r)} \right) = \infty$ for $\frac{2}{s} + \frac{3}{r} \leq 1,\ 3 < r \leq \infty$; (ii) Sun-Wang-Zhang [31], and independently [14], showed that if $7\mu > \lambda$, then $\lim_{T\uparrow T_*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty$; and (iii) Huang-Li-Xin [15] showed that $\lim_{T\uparrow T_*} \|\mathcal{D}(u)\|_{L^1(0,T;L^\infty)} = \infty$.

When dealing the heat flow of harmonic maps (1.3) (with $u = 0$), Wang [32] obtained a Serrin type regularity theorem, which implies that if $0 < T_* < +\infty$ is the first singular time for local smooth solutions, then $\lim_{T\uparrow T_*} \|\nabla d\|_{L^2(0,T;L^\infty)} = \infty$.

When dealing with the incompressible nematic liquid crystal flow, Lin-Lin-Wang [24] and Lin-Wang [22] have established the global existence of a unique "almost strong" solution[2] for the initial-boundary value problem in bounded domains in dimension two, see also Hong [13] and Xu-Zhang [34] for some related works. In dimension three, for the incompressible nematic liquid crystal flow Huang-Wang [16] have obtained a BKM type blow-up criterion very recently, while the existence of global weak solutions still remains to be a largely open question.

Motivated by these works on the blow up criterion of local strong solutions to the Navier-Stokes equation and the incompressible nematic liquid crystal flow, we will establish in this paper the following blow-up criterion of breakdown of local strong solutions in finite time.

[1]namely, $P(\rho) = a\rho^\gamma$ for some $a > 0$ and $\gamma > 1$.
[2]that has at most finitely many possible singular time.

Theorem 1.3 *Let (ρ, u, d) be a strong solution of the initial boundary problem (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10). Assume that P satisfies (1.6), and the initial data (ρ_0, u_0, d_0) satisfies (1.11). If $0 < T_* < +\infty$ is the maximum time of existence, then*

$$\int_0^{T_*} \left(\|\mathcal{D}(u)\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right) dt = \infty. \tag{1.12}$$

We would like to make a few comments now.

Remark 1.4 (a) In [17], we also obtained a blow-up criterion of (1.1)-(1.3) under the initial condition (1.7) and (1.8) or (1.9) in terms of ρ and ∇d. Namely, if $7\mu > 9\lambda$ and $0 < T_* < +\infty$ is the maximum time of existence of strong solutions, then

$$\lim_{T \uparrow T_*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^3(0,T;L^\infty)}\right) = +\infty.$$

(b) For compressible liquid crystal flows without the nematicity constraint $(|d| = 1)^3$, Liu-Liu [21] have recently obtained a Serrin type criterion on the blow-up of strong solutions.
(c) It is a very interesting question to ask whether there exists a global weak solution to the initial-boundary value problem of (1.1)-(1.3) in dimensions at least two. In dimension one, such an existence has been obtained by Ding-Wang-Wen [8].

We conclude this section by introducing the main ideas of the proof, some of which are inspired by some of the arguments on the isentropic compressible Navier-Stokes equation by [15] and [31].
(1) It is well-known that the bound of $\|\mathcal{D}(u)\|_{L^1_t L^\infty_x}$ yields that $\|\rho\|_{L^\infty_t L^\infty_x}$ is bounded from the equation (1.1). See Lemma 2.1.
(2) We observe that in the equation (1.3), the bound of $(\|\mathcal{D}(u)\|_{L^1_t L^\infty_x} + \|\nabla d\|_{L^2_t L^\infty_x})$ yields that $\|\nabla d\|_{L^\infty_t L^r_x}$ is bounded for any $2 \le r < +\infty$, which is a crucial ingredient to obtain higher order estimates of ρ, u, d. See Lemma 2.3.
(3) Due to the possible vacuum state of ρ, the strong nonlinearities of the convection term $u \cdot \nabla u$ and the induced stress tensor $\Delta d \cdot \nabla d$, in order to obtain control of $(\|\nabla \rho\|_{L^\infty_t L^2_x} + \|\nabla u\|_{L^\infty_t L^2_x} + \|\nabla^2 d\|_{L^\infty_t L^2_x})$, we estimate $(\|\sqrt{\rho}\dot{u}\|_{L^2_t L^2_x} + \|\nabla d_t\|_{L^2_t L^2_x})$ by combining an energy estimate of the equation (1.2) in terms of the *material derivative* $\dot{u} \equiv u_t + u \cdot \nabla u$ with second order energy estimates of both (1.2) and (1.3). See Lemma 2.4.
(4) We estimate $(\|\nabla^2 u\|_{L^\infty_t L^2_x} + \|\nabla^3 d\|_{L^\infty_t L^2_x})$ by combining thrid order estimate estimates of (1.2) and (1.3) with H^2-estimate of the Lamé equation and H^3-estimate of the harmonic map equation. See Lemma 2.6.
(5) Finally, we obtain the estimate of $\|\nabla \rho\|_{L^\infty_t L^q_x}$ for $3 < q \le 6$ in terms of $\|u\|_{L^2_t D^{2,q}_x}$. To do it, we employ the elliptic estimate of the equation satisfied by the *effective viscous flux* $G \equiv (2\mu + \lambda)\text{div}u - P(\rho)$ and the bound of $\|\nabla^4 d\|_{L^2_t L^2_x}$ and $\|\nabla u_t\|_{L^2_t L^2_x}$. See Lemma 2.7.

We would like to point out that during all these arguments, specific forms of the pressume function P play no roles, it is the local Lipschitz regularity of P that is relevant.

Acknowledgement. The first two authors are partially supported by NSF grant 1000115. The work was completed during the third author's visit to the University of Kentucky, which is partially supported by the second author's NSF grant 0601162. The third author would like to thank the department of Mathematics for its hospitality.

[3] the right hand side of equation (1.3) is replaced by $\Delta d + f(d)$ for some smooth function $f : \mathbb{R}^3 \to \mathbb{R}^3$, e.g. $f(d) = (|d|^2 - 1)d$.

2 Proof of Theorem 1.3

Let $0 < T_* < \infty$ be the maximum time for the existence of strong solution (ρ, u, d) to (1.1)-(1.3). Namely, (ρ, u, d) is a strong solution to (1.1)-(1.3) in $\Omega \times (0, T]$ for any $0 < T < T_*$, but not a strong solution in $\Omega \times (0, T_*]$. Suppose that (1.12) were false, i.e.

$$M_0 := \int_0^{T_*} \left(\|\mathcal{D}(u)\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2\right) dt < \infty. \tag{2.1}$$

The goal is to show that under the assumption (2.1), there is a bound $C > 0$ depending only on M_0, ρ_0, u_0, d_0, and T_* such that

$$\sup_{0\le t<T_*} \left[\max_{r=2,q}(\|\rho\|_{W^{1,r}} + \|\rho_t\|_{L^r}) + (\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{H^1}) + (\|d_t\|_{H^1} + \|\nabla d\|_{H^2})\right] \le C, \tag{2.2}$$

and

$$\int_0^{T_*} \left(\|u_t\|_{D^1}^2 + \|u\|_{D^{2,q}}^2 + \|d_t\|_{H^2}^2 + \|\nabla d\|_{H^3}^2\right) dt \le C. \tag{2.3}$$

With (2.2) and (2.3), we can then show without much difficulty that T_* is not the maximum time, which is the desired contradiction.

Throughout the rest of the paper, we denote by C a generic constant depending only on ρ_0, u_0, d_0, T_*, M_0, λ, μ, Ω, and P. We denote by

$$A \lesssim B$$

if there exists a generic constant C such that $A \le CB$. For two 3×3 matrices $M = (M_{ij}), N = (N_{ij})$, denote the scalar product between M and N by

$$M : N = \sum_{i,j=1}^{3} M_{ij}N_{ij}.$$

For $d : \Omega \to S^2$, denote by $\nabla d \otimes \nabla d$ as the 3×3 matrix given by

$$(\nabla d \otimes \nabla d)_{ij} = \langle \nabla_i d, \nabla_j d\rangle, \ 1 \le i, j \le 3.$$

The proof is divided into several steps, and we proceed as follows.

Step 1. We will first establish L^∞-control of ρ. More precisely, we have

Lemma 2.1 *Let $0 < T_* < +\infty$ be the maximum time for a strong solution (ρ, u, d) to (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10). If (1.11) and (2.1) hold, then*

$$\sup_{0\le t<T_*} \|\rho\|_{L^\infty} \le C. \tag{2.4}$$

Proof. This estimate is a well-known fact of (1.1) and was proved by Huang-Li-Xin [15] (Lemma 2.1). For the convenience of reader, we sketch it here. For any $1 < r < +\infty$, multiplying (1.1) by $r\rho^{r-1}$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\int \rho^r \, dx &= -\int \left(u \cdot \nabla(\rho^r) + r\rho^r \operatorname{div} u\right) dx \\ &= -\int \left(\operatorname{div}(u\rho^r) + (r-1)\rho^r \operatorname{div} u\right) dx \le (r-1)\|\operatorname{div} u\|_{L^\infty} \int \rho^r \, dx.\end{aligned}$$

Thus

$$\frac{d}{dt}\|\rho\|_{L^r} \leq \frac{r-1}{r}\|\operatorname{div} u\|_{L^\infty}\|\rho\|_{L^r}.$$

This, (2.1), Lemma 2.1, together with Gronwall's inequality, imply

$$\sup_{0\leq t<T_*}\|\rho(t)\|_{L^r} \leq \|\rho_0\|_{L^r}\exp\left(\int_0^{T_*}\|\operatorname{div} u\|_{L^\infty}dt\right) \leq C,$$

which, after sending r to ∞, implies (2.4). This completes the proof. $\square$

Step 2. We next establish the global energy inequality for strong solutions, namely,

Lemma 2.2 *Let $0 < T_* < +\infty$ be the maximum time for a strong solution (ρ, u, d) to (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10). If (1.11) and (2.1) hold, then for any $0 \leq t < T_*$, the following inequality holds:*

$$\begin{aligned}&\int_\Omega\left(\rho|u|^2 + |\nabla d|^2\right)(t)\,dx + \int_0^t\int_\Omega\left(|\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|^2\right)dx\,ds\\ &\leq C\Big[\int_\Omega\left(\rho_0|u_0|^2 + |\nabla d_0|^2\right)\,dx + 1\Big].\end{aligned} \tag{2.5}$$

Furthermore, we have

$$\int_0^{T_*}\int_\Omega|\nabla^2 d|^2\,dx\,dt \leq C. \tag{2.6}$$

Proof. Without loss of generality, we may assume that $P(0) = 0$. Since P is locally Lipschitz by (1.6), it follows that P' is locally bounded on $[0, +\infty)$. Since ρ is bounded in $\Omega \times [0, T_*)$ by (2.4), we then have that, on $\Omega \times [0, T_*)$,

$$\begin{aligned}|P(\rho)| &\leq \left\|P'(\rho)\right\|_{L^\infty}\rho \leq C\rho\ (\ \leq C\ )\\ |\nabla(P(\rho))| &\leq \left\|P'(\rho)\right\|_{L^\infty}|\nabla\rho| \leq C|\nabla\rho|.\end{aligned} \tag{2.7}$$

Since (1.8) and (1.9) are easier to handle[4], we outline the proof for the boundary condition (1.10). Multiplying (1.2) by u and integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\int\rho(\partial_t|u|^2 + u\cdot\nabla|u|^2)\,dx + \int(\mu|\nabla\times u|^2 + (2\mu+\lambda)|\operatorname{div}u|^2)\,dx\\ =&\int P(\rho)\operatorname{div}u\,dx - \int u\cdot\nabla d\cdot\Delta d\,dx.\end{aligned} \tag{2.8}$$

Here we have used the fact that $\Delta u = \nabla\operatorname{div}u - \nabla\times\operatorname{curl}u$, and the Navier-slip boundary condition (1.10) to obtain

$$\begin{aligned}\int\mathcal{L}u\cdot u\,dx &= \int[(2\mu+\lambda)\nabla(\operatorname{div}u)\cdot u - \mu\nabla\times\operatorname{curl}u\cdot u]\,dx\\ &= -\int(2\mu+\lambda)|\operatorname{div}u|^2 + \mu|\operatorname{curl}u|^2)\,dx.\end{aligned}$$

[4]in fact, with respect to the boundary condition (1.8) and (1.9), by integration by parts one has

$$-\int(\mu\Delta u + (\mu+\lambda)\nabla(\operatorname{div}u))\cdot u\,dx = \int(\mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div}u|^2)\,dx \geq \mu\int|\nabla u|^2\,dx.$$

Hereafter we repeatedly use the following identity:

$$\int \langle \nabla \times u, \operatorname{curl} u \rangle \, dx = \int \langle u, \nabla \times (\operatorname{curl} u) \rangle \, dx, \ \forall u \text{ with } \operatorname{curl} u \times \nu = 0 \text{ on } \partial\Omega.$$

By the formula of transportion, we have

$$\int \rho(\partial_t |u|^2 + u \cdot \nabla |u|^2) \, dx = \frac{d}{dt} \int \rho |u|^2 \, dx.$$

By (2.7) and Cauchy's inequality, we have

$$\begin{aligned} |\int P(\rho) \operatorname{div} u \, dx| &\leq \int |P(\rho)| |\operatorname{div} u| \, dx \lesssim \int \rho |\nabla u| \, dx \\ &\leq \epsilon \int |\nabla u|^2 \, dx + C(\epsilon) \int \rho^2 \leq \epsilon \int |\nabla u|^2 \, dx + C(\epsilon), \end{aligned}$$

where we have used (2.4) and the conservation of mass to assure

$$\int \rho^2 \leq \|\rho\|_{L^1} \|\rho\|_{L^\infty} \leq C.$$

Putting these inequalities into (2.8), we obtain

$$\frac{d}{dt} \int \rho |u|^2 \, dx + \int (\mu |\nabla \times u|^2 + (2\mu + \lambda) |\operatorname{div} u|^2) \, dx \leq - \int u \cdot \nabla d \cdot \Delta d \, dx + \epsilon \int |\nabla u|^2 \, dx + C(\epsilon). \quad (2.9)$$

Since Ω is assumed to be simply connected for the boundary condition (1.10), we have the following estimate (see [33] for its proof):

$$\|\nabla u\|_{L^2} \lesssim \|\nabla \times u\|_{L^2} + \|\operatorname{div} u\|_{L^2} \ \forall u \in H^1(\Omega) \text{ with } u \cdot \nu = 0 \text{ on } \partial\Omega. \quad (2.10)$$

This, combined with (1.5), implies that

$$\int (\mu |\nabla \times u|^2 + (2\mu + \lambda) |\operatorname{div} u|^2) dx \geq \frac{\mu}{3} \int (|\nabla \times u|^2 + |\operatorname{div} u|^2) \, dx \geq \frac{1}{C} \int |\nabla u|^2 \, dx. \quad (2.11)$$

Thus, by choosing $\epsilon = \frac{1}{2C}$, (2.9) implies

$$\frac{d}{dt} \int \rho |u|^2 \, dx + \frac{1}{2C} \int |\nabla u|^2 \, dx \leq - \int u \cdot \nabla d \cdot \Delta d \, dx + C. \quad (2.12)$$

Now, multiplying (1.3) by $(\Delta d + |\nabla d|^2 d)$, integrating over Ω and using $\frac{\partial d}{\partial \nu} = 0$ on $\partial\Omega$, we have

$$\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 \, dx + \int \left| \Delta d + |\nabla d|^2 d \right|^2 dx = \int u \cdot \nabla d \cdot \Delta d \, dx, \quad (2.13)$$

where we have used the fact that $|d| = 1$ in Ω and hence

$$\int \langle d_t + u \cdot \nabla d, |\nabla d|^2 d \rangle \, dx = 0.$$

Adding (2.12) and (2.13) together yields (2.5).

To see (2.6), observe that (2.1) implies $\int_0^{T_*} \|\nabla d\|_{L^\infty}^2\,dt \le M_0$ so that

$$\begin{aligned}\int_0^{T_*}\int_\Omega |\nabla d|^4\,dx\,dt &\le M_0\cdot\left(\sup_{0\le t<T_*}\int |\nabla d|^2\,dx\right)\\ &\le CM_0\left[1+\int\left(\rho_0|u_0|^2+|\nabla d_0|^2\right)\,dx\right]\end{aligned}$$

where we have used (2.5) in the last step. This and (2.5) then imply

$$\begin{aligned}\int_0^{T_*}\int_\Omega |\Delta d|^2\,dx\,dt &= \int_0^{T_*}\int_\Omega \left|\Delta d+|\nabla d|^2 d\right|^2\,dx\,dt+\int_0^{T_*}\int_\Omega |\nabla d|^4\,dx\,dt\\ &\le CM_0\Big[1+\int\left(\rho_0|u_0|^2+|\nabla d_0|^2\right)\,dx\Big].\end{aligned}$$

Since $\frac{\partial d}{\partial\nu}=0$ on $\partial\Omega$, the standard L^2-estimate yields

$$\int |\nabla^2 d|^2\,dx \le C\int (|\Delta d|^2+|\nabla d|^2)\,dx.$$

Thus (2.6) follows easily, and the proof is complete. □

Step 3. We will establish $L_t^\infty L_x^r$-control of ∇d for any $2\le r<+\infty$, a key ingredient for the higher order estimates of $u,\nabla d$. More precisely, we have

Lemma 2.3 *Let $0<T_*<+\infty$ be the maximum time for a strong solution (ρ,u,d) to (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10). If (1.11) and (2.1) hold, then for any $2\le r<+\infty$, there exists a $C>0$ depending on $M_0,u_0,d_0,\Omega,n,$ and r such that*

$$\sup_{0\le t<T_*}\|\nabla d\|_{L^r}^r+\int_0^{T_*}\int_\Omega |\nabla d|^{r-2}|\nabla^2 d|^2\,dx\,dt\le C. \tag{2.14}$$

Proof. Here we only consider the Navier-slip boundary condition (1.10), since the argument to deal the first two boundary conditions (1.8) and (1.9) is similar and easier. Differentiating the equation (1.3) with rrespect to x, we have

$$\nabla d_t-\nabla\Delta d=-\nabla(u\cdot\nabla d)+\nabla(|\nabla d|^2 d). \tag{2.15}$$

Multiplying (2.15) by $r|\nabla d|^{r-2}\nabla d$ and integrating over Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\int |\nabla d|^r\,dx+r\int\left(|\nabla d|^{r-2}|\nabla^2 d|^2+(r-2)|\nabla d|^{r-2}|\nabla|\nabla d||^2\right)\,dx\\ =&r\int\nabla(|\nabla d|^2 d)|\nabla d|^{r-2}\nabla d\,dx-r\int\nabla(u\cdot\nabla d)|\nabla d|^{r-2}\nabla d\,dx\\ &+\frac{r}{2}\int_{\partial\Omega}|\nabla d|^{r-2}\langle\nabla(|\nabla d|^2),\nu\rangle\,d\sigma=\sum_{i=1}^3 I_i.\end{aligned} \tag{2.16}$$

We can estimate I_i $(i=1,2,3)$ separately as follows. For I_1, since

$$\nabla(|\nabla d|^2 d)=|\nabla d|^2\nabla d+\nabla(|\nabla d|^2)d \text{ and } d\cdot\nabla d=0,$$

we have

$$I_1 = r \int |\nabla d|^{r+2}\, dx \lesssim \|\nabla d\|_{L^\infty}^2 \int |\nabla d|^r\, dx.$$

For I_2, we have

$$\begin{aligned} I_2 =& -r\int |\nabla d|^{r-2}\nabla_i u^j \langle \nabla_j d, \nabla_i d\rangle\, dx - \int u\cdot\nabla(|\nabla d|^r)\, dx \\ =& -r\int |\nabla d|^{r-2}\mathcal{D}(u) : \nabla d\otimes\nabla d\, dx + \int (\operatorname{div} u)|\nabla d|^r\, dx \lesssim \|\mathcal{D}(u)\|_{L^\infty}\int |\nabla d|^r\, dx. \end{aligned}$$

The estimate of the boundary integral I_3 is more delicate. Let $\mathbb{I}_{\partial\Omega}$ denote the second fundamental form of $\partial\Omega$: for any $x\in\partial\Omega$,

$$\mathbb{I}_{\partial\Omega}(x)(U,V) = -\nabla\nu(x)(U,V),\ \forall U,V\in T_x(\partial\Omega).$$

Let ∇_T denote the tangential derivative on $\partial\Omega$. Since $\frac{\partial d}{\partial\nu} = \langle\nabla d,\nu\rangle = 0$ on $\partial\Omega$, we have $\nabla_T(\frac{\partial d}{\partial\nu}) = 0$ on $\partial\Omega$. Hence we have, on $\partial\Omega$,

$$\frac{1}{2}\langle\nabla(|\nabla d|^2),\nu\rangle = \nabla d\cdot\nabla\langle\nabla d,\nu\rangle - \nabla\nu(\nabla d,\nabla d) = \nabla_T d\cdot\nabla_T(\frac{\partial d}{\partial\nu}) - \nabla\nu(\nabla_T d,\nabla_T d) = \mathbb{I}_{\partial\Omega}(\nabla_T d,\nabla_T d).$$

Therefore we have

$$I_3 = r\int_{\partial\Omega} |\nabla d|^{r-2}\mathbb{I}_{\partial\Omega}(\nabla_T d,\nabla_T d)\, d\sigma \lesssim \int_{\partial\Omega}|\nabla d|^r\, d\sigma.$$

Applying the trace formula $W^{1,1}(\Omega)\subset L^1(\partial\Omega)$ and Hölder's inequality, we obtain

$$\begin{aligned} I_3 \lesssim& \||\nabla d|^r\|_{W^{1,1}} \lesssim \int |\nabla d|^r\, dx + \int |\nabla d|^{r-1}|\nabla^2 d|\, dx \\ \le& C\int |\nabla d|^r\, dx + \frac{r}{4}\int |\nabla d|^{r-2}|\nabla^2 d|^2\, dx. \end{aligned}$$

Putting all these estimates into (2.16), we obtain

$$\frac{d}{dt}\int |\nabla d|^r\, dx + \frac{r}{2}\int |\nabla d|^{r-2}|\nabla^2 d|^2\, dx \lesssim \left(\|\nabla d\|_{L^\infty}^2 + \|\mathcal{D}(u)\|_{L^\infty} + 1\right)\int |\nabla d|^r\, dx.$$

By Gronwall's inequality and (2.1), we obtain that for any $0\le t<T_*$,

$$\begin{aligned} &\int |\nabla d(t)|^r\, dx + \int_0^t\int_\Omega |\nabla d|^{r-2}|\nabla^2 d|^2\, dxds \\ \lesssim& \int |\nabla d_0|^r\, dx\cdot\exp\left(\int_0^{T_*}(1+\|\nabla d\|_{L^\infty}^2 + \|\mathcal{D}(u)\|_{L^\infty})\, dt\right) \le\ C. \end{aligned}$$

This completes the proof. □

Step 4. Estimates of $(\nabla u, \nabla\rho, \nabla^2 d)$ in $L_t^\infty L_x^2(\Omega\times[0,T_*])$. First, for any function f on $\Omega\times(0,T_*)$, let

$$\dot f = f_t + u\cdot\nabla f$$

denote the material derivative of f. Then we have

Lemma 2.4 *Let $0 < T_* < +\infty$ be the maximum time for a strong solution (ρ, u, d) to (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10). If (1.11) and (2.1) hold, then*

$$\sup_{0\le t<T_*}\left(\|\nabla u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\nabla^2 d\|_{L^2}^2\right)+\int_0^{T_*}\int_\Omega\left(\rho|\dot{u}|^2+|\nabla d_t|^2\right)\,dxdt\le C. \tag{2.17}$$

Proof. To make the presentation shorter, here we only consider the difficult case: the Navier-slip boundary condition (1.10). To obtain the estimates of u, we adapt some arguments by [15] (Lemma 2.2). Multiplying (1.2) by $\dot{u}$ and integrating over Ω, we obtain,

$$\begin{aligned}
&\int\rho|\dot{u}|^2\,dx-\int\langle\mathcal{L}u,u_t\rangle\,dx\\
=&\int\langle u\cdot\nabla u,\mathcal{L}u\rangle\,dx-\int u\cdot\nabla u\cdot\nabla(P(\rho))\,dx-\int u_t\cdot\nabla(P(\rho))\,dx\\
&-\int u\cdot\nabla u\cdot\langle\Delta d,\nabla d\rangle\,dx-\int u_t\cdot\langle\Delta d,\nabla d\rangle\,dx.
\end{aligned}\tag{2.18}$$

Similar to the proof of Lemma 2.1, we have

$$\begin{aligned}
-\int\langle\mathcal{L}u,u_t\rangle\,dx=&\int[\mu\langle\nabla\times\operatorname{curl}u,u_t\rangle-(2\mu+\lambda)\langle\nabla(\operatorname{div}u),u_t\rangle]\,dx\\
=&\int[\mu\langle\nabla\times u,\nabla\times u_t\rangle+(2\mu+\lambda)(\operatorname{div}u)(\operatorname{div}u_t)]\,dx\\
=&\frac{1}{2}\frac{d}{dt}\int[\mu|\nabla\times u|^2+(2\mu+\lambda)(\operatorname{div}u)^2]\,dx,
\end{aligned}$$

where we have used the fact that $u_t\cdot\nu=\operatorname{curl}u\times\nu=0$ on $\partial\Omega$ during the integration by parts.

The terms on the right hand side of (2.18) can be estimated as follows.

$$\int\langle u\cdot\nabla u,\mathcal{L}u\rangle\,dx=-\mu\int\langle u\cdot\nabla u,\nabla\times\operatorname{curl}u\rangle\,dx+(2\mu+\lambda)\int\langle u\cdot\nabla u,\nabla(\operatorname{div}u)\rangle\,dx. \tag{2.19}$$

For the first term in the right hand side of (2.19), by using $\operatorname{curl}u\times\nu=0$ on $\partial\Omega$ and the formula

$$u\times\operatorname{curl}u=\frac{1}{2}\nabla(|u|^2)-u\cdot\nabla u,$$

we have

$$\begin{aligned}
&-\mu\int\langle u\cdot\nabla u,\nabla\times\operatorname{curl}u\rangle\,dx=-\mu\int\langle\operatorname{curl}u,\nabla\times(u\cdot\nabla u)\rangle\,dx=\mu\int\operatorname{curl}u\cdot\nabla\times(u\times\operatorname{curl}u)\,dx\\
=&\mu\int\langle\operatorname{curl}u,(\operatorname{curl}u\cdot\nabla)u-(u\cdot\nabla)\operatorname{curl}u+\operatorname{div}(\operatorname{curl}u)u-(\operatorname{div}u)\operatorname{curl}u\rangle\,dx\\
=&\mu\int\left(\mathcal{D}(u):\operatorname{curl}u\otimes\operatorname{curl}u-\frac{1}{2}\operatorname{div}u(\operatorname{curl}u)^2\right)\,dx\lesssim\|\mathcal{D}(u)\|_{L^\infty}\|\nabla u\|_{L^2}^2,
\end{aligned}$$

where we have also used the formulas

$$\nabla\times(a\times b)=(b\cdot\nabla)a-(a\cdot\nabla)b+(\operatorname{div}b)a-(\operatorname{div}a)b,\ \text{ and }\operatorname{div}(\operatorname{curl}u)=0.$$

To estimate the second term in the right hand side of (2.19), denote by $u^\tau = u - (u\cdot\nu)\nu$ the tangential component of u on $\partial\Omega$. Note that (1.10) implies $u = u^\tau$ on $\partial\Omega$. Hence we have

$$\begin{aligned}\int \langle u\cdot\nabla u, \nabla(\operatorname{div} u)\rangle\,dx =& \int_{\partial\Omega}\langle (u\cdot\nabla)u,\nu\rangle \operatorname{div} u\,d\sigma - \int ((\nabla u):(\nabla u)^t\operatorname{div} u + \frac{1}{2}u\cdot\nabla((\operatorname{div} u)^2))\,dx\\ =& \int_{\partial\Omega} u^\tau\cdot\nabla_T(u\cdot\nu)\operatorname{div} u\,d\sigma - \int_{\partial\Omega}\nabla\nu(u^\tau,u^\tau)\operatorname{div} u\,d\sigma\\ &- \int[(\nabla u):(\nabla u)^t\operatorname{div} u - \frac{1}{2}(\operatorname{div} u)^3]\,dx\\ =& \int_{\partial\Omega}\mathbb{I}_{\partial\Omega}(u^\tau,u^\tau)\operatorname{div} u\,d\sigma - \int[(\nabla u):(\nabla u)^t\operatorname{div} u - \frac{1}{2}(\operatorname{div} u)^3]\,dx\\ \lesssim& \|u\|^2_{L^4(\partial\Omega)}\|\operatorname{div} u\|_{L^2(\partial\Omega)} + \|\mathcal{D}(u)\|_{L^\infty}\|\nabla u\|^2_{L^2}.\end{aligned}$$

By the trace formula $H^1(\Omega)\subset L^r(\partial\Omega)$ for $r=2,4$, the Poincaré inequality (see [35]):

$$\|u\|_{L^2}\lesssim\|\nabla u\|_{L^2},\ \forall u\in H^1 \text{ with } u\cdot\nu=0 \text{ on } \partial\Omega,$$

and Hölder's inequality, we have

$$\begin{aligned}\|u\|^2_{L^4(\partial\Omega)}\|\operatorname{div} u\|_{L^2(\partial\Omega)} \lesssim& \|u\|^2_{H^1}\|\nabla u\|_{H^1}\lesssim\|\nabla u\|^2_{L^2}(\|\nabla u\|_{L^2}+\|\nabla^2 u\|_{L^2})\\ \le& C(\epsilon)(1+\|\nabla u\|^4_{L^2})+\epsilon\|\nabla^2 u\|^2_{L^2}\end{aligned}$$

for small $\epsilon>0$ to be determined later. Thus we obtain

$$\int\langle u\cdot\nabla u,\mathcal{L}u\rangle\,dx \le \epsilon\|\nabla^2 u\|^2_{L^2} + C(\epsilon)(1+\|\nabla u\|^4_{L^2}) + C\|\mathcal{D}(u)\|_{L^\infty}\|\nabla u\|^2_{L^2}. \tag{2.20}$$

The remaining terms in the right hand side of (2.18) can be estimated as follows.

$$\begin{aligned}&-\int u\cdot\nabla u\cdot\nabla(P(\rho))\,dx\\ =&-\int_{\partial\Omega}P(\rho)\langle(u\cdot\nabla)u,\nu\rangle\,d\sigma + \int(P(\rho)u\cdot\nabla(\operatorname{div} u) + P(\rho)(\nabla u):(\nabla u)^t)\,dx\\ =&-\int_{\partial\Omega}[P(\rho)(u\cdot\nabla)(u\cdot\nu) - P(\rho)\mathbb{I}_{\partial\Omega}(u^\tau,u^\tau)]\,d\sigma + \int P(\rho)(\nabla u):(\nabla u)^t\,dx\\ &+\int_{\partial\Omega}P(\rho)(u\cdot\nu)\operatorname{div} u\,d\sigma - \int[\nabla(P(\rho))\cdot u\operatorname{div} u + P(\rho)(\operatorname{div} u)^2)]\,dx\\ =&\int_{\partial\Omega}P(\rho)\mathbb{I}_{\partial\Omega}(u^\tau,u^\tau)\,d\sigma + \int[P(\rho)((\nabla u):(\nabla u)^t - (\operatorname{div} u)^2)]\,dx\\ &-\int\nabla(P(\rho))\cdot u\operatorname{div} u\,dx\\ \lesssim&\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2} + \int_\Omega|\nabla\rho||u||\operatorname{div} u|\,dx\\ \lesssim&\|\nabla u\|^2_{L^2} + \|\mathcal{D}(u)\|_{L^3}\|u\|_{L^6}\|\nabla\rho\|_{L^2}\\ \lesssim&\|\nabla u\|^2_{L^2} + \|\mathcal{D}(u)\|^{\frac{1}{3}}_{L^\infty}\|\nabla u\|^{\frac{5}{3}}_{L^2}\|\nabla\rho\|_{L^2}\\ \lesssim&1+(\|\mathcal{D}(u)\|_{L^\infty}+1)\,\|\nabla u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\|\nabla\rho\|^2_{L^2},\end{aligned} \tag{2.21}$$

where we have used (2.7) and the Sobolev and Poincaré inequalities (see [35]):

$$\|u\|_{L^6}\lesssim(\|u\|_{L^2}+\|\nabla u\|_{L^2})\lesssim\|\nabla u\|_{L^2},\ \forall u\in H^1(\Omega) \text{ with } u\cdot\nu=0 \text{ on } \partial\Omega.$$

Since (2.4) and (1.1) also imply

$$|(P(\rho))_t| \le \|P'(\rho)\|_{L^\infty}|\rho_t| \le \|P'(\rho)\|_{L^\infty}(\rho|\nabla u| + |\nabla\rho||u|) \le C(|\nabla u| + |u||\nabla\rho|), \tag{2.22}$$

we have

$$\begin{aligned}
&-\int u_t\cdot\nabla P(\rho)\,dx = \frac{d}{dt}\int P(\rho)\mathrm{div}\,u\,dx - \int (P(\rho))_t \mathrm{div}\,u\,dx\\
\le&\frac{d}{dt}\int P(\rho)\mathrm{div}\,u\,dx + C\int (|u||\nabla\rho||\mathrm{div} u| + |\nabla u|^2)\,dx\\
\le&\frac{d}{dt}\int P(\rho)\mathrm{div}\,u\,dx + C\left[\|\nabla u\|_{L^2}^2 + \|\mathcal{D}(u)\|_{L^3}\|u\|_{L^6}\|\nabla\rho\|_{L^2}\right]\\
\le&\frac{d}{dt}\int P(\rho)\mathrm{div}\,u\,dx + C\left[\|\nabla u\|_{L^2}^2 + \|\mathcal{D}(u)\|_{L^\infty}^{\frac13}\|\nabla u\|_{L^2}^{\frac53}\|\nabla\rho\|_{L^2}\right]\\
\le&\frac{d}{dt}\int P(\rho)\mathrm{div}\,u\,dx + C\left[1 + (1+\|\mathcal{D}(u)\|_{L^\infty})\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\|\nabla\rho\|_{L^2}^2\right],
\end{aligned} \tag{2.23}$$

where we have used the Poincaré inequality and Hölder's inequality.

$$\begin{aligned}
-\int u\cdot\nabla u\cdot\langle\Delta d,\nabla d\rangle\,dx \le&\|u\|_{L^6}\|\nabla u\|_{L^6}\|\Delta d\|_{L^2}\|\nabla d\|_{L^6}\\
\lesssim&\|\nabla u\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2})\|\Delta d\|_{L^2} \quad (\text{by (2.14) with } r=6)\\
\le&\epsilon\|\nabla^2 u\|_{L^2}^2 + C(\epsilon)\|\nabla u\|_{L^2}^2\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^2.
\end{aligned} \tag{2.24}$$

To estimate the last term in the right hand side of (2.18), denote $M(d) = \nabla d\otimes\nabla d - \frac12|\nabla d|^2\mathbb{I}_3$. Then we have $\langle\Delta d,\nabla d\rangle = \mathrm{div}(M(d))$ and

$$\begin{aligned}
-\int u_t\cdot\langle\Delta d,\nabla d\rangle\,dx =& -\int_{\partial\Omega} u_t\cdot(\langle M(d),\nu\rangle)\,d\sigma + \int M(d):\nabla u_t\,dx\\
=&\int M(d):\nabla u_t\,dx \quad (\text{since } u_t\cdot\nu = \frac{\partial d}{\partial\nu} = 0 \text{ on } \partial\Omega)\\
=&\frac{d}{dt}\int M(d):\nabla u\,dx - \int (M(d))_t:\nabla u\,dx\\
\le&\frac{d}{dt}\int M(d):\nabla u\,dx + C\int |\nabla d_t||\nabla d||\nabla u|\,dx\\
\le&\frac{d}{dt}\int M(d):\nabla u\,dx + C(\epsilon)\|\nabla d\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \epsilon\|\nabla d_t\|_{L^2}^2.
\end{aligned} \tag{2.25}$$

Putting (2.20)-(2.25) into (2.18), we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\int\left(\mu|\nabla\times u|^2 + (2\mu+\lambda)|\mathrm{div}\,u|^2\right)\,dx + \int\rho|\dot u|^2\,dx\\
\le&\frac{d}{dt}\int (M(d):\nabla u + P(\rho)\mathrm{div}\,u)\,dx + \epsilon(\|\nabla d_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2)\\
&+ C\left(\|\mathcal{D}(u)\|_{L^\infty} + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2 + 1\right)\|\nabla u\|_{L^2}^2\\
&+ C\|\nabla u\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2)\|\Delta d\|_{L^2}^2 + C(\epsilon).
\end{aligned} \tag{2.26}$$

Now we want to estimate $\|\nabla\rho\|_{L^2}^2$. Differentiating the equation (1.1) with respect to x, multiplying the resulting equation by $2\nabla\rho$ and integrating over Ω, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 =& -\int \left(u\cdot\nabla(|\nabla\rho|^2) + 2\mathcal{D}(u):\nabla\rho\otimes\nabla\rho + 2|\nabla\rho|^2(\operatorname{div} u) + 2\rho\nabla(\operatorname{div} u)\cdot\nabla\rho\right)dx \\
=& -\int |\nabla\rho|^2\operatorname{div} u\,dx - 2\int \mathcal{D}(u):\nabla\rho\otimes\nabla\rho\,dx - 2\int \rho\nabla\operatorname{div} u\cdot\nabla\rho\,dx \\
\lesssim& \left(\|\mathcal{D}(u)\|_{L^\infty}+1\right)\|\nabla\rho\|_{L^2}^2 + \epsilon\|\nabla^2 u\|_{L^2}^2.
\end{aligned}
\tag{2.27}
$$

Next we want to estimate $\|\nabla d_t\|_{L^2}^2$. To do this, we multiply (2.15) by ∇d_t and integrate over Ω and use $\frac{\partial d_t}{\partial\nu}=0$ on $\partial\Omega$ to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int|\Delta d|^2dx + \int|\nabla d_t|^2dx = \int\left(\nabla(|\nabla d|^2 d) - \nabla(u\cdot\nabla d)\right)\nabla d_t dx \\
\le& C(\epsilon)\int\left(|\nabla u|^2|\nabla d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla d|^6 + |\nabla d|^2|\nabla^2 d|^2\right)dx + \epsilon\|\nabla d_t\|_{L^2}^2 \\
\le& \epsilon\|\nabla d_t\|_{L^2}^2 + C + C\|\nabla d\|_{L^\infty}^2\left(\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + C\int|u|^2|\nabla^2 d|^2dx,
\end{aligned}
\tag{2.28}
$$

where we have used (2.14) (with $r=6$) in the last step. For the last term in the right hand side of (2.28), by using Nirenberg's interpolation inequality and (2.14) we have

$$
\begin{aligned}
\int|u|^2|\nabla^2 d|^2dx \le& \|u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2 \lesssim \|\nabla u\|_{L^2}^2\|\nabla d\|_{L^6}\|\nabla^3 d\|_{L^2} + \|\nabla u\|_{L^2}^2 \\
\lesssim& \|\nabla u\|_{L^2}^2\|\nabla^3 d\|_{L^2} + \|\nabla u\|_{L^2}^2 \le C(\epsilon)\|\nabla u\|_{L^2}^4 + \epsilon\|\nabla^3 d\|_{L^2}^2 + C.
\end{aligned}
\tag{2.29}
$$

Applying the standard H^3-estimate to the Neumann boundary value problem of the equation (2.15), and using (2.14), we have

$$
\begin{aligned}
\|\nabla^3 d\|_{L^2}^2 \lesssim& \|\nabla\Delta d\|_{L^2}^2 + \|\nabla d\|_{H^1}^2 \lesssim \|\nabla d_t\|_{L^2}^2 + \|\nabla(u\cdot\nabla d)\|_{L^2}^2 + \|\nabla(|\nabla d|^2 d)\|_{L^2}^2 + \|\nabla d\|_{H^1}^2 \\
\lesssim& \|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right) \\
&+ \|\nabla d\|_{L^6}^6 + \|\nabla d\|_{H^1}^2 + \int|u|^2|\nabla^2 d|^2dx \\
\lesssim& \|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right) + \int|u|^2|\nabla^2 d|^2\,dx + \|\nabla^2 d\|_{L^2}^2 + 1.
\end{aligned}
\tag{2.30}
$$

Substituting (2.29) into (2.30) and choosing ϵ sufficiently small, we have

$$
\|\nabla^3 d\|_{L^2}^2 \lesssim 1 + \|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^2.
\tag{2.31}
$$

Substituting (2.31) into (2.29), we obtain

$$
\int|u|^2|\nabla^2 d|^2 \le C + \epsilon\|\nabla d_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\left(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\right) + C\|\nabla u\|_{L^2}^4.
\tag{2.32}
$$

Putting (2.32) into (2.28) and choosing ϵ sufficiently small, we obtain

$$
\begin{aligned}
\frac{d}{dt}\int|\Delta d|^2\,dx + \int|\nabla d_t|^2\,dx \lesssim& 1 + \|\nabla u\|_{L^2}^4 + \|\nabla d\|_{L^\infty}^2\left(\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + \|\nabla^2 d\|_{L^2}^2 \\
\lesssim& 1 + \|\nabla u\|_{L^2}^4 + \|\nabla d\|_{L^\infty}^2\left(\|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) \\
&+ \|\Delta d\|_{L^2}^2.
\end{aligned}
\tag{2.33}
$$

Putting (2.26), (2.27) and (2.33) together, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int \left(\mu|\nabla\times u|^2+(2\mu+\lambda)|\operatorname{div} u|^2+|\nabla\rho|^2+|\Delta d|^2\right)\,dx+\int \left(2\rho|\dot u|^2+|\nabla d_t|^2\right)\,dx\\
\le& 2\frac{d}{dt}\int (M(d):\nabla u+P(\rho)\operatorname{div} u)\,dx+\epsilon\|\nabla d_t\|_{L^2}^2+\epsilon\|\nabla^2 u\|_{L^2}^2\\
&+C(1+\|\nabla u\|_{L^2}^2+\|\mathcal{D}(u)\|_{L^\infty})\|\nabla\rho\|_{L^2}^2+C\left(1+\|\mathcal{D}(u)\|_{L^\infty}+\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2\right)\|\nabla u\|_{L^2}^2\\
&+C(1+\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2)\|\Delta d\|_{L^2}^2+C\|\nabla d\|_{L^\infty}^2+C.
\end{aligned}
$$

By $W^{2,2}$-estimate of the Lamé equation under the Navier-slip boundary condition (1.10) (see [15] Lemma 2.3 and also the proof of Lemma 2.2), we obtain, by using the equation (1.2) and (2.7),

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2}^2 \lesssim& \|\mathcal{L}u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\lesssim \|\nabla u\|_{L^2}^2+\|\rho\dot u\|_{L^2}^2+\|\nabla(P(\rho))\|_{L^2}^2+\|\Delta d\cdot\nabla d\|_{L^2}^2\\
\lesssim&\|\nabla u\|_{L^2}^2+\|\rho\dot u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2\|\Delta d\|_{L^2}^2.
\end{aligned}
\tag{2.34}
$$

Choosing sufficiently small $\epsilon>0$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int \left(\mu|\nabla\times u|^2+(2\mu+\lambda)|\operatorname{div} u|^2+|\nabla\rho|^2+|\Delta d|^2\right)\,dx+\int \left(2\rho|\dot u|^2+|\nabla d_t|^2\right)\,dx\\
\le& 2\frac{d}{dt}\int (M(d):\nabla u-P(\rho)\operatorname{div} u)\,dx\\
&+C\left[1+\|\mathcal{D}(u)\|_{L^\infty}+\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2\right]\left[\|\nabla u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right]+C(1+\|\nabla d\|_{L^\infty}^2).
\end{aligned}
$$

Integrating from 0 to t, $0<t<T_*$ and applying (2.11) and (2.7), we obtain

$$
\begin{aligned}
&\int \left(|\nabla u|^2+|\nabla\rho|^2+|\Delta d|^2\right)(t)\,dx+\int_0^t\int_\Omega \left(\rho|\dot u|^2+|\nabla d_t|^2\right)\,dxds\\
\lesssim&\int (|M(d)||\nabla u|+|P(\rho)||\operatorname{div} u|)(t)\,dx+\int (|M(d_0)||\nabla u_0|+|P(\rho_0)||\operatorname{div} u_0|)\,dx\\
&+\int \left(|\nabla u_0|^2+|\nabla\rho_0|^2+|\Delta d_0|^2\right)\,dx+C\int_0^t (1+\|\nabla d\|_{L^\infty}^2)\,ds\\
&+\int_0^t \left[1+\|\mathcal{D}(u)\|_{L^\infty}+\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2\right]\left[\|\nabla u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right]\,ds\\
\le& C+\frac{1}{2}\|\nabla u\|_{L^2}^2+C\int (|\nabla d|^4+|\rho|)\,dx\\
&+C\int_0^t \left[1+\|\mathcal{D}(u)\|_{L^\infty}+\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2\right]\left[\|\nabla u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right]\,ds.
\end{aligned}
\tag{2.35}
$$

Since the coefficient function

$$
\left[1+\|\mathcal{D}(u)\|_{L^\infty}+\|\nabla u\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2\right]\in L^1([0,T_*]),
$$

the Gronwall's inequality, Lemma 2.3 and the conservation of mass imply that for any $0\le t<T_*$,

$$
\int (|\nabla u|^2+|\nabla\rho|^2+|\nabla^2 d|^2)(t)\,dx+\int_0^t\int_\Omega (\rho|\dot u|^2+|\nabla d_t|^2)\,dx\,ds\le C.
$$

The proof is complete. □

As an immediate consequence of the proof of Lemma 2.4, we have

Corollary 2.5 *Under the same assumptions as in Lemma 2.4, we have*

$$\sup_{0\le t<T_*}\|d_t\|_{L^2}^2+\int_0^{T_*}\left(\|\rho_t\|_{L^2}^2+\|\nabla u\|_{H^1}^2+\|\nabla d\|_{H^2}^2\right)\,dt\le C. \tag{2.36}$$

Proof. It follows from Lemma 2.4 that $\nabla u, \nabla\rho, \Delta d\in L_t^\infty L_x^2(\Omega\times[0,T_*])$. By Sobolev's inequality, we then have $u\in L_t^\infty L_x^6(\Omega\times[0,T_*])$. On the other hand, Lemma 2.3 implies $\nabla d\in L_t^\infty L_x^r(\Omega\times[0,T_*])$ for $r=3,4$. Therefore, by (1.3), we have

$$|d_t|\lesssim(|u||\nabla d|+|\Delta d|+|\nabla d|^2)\in L_t^\infty L_x^2(\Omega\times[0,T_*]).$$

It is easy to see that $L_t^2L_x^2$-estimate of $\nabla^2 u$ and $\nabla^3 d$ follows from (2.31) , (2.34), (2.1), and Lemma 2.4. To see $L_t^2L_x^2$-estimate of ρ_t, note that (1.1) and Lemma 2.1 imply

$$|\rho_t|\le|\nabla\rho||u|+\rho|\mathrm{div}\,u|\lesssim|u||\nabla\rho|+|\nabla u|.$$

By the Sobolev's embedding, we have $u\in L_t^2L_x^\infty(\Omega\times[0,T_*])$. Hence

$$\||u||\nabla\rho|\|_{L^2(\Omega\times[0,T_*])}\le\|u\|_{L_t^2L_x^\infty(\Omega\times[0,T_*])}\|\nabla\rho\|_{L_t^\infty L_x^2(\Omega\times[0,T_*])}\le C.$$

This clearly implies $\|\rho_t\|_{L^2(\Omega\times[0,T_*])}\le C$. The proof is complete. □

Step 5. Estimates of $(\sqrt{\rho}u_t,\nabla^2u,\nabla d_t,\nabla^3 d)$ in $L_t^\infty L_x^2(\Omega\times[0,T_*])$. More precisely, we have

Lemma 2.6 *Let $0<T_*<+\infty$ be the maximum time for a strong solution (ρ,u,d) to (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10). If (1.11) and (2.1) hold, then*

$$\sup_{0\le t<T_*}\int_\Omega(\rho|u_t|^2+|\nabla^2u|^2+|\nabla d_t|^2+|\nabla^3 d|^2)\,dx+\int_0^{T_*}\int_\Omega(|\nabla u_t|^2+|d_{tt}|^2)dxdt\le C. \tag{2.37}$$

Proof. For simplicity, we only consider the Navier-slip boundary condition (1.10). Differentiating the equation (1.2) with respect to t, we get

$$\begin{aligned}&\rho u_{tt}+\rho_t u_t+\rho u\cdot\nabla u_t+\rho u_t\cdot\nabla u+\rho_t u\cdot\nabla u+\nabla((P(\rho))_t)\\&=\mathcal{L}u_t-\nabla\cdot(\nabla d_t\otimes\nabla d+\nabla d\otimes\nabla d_t-\nabla d\cdot\nabla d_t\mathbb{I}_3)\,.\end{aligned} \tag{2.38}$$

Since $u_t\cdot\nu=0$ and $\mathrm{curl}\,u_t\times\nu=0$ on $\partial\Omega$, as in the proof of Lemma 2.4 we can verify

$$-\int\langle\mathcal{L}u_t,u_t\rangle\,dx=\int(\mu|\nabla\times u_t|^2+(2\mu+\lambda)|\mathrm{div}\,u_t|^2)\,dx.$$

Thus, multiplying (2.38) by u_t and integrating the resulting equation over Ω and using (1.1), we obtain, by using Sobolev's inequality, Hölder's inequality, and (2.22),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + \int \left(\mu|\nabla\times u_t|^2 + (2\mu+\lambda)|\operatorname{div} u_t|^2\right) dx\\
\lesssim&\int \left(\rho|u||\nabla u_t||u_t| + \rho|u||\nabla(u\cdot\nabla u\cdot u_t)| + |(P(\rho))_t||\operatorname{div} u_t|\right)\, dx\\
&+\int \rho|u_t|^2|\nabla u|\, dx + \int |\nabla d_t||\nabla d||\nabla u_t|\, dx\\
\lesssim&\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^\infty} + \|\nabla u\|_{L^3}\|u_t\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u_t\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\\
&+\|\rho_t\|_{L^2}\|\operatorname{div} u_t\|_{L^2} + \int(\rho|u||\nabla u|^2|u_t| + \rho|u|^2|\nabla^2 u||u_t| + \rho|u|^2|\nabla u||\nabla u_t|)\, dx\\
\lesssim&\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{H^1} + \|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\|u\|_{L^6}^2 + \|u_t\|_{L^6}\|\nabla^2 u\|_{L^2}\|u\|_{L^6}^2\\
&+\|u_t\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^3}^2 + \|\rho_t\|_{L^2}\|\operatorname{div} u_t\|_{L^2} + \|\nabla u_t\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\\
\lesssim&\|\nabla u_t\|_{L^2}\left(\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{H^1} + \|\nabla u\|_{H^1} + \|\rho_t\|_{L^2} + \|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\right)\\
\lesssim&\frac{1}{2}\int \mu|\nabla u_t|^2dx + \|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2 + \|\nabla u\|_{H^1}^2 + \|\rho_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2\|\nabla d\|_{L^\infty}^2.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\frac{d}{dt}\int \rho|u_t|^2\, dx + \int \left(\mu|\nabla\times u_t|^2 + (2\mu+\lambda)|\operatorname{div} u_t|^2\right)\, dx\\
\lesssim&\|\nabla u\|_{H^1}^2\int \rho|u_t|^2\, dx + \|\nabla^2 u\|_{L^2}^2 + \|\rho_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2 + 1.
\end{aligned}
\tag{2.39}
$$

Differentiating the equation (1.3) with respect to t, multiplying d_{tt} and integrating over Ω, we obtain, by using $\frac{\partial d_t}{\partial\nu} = 0$ on $\partial\Omega$, Sobolev and Hölder inequalities, Lemma 2.3, and Lemma 2.4,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\nabla d_t|^2\, dx + \int |d_{tt}|^2\, dx =& \int \left\langle \partial_t\left(|\nabla d|^2 d - u\cdot\nabla d\right), d_{tt}\right\rangle dx\\
\lesssim&\|d_{tt}\|_{L^2}\|u_t\|_{L^6}\|\nabla d\|_{L^3} + \|d_{tt}\|_{L^2}\|u\|_{L^6}\|\nabla d_t\|_{L^3}\\
&+\|d_{tt}\|_{L^2}\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|d_{tt}\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\\
\le&\frac{1}{4}\|d_{tt}\|_{L^2}^2 + C[\|\nabla u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}\|\nabla^2 d_t\|_{L^2}\\
&+(1+\|\nabla d\|_{L^\infty}^2)\|\nabla d_t\|_{L^2}^2],
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{d}{dt}\int |\nabla d_t|^2dx + \int |d_{tt}|^2dx\\
\le& C[\|\nabla u_t\|_{L^2}^2 + \|\nabla^2 d_t\|_{L^2}\|\nabla d_t\|_{L^2} + (1+\|\nabla d\|_{L^\infty}^2)\|\nabla d_t\|_{L^2}^2].
\end{aligned}
\tag{2.40}
$$

Now we need to estimate $\|\nabla^2 d_t\|_{L^2}$. In fact, by applying the standard H^2-estimate on the equation (1.3) and Lemma 2.3, we have

$$
\begin{aligned}
\|\nabla^2 d_t\|_{L^2} \lesssim&\|\nabla d_t\|_{L^2} + \|d_{tt}\|_{L^2} + \|\partial_t(u\cdot\nabla d)\|_{L^2} + \|\partial_t(|\nabla d|^2 d)\|_{L^2}\\
\lesssim&\|\nabla d_t\|_{L^2} + \|d_{tt}\|_{L^2} + \|u_t\|_{L^6}\|\nabla d\|_{L^3} + \|u\|_{L^6}\|\nabla d_t\|_{L^3}\\
&+\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|\nabla d_t\|_{L^3}\|\nabla d\|_{L^6}\\
\lesssim&\|d_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla^2 d_t\|_{L^2}^{\frac{1}{2}} + \|\nabla d_t\|_{L^2}\\
\le&\frac{1}{2}\|\nabla^2 d_t\|_{L^2} + C\left[\|d_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla d_t\|_{L^2}\right].
\end{aligned}
$$

Thus

$$\|\nabla^2 d_t\|_{L^2} \lesssim \|d_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla d_t\|_{L^2}. \tag{2.41}$$

Substituting (2.41) into (2.40), and using Cauchy inequality, we obtain

$$\frac{d}{dt}\int |\nabla d_t|^2\,dx + \int |d_{tt}|^2\,dx \le \frac{1}{4}\|d_{tt}\|_{L^2}^2 + C\left[\|\nabla u_t\|_{L^2}^2 + (1+\|\nabla d\|_{L^\infty}^2)\|\nabla d_t\|_{L^2}^2\right].$$

Thus

$$\frac{d}{dt}\int |\nabla d_t|^2 dx + \int |d_{tt}|^2 dx \le C\left[\|\nabla u_t\|_{L^2}^2 + (1+\|\nabla d\|_{L^\infty}^2)\|\nabla d_t\|_{L^2}^2\right]. \tag{2.42}$$

Multiplying (2.42) by $\frac{\mu}{2C}$ and adding the resulting inequality into (2.39), applying Lemma 2.3 and Lemma 2.4, and then employing Gronwall's inequality and applying (2.11) (with u replaced by u_t), we obtain

$$\sup_{0\le t<T_*}\int (\rho|u_t|^2 + |\nabla d_t|^2)\,dx + \int_0^{T_*}\int_\Omega (|\nabla u_t|^2 + |d_{tt}|^2)\,dx\,dt \le C.$$

To estimate $\nabla^3 d$ in $L_t^\infty L_x^2(\Omega\times[0,T_*])$, first observe that by Nirenberg's interpolation inequality, we have

$$\|\nabla d\|_{L^\infty} \lesssim \|\nabla d\|_{L^2} + \|\nabla d\|_{L^2}^{\frac14}\|\nabla^3 d\|_{L^2}^{\frac34}.$$

Putting this inequality into (2.31) and using $L_t^\infty L_x^2$-bounds of $\nabla d_t, \nabla u, \nabla^2 d$, we obtain that for any $0\le t<T_*$,

$$\|\nabla^3 d\|_{L^2}^2 \le C + C\|\nabla^3 d\|_{L^2}^{\frac32} \le \frac12\|\nabla^3 d\|_{L^2}^2 + C,$$

which clearly yields that

$$\sup_{0\le t<T_*}(\|\nabla d\|_{L^\infty} + \|\nabla^3 d\|_{L^2}) \le C.$$

To see $\nabla^2 u \in L_t^\infty L_x^2(\Omega\times[0,T_*])$, observe that the H^2-estimate on the equation (1.2) under (1.10), (2.7), and Lemma 2.4 imply that for any $0\le t<T_*$,

$$\begin{aligned}\|\nabla^2 u\|_{L^2} &\lesssim \|\nabla u\|_{L^2} + \|\mathcal{L}u\|_{L^2}\\ &\lesssim 1 + \|\sqrt{\rho}u_t\|_{L^2} + \|u\cdot\nabla u\|_{L^2} + \|\nabla(P(\rho))\|_{L^2} + \|\nabla^2 d\|_{L^2}\|\nabla d\|_{L^\infty}\\ &\lesssim 1 + \|\sqrt{\rho}u_t\|_{L^2} + \|u\|_{L^6}\|\nabla u\|_{L^3} \lesssim 1 + \|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\\ &\le \frac12\|\nabla^2 u\|_{L^2}^2 + C.\end{aligned}$$

In particular, we have

$$\sup_{0\le t<T_*}\|\nabla^2 u\|_{L^2} \le C.$$

The proof is now complete. □

Step 6. Estimate of $\nabla\rho$ in $L_t^\infty L_x^q(\Omega\times[0,T_*])$ for some $3<q\le 6$. With the estimates already established by the previous Lemmas, we then have the following Lemma.

Lemma 2.7 *Let $0<T_*<+\infty$ be the maximum time for a strong solution (ρ,u,d) to (1.1)-(1.3), (1.7) together with (1.8) or (1.9) or (1.10). If (1.11) and (2.1) hold, then*

$$\sup_{0\le t<T_*}\left(\max_{r=2,q}\|\rho_t\|_{L^r} + \|\rho\|_{W^{1,q}}\right) + \int_0^{T_*}\left(\|u\|_{D^{2,q}}^2 + \|\nabla^2 d_t\|_{L^2}^2 + \|\nabla^4 d\|_{L^2}^2\right)dt \le C, \tag{2.43}$$

for any $3<q\le 6$.

Proof. For $3 < q \le 6$, by the same calculations as in [15] Lemma 2.5, we have

$$(|\nabla\rho|^q)_t + \mathrm{div}\,(|\nabla\rho|^q u) + (q-1)|\nabla\rho|^q \mathrm{div}\, u + q|\nabla\rho|^{q-2}(\nabla\rho)^t\mathcal{D}(u)(\nabla\rho) + q\rho|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla\mathrm{div}\, u = 0,$$

which yields that for (1.8) or (1.9)

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C\left(\|\mathcal{D}(u)\|_{L^\infty}+1\right)\|\nabla\rho\|_{L^q} + C\|\nabla\mathrm{div}\, u\|_{L^q}, \tag{2.44}$$

and that for (1.10)

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C\left(\|\mathcal{D}(u)\|_{L^\infty}+1\right)\|\nabla\rho\|_{L^q} + C\|\nabla G\|_{L^q}, \tag{2.45}$$

where $G \equiv (2\mu+\lambda)\mathrm{div}\, u - P(\rho)$.

For boundary conditions (1.8) or (1.9), by using the L^p-estimate for the elliptic equation and (2.7) we have

$$\begin{aligned}\|\nabla^2 u\|_{L^q} \lesssim& \|\rho u_t\|_{L^q} + \|\rho u\cdot\nabla u\|_{L^q} + \|\nabla(P(\rho))\|_{L^q} + \|\Delta d\cdot\nabla d\|_{L^q} + 1\\ \lesssim& \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|u_t\|_{L^6}^{\frac{3q-6}{2q}} + \|u\|_{L^\infty}\|\nabla u\|_{L^q} + \|\nabla\rho\|_{L^q} + \|\nabla d\|_{L^\infty}\|\Delta d\|_{L^q} + 1\\ \lesssim& \|\nabla u_t\|_{L^2} + \|\nabla\rho\|_{L^q} + 1.\end{aligned} \tag{2.46}$$

Substituting (2.46) into (2.44), and using Gronwall's inequality, we obtain the bound $\sup\limits_{0\le t<T_*}\|\rho\|_{W^{1,q}}$ for the first two boundary conditions (1.8) and (1.9).

For boundary condition (1.10), we rewrite (1.2) as

$$\nabla G = \mu\nabla\times\mathrm{curl} u + \rho u_t + \rho u\cdot\nabla u + \Delta d\cdot\nabla d, \tag{2.47}$$

which yields that G satisfies

$$\begin{cases}\Delta G = \mathrm{div}\,\left(\rho u_t + \rho u\cdot\nabla u + \nabla d\cdot\Delta d\right), \text{ in } \Omega,\\ \nabla G\cdot\nu = -\rho(u\cdot\nabla)\nu\cdot u, \text{ on } \partial\Omega,\end{cases} \tag{2.48}$$

where we have used that $(\nabla\times\mathrm{curl} u)\cdot\nu|_{\partial\Omega} = 0$ $\big(\mathrm{curl} u\times\nu|_{\partial\Omega} = 0$ implies $(\nabla\times\mathrm{curl} u)\cdot\nu|_{\partial\Omega} = 0$, see [15] page 33 or [5, 6]$\big)$, $\nabla d\cdot\nu|_{\partial\Omega} = 0$ and $u\cdot\nu|_{\partial\Omega} = 0$.

Using the L^p-estimate for Neumann problem to the elliptic equation (2.48), we have

$$\begin{aligned}\|\nabla G\|_{L^q} \lesssim& \|\rho u_t\|_{L^q} + \|\rho u\cdot\nabla u\|_{L^q} + \|\nabla d\cdot\Delta d\|_{L^q} + \|\rho|u|^2\|_{C(\overline{\Omega})}\\ \lesssim& \|\nabla u_t\|_{L^2} + 1.\end{aligned} \tag{2.49}$$

Puting (2.49) into (2.45), we obtain the bound $\sup\limits_{0\le t<T_*}\|\rho\|_{W^{1,q}}$ by Gronwall's inequality.

For $r = 2$ or q, (1.1) implies

$$\begin{aligned}\|\rho_t\|_{L^r} &\lesssim \|u\|_{L^\infty}\|\nabla\rho\|_{L^r} + \|\rho\|_{L^\infty}\|\mathrm{div}\, u\|_{L^r}\\ &\lesssim \|\nabla u\|_{H^1}\|\nabla\rho\|_{L^r} + \|\rho\|_{L^\infty}\|\nabla u\|_{H^1} \le C.\end{aligned}$$

It follows from (2.41) that

$$\|\nabla^2 d_t\|_{L^2}^2 \lesssim \|d_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + 1.$$

This, with the help of (2.42), implies, after integrating over $[0, T_*]$,

$$\int_0^T \|\nabla^2 d_t\|_{L^2}^2 dt \le C. \tag{2.50}$$

Applying the standard L^2-estimate to (1.3), we have

$$\begin{aligned}
\|\nabla^4 d\|_{L^2}^2 &\lesssim \|\nabla^2 d_t\|_{L^2}^2 + \|\nabla^2(u\cdot\nabla d)\|_{L^2}^2 + \|\nabla^2(|\nabla d|^2 d)\|_{L^2}^2 + 1\\
&\lesssim \|\nabla^2 d_t\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla^3 d\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\nabla^2 u\|_{L^2}^2 + \|\nabla u\|_{L^6}^2\|\nabla^2 d\|_{L^3}^2 + 1\\
&\lesssim \|\nabla^2 d_t\|_{L^2}^2 + 1.
\end{aligned}$$

Integrating this inequality over $[0, T_*]$, and using (2.50), we get

$$\int_0^T \|\nabla^4 d\|_{L^2}^2 dt \le C.$$

By the bound on $\|\nabla\rho\|_{L^q}$ in (2.43), (2.46) and (2.37), we easily see that

$$\int_0^{T_*} \|\nabla^2 u\|_{L^q}^2\, dt \le C.$$

holds for (1.8) or (1.9). For the boundary condition (1.10), since $u\cdot\nu = 0$ on $\partial\Omega$, it follows from Bourguignon-Brezis [2] (see also [15] Lemma 2.3) and (2.7) that

$$\begin{aligned}
\|\nabla^2 u\|_{L^q} &\lesssim \|\nabla(\operatorname{div} u)\|_{L^q} + \|\nabla(\operatorname{curl} u)\|_{L^q} + \|\nabla u\|_{L^q}\\
&\lesssim \|\nabla G\|_{L^q} + \|\nabla\rho\|_{L^q} + \|\nabla u\|_{H^1} + \|\nabla(\operatorname{curl} u)\|_{L^q}\\
&\lesssim 1 + \|\nabla G\|_{L^q} + \|\nabla(\operatorname{curl} u)\|_{L^q}.
\end{aligned}$$

Since $(\nabla\times u)^\tau = 0$ on $\partial\Omega$, it follows from [33] that

$$\|\nabla(\operatorname{curl} u)\|_{L^q} \lesssim \|\operatorname{div}(\operatorname{curl} u)\|_{L^q} + \|\nabla\times\operatorname{curl} u\|_{L^q} \lesssim \|\nabla\times\operatorname{curl} u\|_{L^q},$$

where we have used the fact that $\operatorname{div}(\operatorname{curl} u) = 0$. On the other hand, since

$$\mu\nabla\times\operatorname{curl} u = \nabla G - \rho u_t - \rho u\cdot\nabla u - \Delta d\cdot\nabla d,$$

(2.49) implies

$$\|\nabla\times\operatorname{curl} u\|_{L^q} \lesssim 1 + \|\nabla u_t\|_{L^2}.$$

Putting these estimates together, we have

$$\|\nabla^2 u\|_{L^q} \lesssim 1 + \|\nabla u_t\|_{L^2} + \|\nabla G\|_{L^q},$$

which clearly implies

$$\int_0^{T_*} \|\nabla^2 u\|_{L^q}^2 \le C.$$

The proof is now complete. □

Step 7. Completion of proof of Theorem 1.3:

With the above established estimates, we obtain (2.2) and (2.3). This implies that T_* is not the maximum time of existence of strong solutions, which contradicts the definition of T_*. Therefore, (2.1) is false. The proof of Theorem 1.3 is now complete. □

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