MULTIPLE VALUED MAPS INTO A SEPARABLE HILBERT SPACE THAT ALMOST MINIMIZE THEIR p DIRICHLET ENERGY OR ARE SQUEEZE AND SQUASH STATIONARY

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ABSTRACT. Let $f : U \subseteq \mathbb{R}^m \to \mathscr{Q}_Q(\ell_2)$ be of Sobolev class $W^{1,p}$, $1 < p < \infty$. If f almost minimizes its p Dirichlet energy then f is Hölder continuous. If $p = 2$ and f is squeeze and squash stationary then f is in VMO.

1. INTRODUCTION

We let ℓ_2 denote the usual infinite dimensional separable Hilbert space. For any positive integer Q, let $\mathscr{Q}_Q(\ell_2)$ denote the space of unordered Q-tuples of elements in ℓ_2. Thus $\mathscr{Q}_Q(\ell_2)$ is the quotient of $(\ell_2)^Q$ under the action of the symmetric group given by

$$\sigma \cdot (u_1, \dots, u_Q) \mapsto (u_{\sigma(1)}, \dots, u_{\sigma(Q)}).$$

The equivalence class of $(u_1, \dots, u_Q)$ will be denoted throughout $u = \oplus_{i=1}^{Q} [\![u_i]\!]$. The distance between two points $u = \oplus_{i=1}^{Q} [\![u_i]\!]$ and $v = \oplus_{i=1}^{Q} [\![v_i]\!]$ in $\mathscr{Q}_Q(\ell_2)$ is defined by

$$\mathscr{G}(u, v) = \min_{\sigma \in S_Q} \sqrt{\sum_{i=1}^{Q} \|u_i - v_{\sigma(i)}\|^2}. \tag{1}$$

For the sake of simplicity, we will often use the notation

$$|u| := \mathscr{G}(u, Q[\![0]\!]).$$

If $e_1, \dots, e_m$ is an orthonormal basis of $\mathbb{R}^m$, then

$$\langle A, B \rangle := \sum_{i=1}^{m} \langle A(e_i), B(e_i) \rangle$$

defines a scalar product in $\mathrm{Hom}(\mathbb{R}^m, \ell^2)$, which is independent of the choice of $e_1, \dots, e_m$. The induced norm $\|A\|_{\mathrm{HS}} = \sqrt{\langle A, A \rangle}$ is the Hilbert-Schmidt norm of A. Whenever $D = \oplus_{i=1}^{Q} [\![D_i]\!] \in \mathscr{Q}_Q(\mathrm{Hom}(\mathbb{R}^m, \ell_2))$ is a unordered Q-tuple of linear maps, we define

$$|\!| D |\!| := \sqrt{\sum_{i=1}^{Q} \|D_i\|_{\mathrm{HS}}^2}.$$

We refer to [4] for the definition of multiple valued Sobolev space $W_p^1(U, \mathscr{Q}_Q(\ell_2))$, $U \subseteq \mathbb{R}^m$ open and $1 < p < \infty$. We merely mention that each $f \in W_p^1(U, \mathscr{Q}_Q(\ell_2))$ is approximately differentiable almost everywhere, its approximate differential Df being itself a Q-valued map $\mathbb{R}^m \to \mathscr{Q}_Q(\mathrm{Hom}(\mathbb{R}^m, \ell_2))$. The p-energy of f is

$$\mathscr{E}_p(f, U) := \left(\int_U |\!| Df |\!|^p \right)^{1/p} < \infty.$$

Existence of minimizers of the p-energy for the Dirichlet problem with Lipschitz boundary data is established in [4]. In the present paper we prove their (interior) Hölder continuity. In fact we work in the more general setting of almost minimizers

which we now recall. Let $\omega : [0,1] \to \mathbb{R}_+$ be a monotone increasing function with $\omega(0) = \lim_{r \downarrow 0^+} \omega(r) = 0$. Such a ω will be referred to as a *modular function*. For $1 < p < +\infty$ and an open set $U \subseteq \mathbb{R}^m$, we say that $u \in W_p^1(U, \mathscr{Q}_Q(\ell_2))$ is a (ω, p)-Dir-minimizing function, if for any ball $B(x,r) \subseteq U$,

$$\mathscr{E}_p(u, B(x,r)) \leqslant (1 + \omega(r))\mathscr{E}_p(v, B(x,r)), \tag{2}$$

whenever $v \in W_p^1(B(x,r), \mathscr{Q}_Q(\ell_2))$ and trace $u_{|B(x,r)} = \text{trace } v_{|B(x,r)}$. When $\omega = 0$ we simply call u a p-Dir-minimizing function. Our first result is the following.

1.1. THEOREM. — *For any modular function ω, there exists $\delta = \delta(p, m, \omega, Q) \in (0,1)$ such that any (ω, p)-Dir-minimizing function $u \in W_p^1(U, \mathscr{Q}_Q(\ell_2))$ is Hölder continuous in U with an exponent δ. Moreover, for any ball $B(x, 2r) \subseteq U$,*

$$\|u\|_{C^\delta(B(x,r))} \leqslant Cr^{p-m}\mathscr{E}_p(u, B(x, 2r)). \tag{3}$$

The Hölder continuity of 2-Dir-minimizing function into $\mathscr{Q}_Q(\ell_2^n)$, i.e. when the target space is finite dimensional, was first proved by F.J. Almgren in his seminal work [1] and proved again by C. De Lellis and E. Spadaro [2] [3] very recently. In order to establish our result we define comparison maps by means of homogeneous extensions of the local boundary data (Lemma 2.1) and an interpolation procedure (Theorem 3.1) inspired by S. Luckhaus [5].

The Dir-minimizing property of f leads to stationarity with respect to domain and range variations: We say f is, respectively, *squeeze* and *squash stationary*. When $Q \geqslant 2$ the squeeze and squash stationarity of f does not imply that it locally minimizes its energy. Thus the above regularity result does not apply to stationary maps. Here we assume $p = 2$ and we contribute the VMO regularity of squeeze and squash stationary maps $f \in W_2^1(U; \mathscr{Q}_Q(\ell_2))$, Proposition 8.3. We observe that the measure $\mu_f : A \mapsto \int_A |Df|^2$ is $m - 2$ monotonic, i.e. that $r \mapsto r^{2-m}\mu_f(B(x,r))$ is nondecreasing, $x \in U$. We also notice, as other authors have, that the monotonicity of frequency, established by F.J. Almgren, depends solely upon the stationary property of f. This in turn shows that $\Theta^{m-2}(\mu_f, x) = 0$ for all $x \in U$, which implies VMO via the Poincaré inequality. Furthermore $\lim_{r \to 0} r^{2-m}\mu_f(B(x,r)) = 0$ uniformly in $x \in U$ according to Dini's Theorem, which is in fact a kind of uniform VMO property. The continuity of f would ensue from a sufficiently fast decay of $\omega(r) = \sup_x r^{2-m}\mu_f(B(x,r))$. We establish the upper bound $\omega(r) \leqslant C|\log r|^{-\alpha}$ for some $0 < \alpha < 1$, which does not verify the suitable Dini growth condition.

2. RADIAL COMPARISON

The symbol $\oplus$ also denotes the concatenation operation

$$\mathscr{Q}_K(\ell_2) \times \mathscr{Q}_L(\ell_2) \to \mathscr{Q}_{K+L}(\ell_2).$$

The barycenter of u is

$$\boldsymbol{\eta}(u) := \frac{1}{Q}\sum_{i=1}^Q u_i \in \ell_2$$

and the translate of u by $a \in \ell_2$ is

$$\tau_a(u) := \bigoplus_{i=1}^Q [\![u_i - a]\!].$$

There are two crucial ingredients in the proof of the Theorem 1.1: a radial comparison lemma and an interpolation lemma. This section is devoted to the radial

comparison lemma. Let $B \subseteq \mathbb{R}^m$ denote the unit open ball. If $u \in W_p^1(B, \mathcal{Q}_Q(\ell_2))$, we let $\mathscr{C}(p, Q, u_{|\partial B})$ be

$$\mathscr{C}(p, Q, u_{|\partial B}) := \inf \left\{ \mathscr{E}_p(v, B) : v \in W_p^1(B, \mathcal{Q}_Q(\ell_2)), v = u \text{ on } \partial B \right\}. \qquad (4)$$

2.1. LEMMA. — *For any $M > 0$, there exists $\eta_0 = \eta_0(p, m, Q, M) > 0$ such that if $u \in W_p^1(B, \mathcal{Q}_Q(\ell_2))$ satisfies*

$$\operatorname{diam}^p(\operatorname{supp} \overline{u}) \leqslant M \mathscr{E}_p(u, \partial B), \qquad (5)$$

where $\overline{u} \in \mathcal{Q}_Q(\ell_2)$ is a mean of u on ∂B, then we have

$$\mathscr{C}(p, Q, u_{|\partial B}) \leqslant \left(\frac{1}{m - p} - 2\eta_0 \right) \mathscr{E}_p(u, \partial B), \qquad (6)$$

In particular, there exists $\varepsilon_0 = \varepsilon_0(p, n, Q, M) > 0$ such that if $u \in W_p^1(B, \mathcal{Q}_Q(\ell_2))$ is an (ω, p)-Dir-minimizing function and $\omega(1) \leqslant \varepsilon_0$, then we have

$$\mathscr{E}_p(u, B) \leqslant \left(\frac{1}{m - p} - \eta_0 \right) \mathscr{E}_p(u, \partial B). \qquad (7)$$

Proof. After multiplying u by a constant if necessary we may assume

$$\mathscr{E}_p(u, \partial B) = 1.$$

Abbreviate $g = u_{|\partial B}$. For $\alpha > 0$ to be chosen later, consider the radial competitor map

$$\forall x \in B, \quad v_\alpha(x) = \|x\|^\alpha g\left(\frac{x}{\|x\|} \right).$$

Since $v_\alpha = u$ on ∂B_1, it follows from the inequality (4) that

$$\mathscr{C}(p, Q, u_{|\partial B}) \leqslant \mathscr{E}_p(v_\alpha, B). \qquad (8)$$

Now we calculate $\mathscr{E}_p(v_\alpha, B)$ as follows. Using the polar coordinates $(r, \theta) \in (0, 1] \times \mathbb{S}^{m-1}$, we have

$$\left\vert Dv_\alpha \right\vert = r^{\alpha-1} \left\{ \alpha^2 |g(\theta)|^2 + \left\vert D_{\mathbb{S}^{m-1}} g(\theta) \right\vert^2 \right\}^{\frac{1}{2}}.$$

Hence

$$\mathscr{E}_p(v_\alpha, B)$$
$$= \int_0^1 r^{m-1+p(\alpha-1)} \int_{\partial B} \left(\alpha^2 |g(\theta)|^2 + \left\vert D_{\mathbb{S}^{m-1}} g(\theta) \right\vert^2 \right)^{\frac{p}{2}} d\mathscr{H}^{m-1} \theta \, dr$$
$$= \frac{1}{m - p + p\alpha} \int_{\partial B} \left(\alpha^2 |g(\theta)|^2 + \left\vert D_{\mathbb{S}^{m-1}} g(\theta) \right\vert^2 \right)^{\frac{p}{2}} d\mathscr{H}^{m-1} \theta \, dr. \qquad (9)$$

Next we want to estimate (9) from above. We distinguish between two cases.
Case 1: $p \in (1, 2]$. Since $0 < \frac{p}{2} \leqslant 1$, applying the elementary inequality

$$(a + b)^{\frac{p}{2}} \leqslant a^{\frac{p}{2}} + b^{\frac{p}{2}}, \ a, \ b > 0,$$

we have

$$\left(\alpha^2 |g(\theta)|^2 + \left\vert D_{\mathbb{S}^{m-1}} g(\theta) \right\vert^2 \right)^{\frac{p}{2}} \leqslant \alpha^p |g(\theta)|^p + \left\vert D_{\mathbb{S}^{m-1}} g(\theta) \right\vert^p,$$

and hence

$$\mathscr{E}_p(v_\alpha, B)$$
$$\leqslant \frac{1}{m - p + p\alpha} \left(\alpha^p \int_{\partial B} |g(\theta)|^p \, d\mathscr{H}^{m-1} \theta + \int_{\partial B} \left\vert D_{\mathbb{S}^{m-1}} g(\theta) \right\vert^p \, d\mathscr{H}^{m-1} \theta \right)$$
$$\leqslant \frac{1}{m - p + p\alpha} \left(1 + \alpha^p \int_{\partial B} |g(\theta)|^p \, d\mathscr{H}^{m-1} \theta \right). \qquad (10)$$

Case 2: $2 < p \leqslant n$. In this case, recall that the following inequality holds: there exists $C = C(p) > 0$ such that for any $\delta > 0$,

$$\forall a, b > 0, \quad (a+b)^{\frac{p}{2}} \leqslant (1+\delta)a^{\frac{p}{2}} + C\delta^{-(\frac{p}{2}-1)}b^{\frac{p}{2}}.$$

Applying this inequality, we have that for any $0 < \delta < 1$,

$$\begin{aligned}
&\mathscr{E}_p(v_\alpha, B) \\
&\leqslant \frac{C\delta^{-(\frac{p}{2}-1)}\alpha^p}{m-p+p\alpha} \int_{\partial B} |g(\theta)|^p \, d\mathscr{H}^{m-1}\theta \\
&\quad + \frac{1+\delta}{m-p+p\alpha} \int_{\partial B} \big| D_{\mathbb{S}^{m-1}} g(\theta) \big|^p \, d\mathscr{H}^{m-1}\theta \\
&\leqslant \frac{1}{m-p+p\alpha} \big(C\delta^{-(\frac{p}{2}-1)}\alpha^p \int_{\partial B} |g(\theta)|^p \, d\mathscr{H}^{m-1}\theta + (1+\delta) \big).
\end{aligned} \tag{11}$$

To estimate $\int_{\partial B} |g(\theta)|^p \, d\mathscr{H}^{m-1}\theta$, we argue as in [2] page 39. Write $\overline{u} =: \oplus_{i=1}^Q [\![\overline{u}_i]\!]$. Let $\hat{u} = \tau_{\boldsymbol{\eta}(\overline{u})}(u)$ and $\hat{g} = \tau_{\boldsymbol{\eta}(\overline{u})}(g)$ denote the translations of u and g by $\boldsymbol{\eta}(\overline{u})$. It is easy to see that $\hat{g} = \hat{u}|_{\partial B}$. It is clear that $\overline{\hat{g}} := \tau_{\boldsymbol{\eta}(\overline{u})}(\overline{u})$ is a mean of $\hat{g}$, and

$$|\overline{\hat{g}}|^2 = \sum_{i=1}^Q \|\overline{u}_i - \boldsymbol{\eta}(\overline{u})\|^2 \leqslant Q\mathrm{diam}^2 \, \mathrm{supp} \, \overline{u} \leqslant QM^2.$$

By the Poincaré inequality, we have

$$\begin{aligned}
\int_{\partial B} |\hat{g}|^p d\mathscr{H}^{m-1} &\leqslant 2^p \big(\int_{\partial B} \mathscr{G}(\hat{g}, \overline{\hat{g}})^p d\mathscr{H}^{m-1} + \int_{\partial B} |\overline{\hat{g}}|^p d\mathscr{H}^{m-1} \big) \\
&\leqslant C \, (\mathscr{E}_p(\hat{u}, \partial B) + M^p) \leqslant C(1 + M^p).
\end{aligned} \tag{12}$$

Since $\mathscr{C}(p, Q, u_{|\partial B})$ is invariant under translations of $u_{|\partial B}$ we conclude that

$$\begin{aligned}
\mathscr{C}(p, Q, u_{|\partial B}) &= \mathscr{C}(p, Q, \tau_{\boldsymbol{\eta}(\overline{u})}(u)_{|\partial B}) \\
&\leqslant \mathscr{E}_p(\tau_{\boldsymbol{\eta}(\overline{u})}(v_\alpha), B) \\
&\leqslant \mathscr{M}(m, p, M, \alpha, \delta)
\end{aligned}$$

where

$$\mathscr{M}(m, p, M, \alpha, \delta) := \begin{cases} \frac{1}{m-p+p\alpha} \big(1 + C(1+M^p)\alpha^p \big) & 1 < p \leqslant 2 \\ \frac{1}{m-p+p\alpha} \big((1+\delta) + C\delta^{-(p/2-1)}(1+M^p)\alpha^p \big) & 2 < p \leqslant n. \end{cases} \tag{13}$$

Now we need to show the following claim:
Claim. There exist $\alpha_0 > 0$ and $\delta_0 > 0$ depending only on p, m, M such that

$$\mathscr{M}(m, p, M, \alpha_0, \delta_0) < \frac{1}{m-p}. \tag{14}$$

To establish (14), we first consider the case $1 < p \leqslant 2$. By the definition, we have

$$\mathscr{M}(m, p, M, \alpha, \delta) = \frac{1}{m-p+p\alpha} \big(1 + C(1+M^p)\alpha^p \big)$$

is independent of δ. It is readily seen that

$$\mathscr{M}(m, p, M, \alpha, \delta) < \frac{1}{m-p} \Leftrightarrow C(1+M^p)\alpha^p < \frac{p}{m-p}\alpha. \tag{15}$$

Since $p > 1$ it is most obvious that (15) holds provided $0 < \alpha \leqslant \alpha_0(m, p, M)$ is small enough. Next we consider the case $2 < p \leqslant n$. In this case, we have

$$\mathscr{M}(m, p, M, \alpha, \delta) = \frac{1}{m-p+p\alpha} \big((1+\delta) + C\delta^{-(\frac{p}{2}-1)}(1+M^p)\alpha^p \big).$$

Again it is easy to see that

$$\mathscr{M}(m, p, M, \alpha, \delta) < \frac{1}{m-p} \Leftrightarrow \delta + C\delta^{-(\frac{p}{2}-1)}(1+M^p)\alpha^p < \frac{p}{m-p}\alpha. \qquad (16)$$

Letting $\delta = \alpha^2$ the left member of (16) becomes a constant multiple of α^2. Thus the inequality is verified provided $0 < \alpha \leqslant \alpha_0(m, p, M)$ is small enough. Combining together both cases, we see that there exists $\eta_0 > 0$ depending only on p, m, Q, M such that (6) holds. To show (7), first observe that the (ω, p)-Dir-minimality of u and (6) imply

$$\mathscr{E}_p(u, B) \leqslant (1 + \omega(1))\mathscr{C}(p, Q, u_{|\partial B})$$

$$\leqslant (1 + \varepsilon_0)\big(\frac{1}{m-p} - 2\eta_0\big)$$

$$\leqslant \frac{1}{m-p} - \eta_0 + \big(\frac{\varepsilon_0}{m-p} - \eta_0\big)$$

$$\leqslant \frac{1}{m-p} - \eta_0,$$

provided $\varepsilon_0 \leqslant (m-p)\eta_0$. Hence the proof is complete. $\qquad \square$

An immediate consequence of the radial comparison lemma is the Hölder continuity of (ω, p)-Dir-minimizing functions to ℓ_2, which implies Theorem 1.1 holds for the case $Q = 1$. More precisely, we have

2.2. COROLLARY. — *There exists $\eta_0 > 0$ depending only on m, p such that for any $u \in W_p^1(\partial B(0, r), \ell_2)$, one has*

$$\mathscr{C}(p, 1, u_{|\partial B(0,r)}) \leqslant \big(\frac{1}{m-p} - 2\eta_0\big) r \mathscr{E}_p(u, \partial B(0, r)). \qquad (17)$$

Proof. By scaling, it suffices to show (17) for $r = 1$. This follows from Lemma 2.1. In fact, for $Q = 1$, one has that the diameter of supp $\overline{u}$ equals to 0, *i.e.* $M = 0$ in the condition (5). Hence $\mathscr{M}(m, p, M, \alpha, \delta) = \mathscr{M}(m, p, \alpha, \delta)$, given by (13). We define $\alpha_0, \delta_0, \eta_0$ as in (14) when M is replaced by 0. Clearly, α_0 and δ_0 depend only on p, m and

$$\mathscr{M}(m, p, \alpha_0, \delta_0) \leqslant \frac{1}{m-p} - 2\eta_0.$$

This immediately implies (17). $\qquad \square$

2.3. COROLLARY. — *For any given modular function ω, there exists $\delta = \delta(p, \omega, m) \in (0, 1)$ such that any (ω, p)-Dir-minimizing function $u \in W_p^1(U, \ell_2)$ is Hölder continuous in U with an exponent δ. Moreover, for any ball $B(x, 2r) \subseteq U$, we have*

$$\|u\|_{C^\delta(B(x,r))} \leqslant C r^{p-m} \mathscr{E}_p(u, B(x, 2r)). \qquad (18)$$

Proof. Since $u \in W_p^1(U, \ell_2)$ is (ω, p)-Dir-minimizing in U, by (17) we have that for any ball $B(x, r) \subseteq U$,

$$\mathscr{E}_p(u, B(x, r)) \leqslant (1 + \omega(r))\mathscr{C}(p, 1, u_{|\partial B(x,r)})$$

$$\leqslant (1 + \omega(r))\big(\frac{1}{m-p} - 2\eta_0\big) r \mathscr{E}_p(u, \partial B(x, r)). \qquad (19)$$

Since $\lim_{r \downarrow 0} \omega(r) = 0$, there exists $r_0 > 0$ such that

$$\forall r \in (0, r_0], \quad (1 + \omega(r))\big(\frac{1}{m-p} - 2\eta_0\big) \leqslant \frac{1}{m-p} - \eta_0.$$

Thus we have that

$$\mathscr{E}_p(u, B(x, r)) \leqslant \big(\frac{1}{m-p} - \eta_0\big) r \mathscr{E}_p(u, \partial B(x, r)) \qquad (20)$$

holds for all $B(x, r) \subseteq U$ with $0 < r \leqslant r_0$. It is standard that integrating (20) over r yields that

$$\frac{1}{r^{m-p+\eta_0}} \mathscr{E}_p(u, B(x, r)) \leqslant \frac{1}{r_0^{m-p+\eta_0}} \mathscr{E}_p(u, B(x, r_0)), \tag{21}$$

for all balls $B(x, r_0) \subseteq U$ and $r \in (0, r_0]$. This, combined with the Morrey decay lemma for ℓ_2-valued functions, implies that $u \in C^{\eta_0/p}(U)$ and

$$\|u\|_{C^{\eta_0/p}(B(x,r))} \leqslant C \frac{1}{r_0^{m-p+\eta_0}} \mathscr{E}_p(u, B(x, r_0))$$

holds for $B(x, r_0) \subseteq U$ and $0 < r \leqslant r_0$. This completes the proof.   □

3. INTERPOLATION

In this section, we will establish an interpolation lemma. Such an interpolation property has been established in $\mathscr{Q}_Q(\ell_2^n)$ by F. Almgren [1]. However, the original proof by [1] is of extrinsic nature, *i.e.* it depends on the existence of a Lipschitz embedding of $\mathscr{Q}_Q(\ell_2^n)$ into ℓ_2^N for some large positive integer $N = N(m, n, Q)$. There seems to be no useful ersatz of this embedding in the case of $\mathscr{Q}_Q(\ell_2)$.

3.1. THEOREM. — *For any $1 < p \leqslant m$ and $\varepsilon > 0$, there exists $C = C(m, p, Q) > 0$ such that if $g_1, g_2 \in W_p^1(\partial B, \mathscr{Q}_Q(\ell_2))$, then there exists $h \in W_p^1(B \backslash B(0, 1 - \varepsilon), \mathscr{Q}_Q(\ell_2))$ such that*

$$\forall x \in \partial B, \quad h(x) = g_1(x), \qquad \forall x \in \partial B(0, 1 - \varepsilon), \quad h(x) = g_2\left(\frac{x}{1 - \varepsilon}\right), \tag{22}$$

and

$$\mathscr{E}_p(h, B \backslash B(0, 1 - \varepsilon)) \leqslant C\left(\varepsilon \sum_{i=1}^2 \mathscr{E}_p(g_i, \partial B) + \varepsilon^{1-p} \int_{\partial B} \mathscr{G}^p(g_1, g_2) \, d\mathscr{H}^{m-1}\right). \tag{23}$$

For $1 < p < +\infty$, set

$$m_p = \begin{cases} p - 1 & \text{if } p \in \mathbb{Z}_+ \\ \lfloor p \rfloor & \text{if } p \notin \mathbb{Z}_+, \end{cases} \tag{24}$$

where $\lfloor p \rfloor$ denotes the integer part of p.

Here we provide an intrinsic proof of Theorem 3.1, analogous to that by S. Luckhaus [5]. The rough idea is first to find a suitable triangulation of ∂B_1 and then do interpolations up to m_p-dimensional skeletons by first suitably approximating g_1, g_2 by Lipschitz maps and perform suitable Lipschitz extensions from 0-dimensional skeletons for all m_p-dimensional skeletons, here we need an important compactness theorem similar to Kolmogorov's theorem in our context. Finally we perform homogenous of degree zero extensions in skeletons of dimensions higher than m_p. We denote the unit interval by $I := [-1, 1]$.

For this, we need to establish the following lemma.

3.2. LEMMA. — *For any $1 < p < \infty$ and $\varepsilon > 0$, assume $m \leqslant m_p$. There exists a constant $C = C(p, Q) > 0$ such that if $g_1, g_2 \in W_p^1(I^m, \mathscr{Q}_Q(\ell_2))$, then there is a map $h \in W_p^1(I^m \times [-\varepsilon, \varepsilon], \mathscr{Q}_Q(\ell_2))$ such that*

$$\begin{cases} h(x, \ \varepsilon) = g_1(x) & x \in I^m, \\ h(x, -\varepsilon) = g_2(x) & x \in I^m, \end{cases} \tag{25}$$

and

$$\mathscr{E}_p(h, I^m \times [-\varepsilon, \varepsilon]) \leqslant C\left(\varepsilon \sum_{i=1}^2 \mathscr{E}_p(g_i, I^m) + \varepsilon^{1-p} \int_{I^m} \mathscr{G}^p(g_1, g_2) \, d\mathscr{H}^m\right). \tag{26}$$

Proof. We adapt some notations from [2] page 62. Let us introduce $I_k := [-1 - \frac{1}{k}, 1 + \frac{1}{k}]$. For sufficiently large $k \in \mathbb{Z}_+$, decompose I_k^m into the union of $(k+1)^m$ cubes $\{C_{k,l}\}$, $1 \leq l \leq (k+1)^m$, with disjoint interiors, side length equal to $2/k$ and faces parallel to the coordinate hyperplane. Let $x_{k,l}$ denote their centers so that

$$C_{k,l} = x_{k,l} + \left[-\frac{1}{k}, \frac{1}{k} \right]^m, \qquad 1 \leq l \leq (k+1)^m.$$

We also decompose I^m into the union of k^m cubes $\{D_{k,l}\}$, $1 \leq l \leq k^m$ and of side length $2/k$. Note that the centers of cubes in the collection $\{C_{k,l} : 1 \leq l \leq (k+1)^m\}$ are precisely the vertices of cubes in the collection $\{D_{k,l} : 1 \leq l \leq k^m\}$. Now we define two functions on the set of vertices $h_1^k, h_2^k : \{x_{k,1}, \ldots, x_{k,(k+1)^m}\} \to \mathscr{Q}_Q(\ell_2)$ by letting

$$h_i^k(x_{k,l}) = \text{a mean of } g_i \text{ on } \left(\{x_{k,l}\} + \left[-\frac{2}{k}, \frac{2}{k} \right]^m \right) \cap I^m, \ 1 \leq l \leq (k+1)^m,$$

for $i = 1, 2$. Now we want to extend both h_1^k and h_2^k from the set of vertices to the cube I^m. For each cube $D_{k,l}$, we let $V_{k,l}$ denote the set of vertices of $D_{k,l}$, consisting of 2^m points extracted from $\{x_{k,l'}\}_{1 \leq l' \leq (k+1)^m}$, and let $F_{k,l}^j$ denote the set of all faces of $D_{k,l}$ of dimension j for $1 \leq j \leq m-1$. On the first cube $D_{k,1}$ we claim that there exist Lipschitz functions $h_1^{k,1}, h_2^{k,1} : D_{k,1} \to \mathscr{Q}_Q(\ell_2)$ that are extensions of $h_{1|V_{k,1}}^k, h_{2|V_{k,1}}^k : V_{k,1} \to \mathscr{Q}_Q(\ell_2)$, respectively, such that for $i = 1, 2$,

$$\text{Lip}\left(h_i^{k,1}, F\right) \leq C\text{Lip}\left(h_i^k, V_{k,1} \cap F\right), \ \forall F \in F_{k,1}^j, 1 \leq j \leq m. \tag{27}$$

In particular, for $j = m$, (27) yields

$$\text{Lip}\left(h_i^{k,1}, D_{k,1}\right) \leq C\text{Lip}\left(h_i^k, V_{k,1}\right). \tag{28}$$

Indeed, we apply finitely many times Theorem 2.4.3 of [4]: first extend the maps $h_{i|V_{k,1}}^k$ to *each* edge in $F_{k,1}^1$ (thus apply Theorem 2.4.3 Card $F_{k,1}^1$ times), then to *each* j-dimensional face in $F_{k,1}^j$, for $j = 2, \ldots, m$, by induction on j.

On all those cubes $D_{k,l}$ that are adjacent to $D_{k,1}$ (i.e., share a common $(m-1)$-dimensional face $\partial D_{k,l} \cap \partial D_{k,1}$ with $D_{k,1}$), we proceed similarly to find Lipschitz functions $h_1^{k,l}, h_2^{k,l} : D_{k,l} \to \mathscr{Q}_Q(\ell_2)$ that are Lipschitz extensions of $\widetilde{h_1^{k,l}}, \widetilde{h_2^{k,l}} : (\partial D_{k,l} \cap \partial D_{k,1}) \cup V_{k,l} \to \mathscr{Q}_Q(\ell_2)$ respectively, where

$$\widetilde{h_i^{k,l}}(x) = \begin{cases} h_i^{k,1}(x) & \text{if } x \in \partial D_{k,l} \cap \partial D_{k,1} \\ h_i^k(x) & \text{if } x \in V_{k,l} \end{cases}$$

for $i = 1, 2$. Moreover, for $i = 1, 2$ the following estimates hold:

$$\text{Lip}(h_i^{k,l}, F) \leq C\text{Lip}(h_i^k, V_{k,l} \cap F), \ \forall F \in F_{k,l}^j, \ 1 \leq j \leq m, \tag{29}$$

In particular, for $j = m$, (29) yields

$$\text{Lip}\left(h_i^{k,l}, D_{k,l}\right) \leq C\text{Lip}(h_i^k, V_{k,l}). \tag{30}$$

Repeating the above procedure with all subcubes $D_{k,l}$ for $1 \leq l \leq k^m$, we will eventually obtain two Lipschitz functions $\mathbf{h}_1^k, \mathbf{h}_2^k : I^m \to \mathscr{Q}_Q(\ell_2)$ such that

$$\mathbf{h}_1^k(x_{k,l}) = h_1^k(x_{k,l}); \ \mathbf{h}_2^k(x_{k,l}) = h_2^k(x_{k,l}), \ \forall 1 \leq l \leq (k+1)^m, \tag{31}$$

and for $i = 1, 2$,

$$\text{Lip}(\mathbf{h}_i^k, D_{k,l}) \leq C\text{Lip}(h_i^k, V_{k,l}), \quad \forall 1 \leq l \leq k^m. \tag{32}$$

Now we want to find a Lipschitz map $h^k : I^m \times [-\varepsilon, \varepsilon] \to \mathscr{Q}_Q(\ell_2)$ that is a suitable extension of

$$(x, \varepsilon) \in I^m \times \{\varepsilon\} \mapsto \mathbf{h}_1^k(x), \quad (x, -\varepsilon) \in I^m \times \{-\varepsilon\} \mapsto \mathbf{h}_2^k(x).$$

This can be done as follows. For each cube $D_{k,l}$, $1 \leqslant l \leqslant k^m$, we let $h^{k,l}$: $D_{k,l} \times [-\varepsilon, \varepsilon] \to \mathscr{Q}_Q(\ell_2)$ be a Lipschitz extension of

$$(x, \varepsilon) \in D_{k,l} \times \{\varepsilon\} \mapsto \mathbf{h}_1^k(x), \quad (x, -\varepsilon) \in D_{k,l} \times \{-\varepsilon\} \mapsto \mathbf{h}_2^k(x)$$

such that

- if two cubes $D_{k,l}$ and $D_{k,l'}$ $(l \neq l')$ share a common $(m-1)$-dimensional face $\partial D_{k,l} \cap \partial D_{k,l'}$, then $h^{k,l}$ and $h^{k,l'}$ take the same value on the common m-dimensional face $(\partial D_{k,l} \cap \partial D_{k,l'}) \times [-\varepsilon, \varepsilon]$.
- the following inequalities hold:

$$\mathrm{Lip}(h^{k,l}, \partial D_{k,l} \times [-\varepsilon, \varepsilon]) \leqslant C \left(\mathrm{Lip}(\mathbf{h}_1^k, \partial D_{k,l}) + \mathrm{Lip}(\mathbf{h}_2^k, \partial D_{k,l}) \right)$$
$$+ C\varepsilon^{-1} \sum_{x \in V_{k,l}} \mathscr{G}(\mathbf{h}_1^k(x), \mathbf{h}_2^k(x))$$
$$\leqslant C \left(\mathrm{Lip}(h_1^k, V_{k,l}) + \mathrm{Lip}(h_2^k, V_{k,l}) \right)$$
$$+ C\varepsilon^{-1} \sum_{x \in V_{k,l}} \mathscr{G}(h_1^k(x), h_2^k(x)). \tag{33}$$

and

$$\mathrm{Lip}(h^{k,l}, D_{k,l} \times [-\varepsilon, \varepsilon]) \leqslant C\mathrm{Lip}(h^{k,l}, \partial(D_{k,l} \times [-\varepsilon, \varepsilon]))$$
$$\leqslant C \left(\mathrm{Lip}(\mathbf{h}_1^k, D_{k,l}) + \mathrm{Lip}(\mathbf{h}_2^k, D_{k,l}) \right)$$
$$+ C\mathrm{Lip}(h^{k,l}, \partial D_{k,l} \times [-\varepsilon, \varepsilon])$$
$$\leqslant C \left(\mathrm{Lip}(h_1^k, V_{k,l}) + \mathrm{Lip}(h_2^k, V_{k,l}) \right)$$
$$+ C\varepsilon^{-1} \sum_{x \in V_{k,l}} \mathscr{G}(h_1^k(x), h_2^k(x)). \tag{34}$$

Finally we define $h^k : I^m \times [-\varepsilon, \varepsilon] \to \mathscr{Q}_Q(\ell_2)$ by simply letting

$$h^k_{|D_{k,l} \times [-\varepsilon, \varepsilon]} = h^{k,l}, \quad \forall \, 1 \leqslant l \leqslant k^m.$$

Obviously h^k satisfies that $h^k(x, \varepsilon) = \mathbf{h}_1^k(x)$ and $h^k(x, -\varepsilon) = \mathbf{h}_2^k(x)$ for $x \in I^m$.

We want to estimate the terms in the right hand side of (34). It is easy to see that for any $1 \leqslant l \leqslant k^m$,

$$\mathrm{Lip}(h_1^k, V_{k,l}) \leqslant C \max \left\{ k\mathscr{G}(h_1^k(x), h_1^k(x')) : \; x, x' \in V_{k,l} \text{ are two adjacent vertices} \right\}.$$

On the other hand, for two adjacent vertices $x, x' \in V_{k,l}$, by the definition of $h_1^k(x)$ and $h_1^k(x')$ and Poincaré's inequality we have

$$\mathscr{G}(h_1^k(x), h_1^k(x'))^p$$
$$\leqslant Ck^m \int_{((\{x\} + [-2k^{-1}, 2k^{-1}]^m) \cap (\{x'\} + [-2k^{-1}, 2k^{-1}]^m)) \cap I^m} \mathscr{G}(h_1^k(x), h_1^k(x'))^p$$
$$\leqslant Ck^m \int_{(\{x\} + [-2k^{-1}, 2k^{-1}]^m) \cap I^m} \mathscr{G}(g_1(y), h_1^k(x))^p dy$$
$$+ Ck^m \int_{(\{x'\} + [-2k^{-1}, 2k^{-1}]^m) \cap I^m} \mathscr{G}(g_1(y), h_1^k(x'))^p dy$$
$$\leqslant Ck^{m-p} \mathscr{E}_p(g_1, (\{x\} + [-2k^{-1}, 2k^{-1}]^m) \cap I^m)$$
$$+ Ck^{m-p} \mathscr{E}_p(g_1, (\{x'\} + [-2k^{-1}, 2k^{-1}]^m) \cap I^m)$$
$$\leqslant Ck^{m-p} \mathscr{E}_p(g_1, \widetilde{D_{k,l}} \cap I^m),$$

where $\widetilde{D_{k,l}}$ denotes cube:

$$\widetilde{D_{k,l}} = \{x_{k,l}\} + \left[-\frac{3}{k}, \frac{3}{k} \right]^m.$$

Thus we have

$$\text{Lip}^p(h_1^k, V_{k,l}) \leqslant Ck^m \mathscr{E}_p(g_1, \widetilde{D_{k,l}} \cap I^m). \tag{35}$$

Similarly, we have

$$\text{Lip}^p(h_2^k, V_{k,l}) \leqslant Ck^m \mathscr{E}_p(g_2, \widetilde{D_{k,l}} \cap I^m). \tag{36}$$

While for $x \in V_{k,l}$, we have

$$\begin{aligned}
\mathscr{G}(h_1^k(x), h_2^k(x))^p &\leqslant C \max_{y \in D_{k,l}} \left(\mathscr{G}(\mathbf{h}_1^k(y), h_1^k(x))^p + \mathscr{G}(\mathbf{h}_2^k(y), h_2^k(x))^p \right) \\
&\quad + Ck^m \int_{D_{k,l}} \mathscr{G}(\mathbf{h}_1^k(y), \mathbf{h}_2^k(y))^p dy \\
&\leqslant C \left(k^{-p} \text{Lip}^p(\mathbf{h}_1^k, D_{k,l}) + k^{-p} \text{Lip}^p(\mathbf{h}_2^k, D_{k,l}) \right] \\
&\quad + Ck^m \int_{D_{k,l}} \mathscr{G}(\mathbf{h}_1^k(y), \mathbf{h}_2^k(y))^p dy \\
&\leqslant Ck^{m-p} \left(\mathscr{E}_p(g_1, \widetilde{D_{k,l}} \cap I^m) + \mathscr{E}_p(g_2, \widetilde{D_{k,l}} \cap I^m) \right) \\
&\quad + Ck^m \int_{D_{k,l}} \mathscr{G}(\mathbf{h}_1^k(y), \mathbf{h}_2^k(y))^p dy. \tag{37}
\end{aligned}$$

With all these estimates, we can bound $\mathscr{E}_p\left(h^k, I^m \times [-\varepsilon, \varepsilon]\right)$ as follows:

$$\begin{aligned}
&\mathscr{E}_p(h^k, I^m \times [-\varepsilon, \varepsilon]) \\
&= \sum_{l=1}^{k^m} \mathscr{E}_p(h^{k,l}, D_{k,l} \times [-\varepsilon, \varepsilon]) \\
&\leqslant \frac{C\varepsilon}{k^m} \sum_{l=1}^{k^m} \text{Lip}^p(h^{k,l}, D_{k,l} \times [-\varepsilon, \varepsilon]) \\
&\leqslant C\varepsilon(1 + \varepsilon^{-p} k^{-p}) \sum_{l=1}^{k^m} \left(\mathscr{E}_p(g_1, \widetilde{D_{k,l}} \cap I^m) + \mathscr{E}_p(g_2, \widetilde{D_{k,l}} \cap B) \right) \\
&\quad + C\varepsilon^{1-p} \sum_{l=1}^{k^m} \int_{D_{k,l}} \mathscr{G}(\mathbf{h}_1^k(y), \mathbf{h}_2^k(y))^p dy \\
&\leqslant C\varepsilon(1 + (k\varepsilon)^{-p}) \left(\mathscr{E}_p(g_1, I^m) + \mathscr{E}_p(g_2, I^m) \right) \\
&\quad + C\varepsilon^{1-p} \int_{I^m} \mathscr{G}(\mathbf{h}_1^k(y), \mathbf{h}_2^k(y))^p dy. \tag{38}
\end{aligned}$$

Observe that we have for $i = 1, 2$,

$$\begin{aligned}
\int_{I^m} \mathscr{G}(\mathbf{h}_i^k(y), g_i(y))^p dy &\leqslant C \sum_{l=1}^{k^m} \int_{D_{k,l}} \left(\mathscr{G}(\mathbf{h}_i^k, h_i^k(x_{k,l}))^p + \mathscr{G}(h_i^k(x_{k,l}), g_i)^p \right) \\
&\leqslant Ck^{-p} \sum_{l=1}^{k^m} \mathscr{E}_p(g_i, \widetilde{D_{k,l}} \cap I^m) \\
&\quad + C \sum_{l=1}^{k^m} \int_{(\{x\} + [-2k^{-1}, 2k^{-1}]) \cap I^m} \mathscr{G}(h_i^k(x_{k,l}), g_i)^p \\
&\leqslant Ck^{-p} \sum_{l=1}^{k^m} \mathscr{E}_p(g_i, \widetilde{D_{k,l}} \cap I^m) \\
&\leqslant Ck^{-p} \mathscr{E}(g_i, I^m),
\end{aligned}$$

which converges to 0 as k goes to ∞. We would establish (26) if we can show that there exists $h \in W_p^1(I^m \times [-\varepsilon, \varepsilon], \mathcal{Q}_Q(\ell_2))$ such that after passing to possible subsequences, $h^k \to h$ in $L^p(I^m \times [-\varepsilon, \varepsilon])$.

To see this, since $p > m$, it follows from Sobolev's embedding theorem that $g_i \in C^{1-m/p}(I^m, \mathcal{Q}_Q(\ell_2))$ for $i = 1, 2$. If we define

$$\mathscr{C}_i = \left\{ y \in \ell_2 \; : \; y \in \mathrm{supp}(g_i(x)) \text{ for some } x \in I^m \right\}, \; i = 1, 2,$$

then $\mathscr{C}_i \subseteq \ell_2$ is a compact set for $l = 1, 2$. Hence

$$\mathscr{C} := \mathscr{C}_1 \cup \mathscr{C}_2$$

is also a compact set in ℓ_2. Let $\mathscr{D} \subseteq \ell_2$ be the convex hull of $\mathscr{C} \cup \{0\}$. Then both $\mathscr{D}$ and $\mathcal{Q}_Q(\mathscr{D})$ are compact sets. By checking the proof of the Lipschitz extension theorem, we can see that $h^k(I^m \times [-\varepsilon, \varepsilon]) \subseteq \mathcal{Q}_Q(\mathscr{D})$. Now we can apply [4] Theorem 4.8.2 to conclude that there exists $h \in W_p^1(I^m \times [-\varepsilon, \varepsilon], \mathcal{Q}_Q(\ell_2))$ and integers $k_1 < k_2 < \cdots$ such that

$$\lim_{j \to \infty} \int_{I^m \times [-\varepsilon, \varepsilon]} \mathscr{G}(h^{k_j}, h)^p = 0. \tag{39}$$

By the lower semicontinuity of $\mathscr{E}_p$, we have

$$\mathscr{E}_p(h, I^m \times [-\varepsilon, \varepsilon]) \leqslant \liminf_{j \to \infty} \mathscr{E}_p(h^{k_j}, I^m \times [-\varepsilon, \varepsilon]). \tag{40}$$

Since $h^k(\cdot, \varepsilon) = \mathbf{h}_1^k \to g_1$ in $L^p(I^m)$ and $h^k(\cdot, -\varepsilon) = \mathbf{h}_2^k \to g_2$ in $L^p(I^m)$ as $k \to \infty$, it follows from [4] Theorem 4.7.3 that $h(\cdot, \varepsilon) = g_1$ on $B \times \{\varepsilon\}$ and $h(\cdot, -\varepsilon) = g_2$ on $B \times \{-\varepsilon\}$ in the sense of traces. Finally, by sending $k = k_j$ to ∞ in (38), we see that h satisfies the inequality (26). The proof is now complete. $\qquad\square$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. The idea is motivated by Luckhaus [5]. We first decompose ∂B into "cells" of diameter ε as follows. Since B is bilipschitz isomorphic to the open unit cube its boundary can be decomposed into open cubes of side length less than ε and dimension ranging between 0 and $m - 1$. This gives a partition

$$\partial B = \bigcup_{j=1}^{m-1} \bigcup_{i=1}^{k_j} e_i^j, \; e_i^j \cap e_{i'}^{j'} = \varnothing \text{ if } i \neq i' \text{ or } j \neq j',$$

and for each e_i^j we have a bilipschitz isomorphism

$$\Phi_i^j : e_i^j \to B^j(0, \varepsilon) \text{ with } \|\nabla \Phi_i^j\|_{L^\infty} + \|\nabla(\Phi_i^j)^{-1}\|_{L^\infty} \leqslant c(m).$$

Here B_ε^j is the j-dimensional open ball centered at 0 and of radius ε.

Denote by $Q_j := \cup_{i=1}^{k_j} e_i^j$ the union of j cells. We next use the fact that for any nonnegative measurable function f

$$\int_{SO(m)} d\sigma \int_{\sigma(Q_j)} f \, d\mathscr{H}^j = \frac{\mathscr{H}^j(Q_j)}{\mathscr{H}^{m-1}(\partial B)} \int_{\partial B} f \, d\mathscr{H}^{m-1},$$

where $d\sigma$ is the Haar measure on $SO(m)$. By Fubini's theorem, we can further choose a rotation $\sigma \in SO(m)$ such that for all j

$$\mathscr{E}_p(g_1, \sigma(Q_j)) + \mathscr{E}_p(g_2, \sigma(Q_j)) + \int_{\sigma(Q_j)} \varepsilon^{-p} \mathscr{G}^p(g_1, g_2) d\mathscr{H}^j \leqslant c(m) \mathbf{K}^p \varepsilon^{j+1-m}, \tag{41}$$

where $\mathbf{K} > 0$ is the constant defined by

$$\mathbf{K}^p := \mathscr{E}_p(g_1, \partial B) + \mathscr{E}_p(g_2, \partial B) + \int_{\partial B} \varepsilon^{-p} \mathscr{G}(g_1, g_2)^p d\mathscr{H}^{m-1}.$$

To avoid extra notation, assume $\sigma = \mathrm{id}$. Now decompose $B\backslash B(0, 1-\varepsilon)$ into cells

$$\hat{e}_i^j = \left\{z \in \mathbb{R}^n \ : \ \frac{z}{\|z\|} \in e_i^j, \ 1-\varepsilon < \|z\| < 1\right\}, \ 0 \leqslant j \leqslant m-1, \ 1 \leqslant i \leqslant k_j.$$

For $j \leqslant m_p$, we use the fact that $\hat{e}_i^j$ is bilipschitz isomorphic to $B^j(0, \varepsilon) \times [-\varepsilon, \varepsilon]$, i.e. there exist

$$\Psi_i^j : \hat{e}_i^j \to B^j(0, \varepsilon) \times [-\varepsilon, \varepsilon] \text{ with } \|\nabla \Psi_i^j\|_{L^\infty} + \|\nabla (\Psi_i^j)^{-1}\|_{L^\infty} \leqslant c(m).$$

We can apply Lemma 3.2 to find an extension map $h_i^j \in W_p^1\big(\hat{e}_i^j, \mathscr{Q}_Q(\ell_2)\big)$ such that

$$h_i^j(x, 1) = g_1(x), \ h_i^j\big((1-\varepsilon)x\big) = g_2(x), \ \forall x \in e_i^j, \tag{42}$$

and

$$\mathscr{E}_p(h_i^j, \hat{e}_i^j) \leqslant C\varepsilon\Big(\mathscr{E}_p(g_1, e_i^j) + \mathscr{E}_p\big(g_2, e_i^j\big) + \int_{e_i^j} \varepsilon^{-p}\mathscr{G}^p(g_1(x), g_2(x))d\mathscr{H}^j\Big). \tag{43}$$

Moreover, we can see from the proof of Lemma 3.2 that if for $1 \leqslant i < i' \leqslant k_j$, the two cells $\hat{e}_i^j$ and $\hat{e}_{i'}^j$ share a common j-face, then one can ensure from the construction of extensions that $h_i^j = h_{i'}^j$ on $\partial\hat{e}_i^j \cap \partial\hat{e}_{i'}^j$ in the sense of traces. Denote $\hat{Q}^j = \cup_{i=1}^{k_j}\hat{e}_i^j$. Then we can glue all h_i^j together by letting $h_{|\hat{e}_i^j}^j = h_i^j$ for $1 \leqslant i \leqslant k_j$ to obtain an extension map $h^j \in W_p^1\big(\hat{Q}^j, \mathscr{Q}_Q(\ell_2)\big)$ such that

$$h^j(x, 1) = g_1(x), \ h^j\big(\frac{x}{1-\varepsilon}\big) = g_2(x), \ \forall x \in Q^j, \tag{44}$$

and

$$\mathscr{E}_p(h^j, \hat{Q}^j) \leqslant C\varepsilon\Big(\mathscr{E}_p(g_1, Q^j) + \mathscr{E}_p(g_2, Q^j) + \int_{Q^j} \varepsilon^{-p}\mathscr{G}^p(g_1(x), g_2(x))d\mathscr{H}^j\Big). \tag{45}$$

For $j \geqslant m_p + 1$, we use the fact that $\hat{e}_i^j$ is bilipschitz isomorphic to $B^{j+1}(0, \varepsilon)$, i.e. there exist

$$F_i^j : \hat{e}_i^j \to B^{j+1}(0, \varepsilon) \text{ with } \|\nabla F_i^j\|_{L^\infty} + \|\nabla (F_i^j)^{-1}\|_{L^\infty} \leqslant c(m).$$

Since $j + 1 \geqslant m_p + 2 > p$, we can extend h_i^j inductively from the boundary, homogeneous of degree 0:

$$h_i^j\big((F_i^j)^{-1}(z)\big) = h_i^j\big((F_i^j)^{-1}(\frac{\varepsilon z}{\|z\|})\big), \ z \in B^{j+1}(0, \varepsilon).$$

Then we have

$$\mathscr{E}_p\big(h_i^j, \hat{e}_i^j\big) \leqslant C\Big(\int_0^\varepsilon \big(\frac{r}{\varepsilon}\big)^{(j+1)-1-p} dr\Big)\mathscr{E}_p\big(h_i^j, \partial\hat{e}_i^j\big) \leqslant C\varepsilon\mathscr{E}_p\big(h_i^j, \partial\hat{e}_i^j\big). \tag{46}$$

By adding (46) over all $1 \leqslant i \leqslant k_j$ and applying (41), we then obtain that for any $j \geqslant m_p + 1$, h^j satisfies (44) and

$$\mathscr{E}_p(h^j, \hat{Q}^j) \leqslant C\varepsilon^{j+2-m}\mathbf{K}^p. \tag{47}$$

Since $\hat{Q}^{m-1} = B\backslash B(0, 1-\varepsilon)$, if we define $h = h^{m-1}$ then h satisfies

$$h_{|\partial B} = g_1, \ h_{|\partial B(0, 1-\varepsilon)} = g_2\big(\frac{\cdot}{1-\varepsilon}\big),$$

and

$$\mathscr{E}_p(h, B\backslash B(0, 1-\varepsilon)) \leqslant C\varepsilon\mathbf{K}^p.$$

Hence the conclusions hold. The proof is complete. $\qquad\square$

4. Proof of Theorem 1.1

In this section, we will provide a proof of Theorem 1.1 by induction on $Q \in \mathbb{Z}_+$. The idea is similar to that by Almgren [1], but we follow closely the presentation by [2].

For $v = \oplus_{i=1}^{Q} [\![v_i]\!] \in \mathscr{Q}_Q(\ell_2)$, we define the diameter and splitting distance of v as

$$d(v) = \max_{1 \leqslant i,j \leqslant Q} \|v_i - v_j\|, \quad s(v) = \min_{1 \leqslant i,j \leqslant Q} \{|v_i - v_j| : v_i \neq v_j\}.$$

If $v = Q[\![v_0]\!]$ for some $v_0 \in \ell_2$, then we define $s(v) = +\infty$.

First we need to construct a Lipschitz retraction map from $\mathscr{Q}_Q(\ell_2)$ to $B_{\mathscr{G}}(v, r)$, with Lipschitz norm no more than 1.

4.1. PROPOSITION. — *For $v \in \mathscr{Q}_Q(\ell_2)$ and $0 < r < s(v)/4 < \infty$, there exists a Lipschitz map $\Phi : \mathscr{Q}_Q(\ell_2) \to B_{\mathscr{G}}(v, r)$ such that $\Phi(u) = u$ for any $u \in B_{\mathscr{G}}(u, r)$ and $\mathrm{Lip}(\Phi) \leqslant 1$.*

Proof. Write $v = \oplus_{j=1}^{J} k_j [\![v_j]\!]$ such that $J \geqslant 2$ and $\|v_i - v_j\| > 4r$ for $i \neq j$. If $\mathscr{G}(u, v) < 2r$, then we have that $u = \oplus_{j=1}^{J} u_j$ with $u_j = \oplus_{l=1}^{k_j} [\![u_{l,j}]\!] \in B_{\mathscr{G}}(k_j[\![v_j]\!], 2r) \subseteq \mathscr{Q}_{k_j}(\ell_2)$ for $1 \leqslant j \leqslant J$. Now we can define a Lipschitz retraction map $\Phi : \mathscr{Q}_Q(\ell_2) \to B_{\mathscr{G}}(v, r)$ by letting

$$\Phi(u) = \begin{cases} \bigoplus_{j=1}^{J} \bigoplus_{l=1}^{k_j} [\![\dfrac{2r - \mathscr{G}(u,v)}{\mathscr{G}(u,v)}(u_{l,j} - v_j) + v_j]\!], & u \in B_{\mathscr{G}}(v, 2r) \backslash B_{\mathscr{G}}(v, r), \\ v, & u \in \mathscr{Q}_Q(\ell_2) \backslash B_{\mathscr{G}}(v, 2r), \\ u, & u \in B_{\mathscr{G}}(v, r). \end{cases} \quad (48)$$

It is readily seen that Φ is an identity map in $B_{\mathscr{G}}(v, r)$ and satisfies

$$\mathscr{G}(\Phi(u_1), \Phi(u_2)) \leqslant \mathscr{G}(u_1, u_2), \quad \forall u_1, u_2 \in \mathscr{Q}_Q(\ell_2).$$

Thus Φ has Lipschitz norm at most 1. □

4.2. LEMMA. — *For any $0 < \varepsilon < 1$, set $\beta(\varepsilon, Q) = \left(\dfrac{\varepsilon}{3}\right)^{3^Q}$. Then, for any $P \in \mathscr{Q}_Q(\ell_2)$ with $s(P) < +\infty$, there exists a point $\widetilde{P} \in \mathscr{Q}_Q(\ell_2)$ such that*

$$\begin{cases} \beta(\varepsilon, Q)d(P) \leqslant s(\widetilde{P}) < +\infty, \\ \mathcal{G}_2(\widetilde{P}, P) \leqslant \varepsilon s(\widetilde{P}). \end{cases} \quad (49)$$

Proof. The proof can be done exactly in the same way as Lemma 3.8 of [2] page 35. Here we omit it. □

4.3. PROPOSITION. — *Assume $Q \geqslant 2$. There exists $\alpha(Q) > 0$ such that if $u \in W_p^1(\partial B(0, r), \mathscr{Q}_Q(\ell_2))$ satisfies that for some $P \in \mathscr{Q}_Q(\ell_2)$, $\mathscr{G}(u(x), P) \leqslant \alpha(Q)d(P)$ for $\mathscr{H}^{m-1}$ a.e. $x \in \partial B(0, r)$, then there exist $1 \leqslant K, L \leqslant Q - 1$ with $K + L = Q$ and two functions $v \in W_p^1(\partial B(0, r), \mathscr{Q}_K(\ell_2))$ and $w \in W_p^1(\partial B(0, r), \mathscr{Q}_L(\ell_2))$ so that $u = v \oplus w$ a.e. in $\partial B(0, r)$.*

Proof. Set $\varepsilon = 1/9$ and $\alpha(Q) = \varepsilon\beta(\varepsilon, Q) = \frac{1}{9}(27)^{-3^Q}$. From Lemma 4.2, we find a point $\widetilde{P} \in \mathscr{Q}_Q(\ell_2)$ satisfying (49). Hence we have that for $\mathscr{H}^{m-1}$ a.e. $x \in \partial B(0, r)$,

$$\mathscr{G}(u(x), \widetilde{P}) \leqslant \mathscr{G}(u(x), P) + \mathscr{G}(P, \widetilde{P}) \leqslant \alpha(Q)d(P) + \frac{s(\widetilde{P})}{9} \leqslant \frac{2s(\widetilde{P})}{9} < \frac{s(\widetilde{P})}{4}.$$

Since $s(\widetilde{P}) < +\infty$, there exists $2 \leqslant J \leqslant Q$ such that $\widetilde{P} = \oplus_{j=1}^{J} k_j [\![\widetilde{P}_j]\!] \in \mathscr{Q}_Q(\ell_2)$ with the $\widetilde{P}_j$'s all different. Therefore, there exists J functions

$$u_j : \partial B(0, r) \to B_{\mathscr{G}}(k_j[\![\widetilde{P}_j]\!], 2r) \subseteq \mathscr{Q}_{k_j}(\ell_2)$$

such that $u = \oplus_{j=1}^{J} u_j$ holds $\mathscr{H}^{m-1}$ a.e. in $\partial B(0,r)$. Since $u \in W_p^1(\partial B(0,r), \mathscr{Q}_Q(\ell_2))$, it follows that $u_j \in W_p^1(\partial B(0,r), \mathscr{Q}_{k_j}(\ell_2))$ for $1 \leqslant j \leqslant J$. The proof is complete. $\quad\square$

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. The key step is to establish the following decay property: there exists $\eta_0 > 0$ depending on p, m, Q such that for any $u \in W_p^1(\partial B(0,r), \mathscr{Q}_Q(\ell_2))$,

$$\mathscr{C}(p, Q, u_{|\partial B(0,r)}) \leqslant \left(\frac{1}{m-p} - 2\eta_0\right) r \mathscr{E}_p(u, \partial B(0,r)). \tag{50}$$

By scalings, one can see that if (50) holds for $r = 1$, then it holds for all $r > 0$. We will prove (50) based on an induction on Q. For $Q = 1$, it is clear that (50) follows from Corollary 2.3. Let $Q \geqslant 2$ be fixed and assume that (50) holds for every $Q^* < Q$. Assume, furthermore, that

$$d(\overline{u})^p > M\mathscr{E}_p(u, \partial B)$$

for some large constant $M > 1$, which will be chosen later. Apply Lemma 4.2 with $\varepsilon = \frac{1}{16}$ and $P = \overline{u}$, we obtain that there are $2 \leqslant J \leqslant Q$ and a point $\widetilde{P} = \oplus_{j=1}^{J} k_j [\![Q_j]\!] \in \mathscr{Q}_Q(\ell_2)$ such that

$$\beta d(\overline{u}) < s(\widetilde{P}) = \min\{\|Q_i - Q_j\| : \ i \neq j\}, \tag{51}$$

$$\mathscr{G}(\widetilde{P}, \overline{u}) \leqslant \frac{s(\widetilde{P})}{16}, \tag{52}$$

where $\beta = \beta(1/16, Q)$ is the constant given by Lemma 4.2. Let $\Phi : \mathscr{Q}_Q(\ell_2) \to B_{\mathscr{G}}(\widetilde{P}, s(\widetilde{P})/8)$ be the Lipschitz contraction map given by Proposition 4.1. For a small $\eta > 0$, define

$$h : x \in B(0, 1-\eta) \mapsto \Phi\left(u\left(\frac{x}{1-\eta}\right)\right) \in \mathscr{Q}_Q(\ell_2).$$

Then we have that $\Phi(u) \in W_p^1\left(\partial B(0, 1-\eta), B_{\mathscr{G}}(\widetilde{P}, s(\widetilde{P})/8)\right)$. Apply Proposition 4.3, we conclude that there exist $1 \leqslant K, L \leqslant Q-1$, with $K + L = Q$, and $h_1 \in W_p^1(\partial B(0, 1-\eta), \mathscr{Q}_K(\ell_2))$, $h_2 \in W_p^1(\partial B(0, 1-\eta), \mathscr{Q}_L(\ell_2))$ such that $h = h_1 \oplus h_2$ in $\partial B(0, 1-\eta)$. By the induction hypothesis, we have

$$\begin{cases} \mathscr{C}(p, K, h_{1|\partial B(0,1-\eta)}) \leqslant \left(\frac{1}{m-p} - 6\eta_0\right)(1-\eta)\mathscr{E}_p(h_1, \partial B(0, 1-\eta)), \\ \mathscr{C}(p, L, h_{2|\partial B(0,1-\eta)}) \leqslant \left(\frac{1}{m-p} - 6\eta_0\right)(1-\eta)\mathscr{E}_p(h_2, \partial B(0, 1-\eta)). \end{cases} \tag{53}$$

By (53), there exist $\hat{h}_1 \in W_p^1(B(0, 1-\eta), \mathscr{Q}_K(\ell_2))$ and $\hat{h}_2 \in W_p^1(B(0, 1-\eta), \mathscr{Q}_L(\ell_2))$ such that

$$\begin{cases} \mathscr{E}_p(\hat{h}_1, B(0, 1-\eta)) \leqslant \left(\frac{1}{m-p} - 6\eta_0\right)(1-\eta)\mathscr{E}_p(h_1, \partial B(0, 1-\eta)) + \eta_0 \mathscr{E}_p(u, \partial B), \\ \mathscr{E}_p(\hat{h}_2, B(0, 1-\eta)) \leqslant \left(\frac{1}{m-p} - 6\eta_0\right)(1-\eta)\mathscr{E}_p(h_2, \partial B(0, 1-\eta)) + \eta_0 \mathscr{E}_p(u, \partial B). \end{cases} \tag{54}$$

Define $\hat{h} = \hat{h}_1 \oplus \hat{h}_2$. Then $\hat{h} \in W^1_p(B(0, 1-\eta), \mathscr{Q}_Q(\ell_2))$ satisfies $\hat{h} = h$ on $\partial B(0, 1-\eta)$ and

$$
\begin{aligned}
\mathscr{E}_p(\hat{h}, B_{1-\eta}) &\leqslant \left(\frac{1}{m-p} - 6\eta_0\right)(1-\eta)\left(\mathscr{E}_p(h_1, \partial B(0, 1-\eta)) + \mathscr{E}_p(h_2, \partial B(0, 1-\eta))\right) \\
&\qquad + 2\eta_0\mathscr{E}_p(u, \partial B) \\
&= \left(\frac{1}{m-p} - 6\eta_0\right)(1-\eta)\mathscr{E}_p(h, \partial B(0, 1-\eta)) + 2\eta_0\mathscr{E}_p(u, \partial B) \\
&= \left(\frac{1}{m-p} - 6\eta_0\right)\mathscr{E}_p(\Phi(u), \partial B) + 2\eta_0\mathscr{E}_p(u, \partial B) \\
&\leqslant \left(\frac{1}{m-p} - 6\eta_0\right)\mathscr{E}_p(u, \partial B) + 2\eta_0\mathscr{E}_p(u, \partial B) \\
&= \left(\frac{1}{m-p} - 4\eta_0\right)\mathscr{E}_p(u, \partial B)
\end{aligned}
\tag{55}
$$

where we have used the fact that $\mathrm{Lip}(\Phi) \leqslant 1$ in the last inequality.

Now let $\hat{g} \in W^1_p(B\backslash B(0, 1-\eta), \mathscr{Q}_Q(\ell_2))$ be an extension of

$$
\Phi\left(u\left(\frac{\cdot}{1-\eta}\right)\right) \in W^1_p(\partial B(0, 1-\eta), \mathscr{Q}_Q(\ell_2))
$$

and $u \in W^1_p(\partial B, \mathscr{Q}_Q(\ell_2))$ as in Lemma 3.2. Define $\hat{u} \in W^1_p(B, \mathscr{Q}_Q(\ell_2))$ by

$$
\hat{u} = \begin{cases} \hat{h} & \text{in } B(0, 1-\eta) \\ \hat{g} & \text{in } B\backslash B(0, 1-\eta). \end{cases}
$$

Then we have

$$
\begin{aligned}
&\mathscr{E}_p(\hat{u}, B) \\
&= \mathscr{E}_p(\hat{h}, B(0, 1-\eta)) + \mathscr{E}_p(\hat{g}, B\backslash B(0, 1-\eta)) \\
&\leqslant \left(\frac{1}{m-p} - 4\eta_0 + C\eta\right)\mathscr{E}_p(u, \partial B) + \frac{C}{\eta}\int_{\partial B} \mathscr{G}^p(u, \Phi(u))d\mathscr{H}^{m-1}.
\end{aligned}
\tag{56}
$$

Now we need to estimate $\int_{\partial B} \mathscr{G}^p(u, \Phi(u))d\mathscr{H}^{m-1}$. Define

$$
E := \{x \in \partial B : u(x) \neq \Phi(u(x))\} = \left\{x \in \partial B : u(x) \notin B_{\mathscr{G}}\left(\widetilde{P}, \frac{s(\widetilde{P})}{8}\right)\right\}.
$$

Since $\Phi(\overline{u}) = \overline{u}$, we have

$$
\mathscr{G}(\overline{u}, \Phi(u(x))) \leqslant \mathscr{G}(\overline{u}, u(x)), \ \forall x \in \partial B.
$$

Hence we have

$$
\begin{aligned}
\int_{\partial B} \mathscr{G}^p(u, \Phi(u))d\mathscr{H}^{m-1} &= \int_E \mathscr{G}^p(u, \Phi(u))d\mathscr{H}^{m-1} \\
&\leqslant C\int_E \left(\mathscr{G}^p(u, \overline{u}) + \mathscr{G}^p(\overline{u}, \Phi(u))\right)d\mathscr{H}^{m-1} \\
&\leqslant C\int_E \mathscr{G}^p(u, \overline{u})\,d\mathscr{H}^{m-1} \\
&\leqslant C\|\mathscr{G}(u, \overline{u})\|_{L^{p^*}(\partial B)}^{\frac{p}{p^*}}\left(\mathscr{H}^{m-1}(E)\right)^{1-\frac{p}{p^*}} \\
&\leqslant C\mathscr{E}_p(u, \partial B)\left(\mathscr{H}^{m-1}(E)\right)^{1-\frac{p}{p^*}},
\end{aligned}
\tag{57}
$$

where p^* is the Sobolev exponent of p in $\mathbb{R}^{n-1}$:

$$
p^* = \begin{cases} \frac{(m-1)p}{m-1-p} & \text{if } p < m-1, \\ \text{any } q \in (p, +\infty) & \text{if } p \geqslant m-1. \end{cases}
$$

For any $x \in E$, we have

$$\mathscr{G}(u(x), \overline{u}) \geqslant \mathscr{G}(u(x), \widetilde{P}) - \mathscr{G}(\widetilde{P}, \overline{u}) \geqslant \frac{s(\widetilde{P})}{8} - \frac{s(\widetilde{P})}{16} = \frac{s(\widetilde{P})}{16}.$$

So we have that

$$\begin{aligned}
\mathscr{H}^{m-1}(E) &\leqslant \mathscr{H}^{m-1}\Big(\big\{x \in \partial B : \mathscr{G}(u(x), \overline{u}) \geqslant \frac{s(\widetilde{P})}{16}\big\}\Big) \\
&\leqslant \frac{C}{s^p(\widetilde{P})} \int_{\partial B} \mathscr{G}^p(u(x), \overline{u}) \\
&\leqslant \frac{C}{d^p(\overline{u})} \mathscr{E}_p(u, \partial B) \\
&\leqslant \frac{C}{M}.
\end{aligned} \tag{58}$$

Therefore we obtain

$$\int_{\partial B} \mathscr{G}^p(u, \Phi(u)) d\mathscr{H}^{m-1} \leqslant C\left(\frac{C}{M}\right)^{1-\frac{p}{p\ast}} \mathscr{E}_p(u, \partial B). \tag{59}$$

Substituting (59) into (56), we find that

$$\mathscr{E}_p(\hat{u}, B) \leqslant \left(\frac{1}{m-p} - 4\eta_0 + C\eta + \frac{C}{\eta} M^{\frac{p}{p\ast}-1}\right) \mathscr{E}_p(u, \partial B). \tag{60}$$

Now we first choose $\eta = \eta_0/C$ and then choose

$$M = \big(\frac{C^2}{\eta_0^2}\big)^{\frac{p\ast}{p\ast - p}}$$

so that (60) yields

$$\mathscr{C}(p, Q, u_{|\partial B}) \leqslant \mathscr{E}_p(\hat{u}, B) \leqslant \big(\frac{1}{m-p} - 2\eta_0\big) \mathscr{E}_p(u, \partial B), \text{ if } d^p(\overline{u}) > M\mathscr{E}_p(u, \partial B). \tag{61}$$

On the other hand, Lemma 2.1 implies that

$$\mathscr{C}(p, Q, u_{|\partial B}) \leqslant \mathscr{E}_p(\hat{u}, B) \leqslant \big(\frac{1}{m-p} - 2\eta_0\big) \mathscr{E}_p(u, \partial B), \text{ if } d^p(\overline{u}) \leqslant M\mathscr{E}_p(u, \partial B). \tag{62}$$

Combining (61) and (62) yields that (50) holds for $r = 1$. Note that (50) for all $r \neq 1$ follows from (50) for $r = 1$ by simple scalings.

Since u is (ω, p)-Dir-minimizing in U, by (50) we have that for any ball $B(x, r) \subseteq U$,

$$\begin{aligned}
\mathscr{E}_p(u, B(x, r)) &\leqslant (1 + \omega(r))\mathscr{C}(p, Q, u_{|\partial B(x,r)}) \\
&\leqslant (1 + \omega(r))\big(\frac{1}{m-p} - 2\eta_0\big) r \mathscr{E}_p(u, \partial B(x, r)).
\end{aligned} \tag{63}$$

Since $\lim_{r \downarrow 0} \omega(r) = 0$, there exists $r_0 > 0$ such that

$$(1 + \omega(r))\big(\frac{1}{m-p} - 2\eta_0\big) \leqslant \frac{1}{m-p} - \eta_0, \ \forall 0 < r \leqslant r_0.$$

Thus we have that

$$\mathscr{E}_p(u, B(x, r)) \leqslant \big(\frac{1}{m-p} - \eta_0\big) r \mathscr{E}_p(u, \partial B(x, r)) \tag{64}$$

holds for all $B(x, r) \subseteq U$ with $0 < r \leqslant r_0$. It is standard that integrating (64) over r yields that

$$\frac{1}{r^{m-p+\eta_0}} \mathscr{E}_p(u, B(x, r)) \leqslant \frac{1}{r_0^{m-p+\eta_0}} \mathscr{E}_p(u, B(x, r_0)), \ \forall B(x, r_0) \subseteq U, \ 0 < r \leqslant r_0. \tag{65}$$

This, combined with the Morrey decay lemma [6] for $\mathcal{Q}_Q(\ell_2)$-valued functions, implies that $u \in C^{\eta_0/p}(U)$ and

$$\|u\|_{C^{\eta_0/p}(B(x_0,r))} \leqslant C \frac{1}{r_0^{m-p+\eta_0}} \mathscr{E}_p(u, B(x_0, r_0))$$

holds for $B(x_0, r_0) \subseteq U$ and $0 < r \leqslant r_0$. This completes the proof of Theorem 1.1. $\qquad\square$

5. Squeeze stationary maps

We say that a map $f \in W_2^1(U, \mathcal{Q}_Q(\ell_2))$ is squeeze stationary in U whenever for every $X \in C_c^\infty(U, \mathbb{R}^m)$, we have

$$\frac{d}{dt}\Big|_{t=0} \int_U \big| D(f \circ \Phi_t) \big|^2 = 0, \tag{66}$$

where $\Phi_t(x) = x + tX(x)$ is a diffeomorphism of U to itself for small values of t.

5.1. Proposition. — *A map $f \in W_2^1(U, \mathcal{Q}_Q(\ell_2))$ is squeeze stationary if and only if for every vector field $X \in C_c^\infty(U, \mathbb{R}^m)$ one has:*

$$2\sum_{i=1}^Q \int_U \langle Df(y), Df(y) \circ DX(y) \rangle dy - \int_U \big| Df(y) \big|^2 \operatorname{div} X(y) dy = 0. \tag{67}$$

Proof. We prove (67) by computing (66). Note that

$$\big| D(f \circ \Phi_t)(x) \big|^2 = \sum_{i=1}^Q \|D(f_i \circ \Phi_t)(x)\|_{\mathrm{HS}}^2$$

$$= \sum_{i=1}^Q \|Df_i(\Phi_t(x)) \circ (D\Phi_t(x))\|_{\mathrm{HS}}^2$$

$$= \sum_{i=1}^Q \|Df_i(\Phi_t(x)) \circ (\mathrm{id}_{\mathbb{R}^m} + tDX(x))\|_{\mathrm{HS}}^2$$

$$= \sum_{i=1}^Q \|Df_i(\Phi_t(x)) + tDf_i(\Phi_t(x)) \circ (DX(x))\|_{\mathrm{HS}}^2.$$

We now change variable $y = \Phi_t(x)$ in (66):

$$\int_U \big| D(f \circ \Phi_t)(x) \big|^2 dx$$

$$= \sum_{i=1}^Q \|Df_i(y) + tDf(y) \circ DX(\Phi_t^{-1}(y))\|^2 |\det(D\Phi_t^{-1}(y)|dy$$

$$= \sum_{i=1}^Q \int_U \|A_i(y) + tB_i(y,t)\|_{\mathrm{HS}}^2 C_i(y,t) dy,$$

where

$$A_i(y) = Df_i(y), B_i(y,t) = Df(y) \circ DX(\Phi_t^{-1}(y)), C_i(y,t) = |\det(D\Phi_t^{-1}(y)|.$$

It remains to differentiate with respect to t:

$$\|A_i(y) + tB_i(y,t)\|_{\mathrm{HS}}^2 = \langle A_i(y) + tB_i(y,t), A_i(y) + tB_i(y,t) \rangle$$
$$= \|A_i(y)\|_{\mathrm{HS}}^2 + 2t\langle A_i(y), B_i(y,t) \rangle + t^2 \langle B_i(y) \rangle_{\mathrm{HS}}^2$$
$$= \|A_i(y)\|_{\mathrm{HS}}^2 + 2t\langle A_i(y), B_i(y,0) \rangle + o(t),$$

thus

$$\frac{d}{dt}\Big|_{t=0} \|A_i(y) + tB_i(y)\|_{\mathrm{HS}}^2 = 2\langle A_i(y), B_i(y,0)\rangle$$
$$= 2\langle Df_i(y), Df_i(y) \circ DX(y)\rangle.$$

Furthermore, $y = \Phi_t(\Phi_t^{-1}(y))$ so that

$$\mathrm{id}_{\mathbb{R}^m} = D\Phi_t(\Phi_t^{-1}(y)) \circ D\Phi_t^{-1}(y),$$

and hence

$$\det D\Phi_t^{-1}(y) = \frac{1}{h(y,t)} \qquad \text{where} \qquad h(y,t) := \det D\Phi_t(\Phi_t^{-1}(y)).$$

It follows that

$$\frac{d}{dt}\Big|_{t=0} \left(\det D\Phi_t^{-1}(y)\right) = -\frac{1}{h(y,0)}\frac{\partial h}{\partial t}(y,0).$$

Clearly $h(0) = 1$. Next,

$$\det D\Phi_t(\Phi_t^{-1}(y)) = \det(\mathrm{id}_{\mathbb{R}^m} + tDX(\Phi_t^{-1}(y)))$$
$$= 1 + t\,\mathrm{tr}\,DX(\Phi_t^{-1}(y)) + o(t),$$

whence

$$\frac{\partial h}{\partial t}(y,0) = \mathrm{tr}\,DX(y) = \mathrm{div}\,X(y).$$

Putting together everything, we obtain Equation (67). $\qquad\square$

5.2. PROPOSITION. — *Let $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$ be squeeze stationary and $a \in U$. It follows that the function $\Theta_a : (0, \mathrm{dist}(a, \partial U)) \to \mathbb{R}_+$ defined by*

$$\Theta_a(r) := \frac{1}{r^{m-2}}\int_{B(a,r)} |Df|^2$$

is absolutely continuous and nondecreasing. In fact,

$$\Theta_a'(r) = \frac{2}{r^{m-2}}\int_{\partial B(a,r)}\sum_{i=1}^{Q}\|\frac{\partial f_i}{\partial \nu}\|^2 d\mathscr{H}^{m-1}. \tag{68}$$

In other words, the Radon measure

$$A \mapsto \int_A |Df|^2 \tag{69}$$

is $(m-2)$ monotonic.

Proof. It is clear that Θ_a is absolutely continuous because the measure in (69) is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^m$ and $\mathscr{L}^m(B(a, r + h)\backslash B(a,r)) \leqslant Ch$. For simplicity, assume $a = 0$ and write $\Theta(r)$ for $\Theta_a(r)$. We now plug in equation (67) a vector field

$$X(x) = \chi(\|x\|)x,$$

where $\chi \in C_c^\infty(\mathbb{R})$ is constant in a neighborhood of 0. For every $i, j = 1, \ldots, m$ one has

$$\frac{\partial}{\partial x_j}\langle X(x), e_i\rangle = \frac{\partial}{\partial x_j}\left(\chi(\|x\|)x_i\right)$$
$$= \chi(\|x\|)\delta_{ij} + \chi'(\|x\|)\frac{x_i x_j}{\|x\|}.$$

In particular,

$$\mathrm{div}\,X(x) = m\chi(\|x\|) + \chi'(\|x\|)\|x\|.$$

We now compute the first term in (67):

$$\sum_{i=1}^{Q}\langle Df_i(x), Df_i(x)\circ DX(x)\rangle = \chi(\|x\|)\sum_{i=1}^{Q}\|Df_i(x)\|_{\mathrm{HS}}^2$$
$$+\chi'(\|x\|)\sum_{i=1}^{Q}\langle Df_i(x), Df_i(x)\circ\big(x\otimes\frac{x}{\|x\|}\big)\rangle$$
$$= \chi(\|x\|)\,\big|\,Df(x)\,\big|^2$$
$$+\chi'(\|x\|)\|x\|\sum_{i=1}^{Q}\langle Df_i(x), Df_i(x)\circ A\rangle,$$

where A is the matrix $u_1\otimes u_1$ for $u_1 := x/\|x\| \in \mathbb{S}^{m-1}$, i.e. $A_{ij} = \langle u_1, e_i\rangle\langle u_1, e_j\rangle$. One easily sees that $Au_1 = u_1$ and that $Av = 0$ whenever $\langle u_1, v\rangle = 0$. Now we complete u_1 to an orthonormal basis $(u_1,\ldots,u_m)$. It follows that

$$\langle Df_i(x), Df_i(x)\circ A\rangle = \sum_{j=1}^{m}\langle Df_i(x)(u_j), Df_i(x)(A(u_j))\rangle$$
$$= \langle Df_i(x)(u_1), Df_i(x)(u_1)\rangle$$
$$= \left\|\frac{\partial f_i}{\partial\nu}(x)\right\|^2.$$

We infer from (67) that

$$2\int_U\big(\,\big|\,Df(x)\,\big|^2\chi(\|x\|)+\chi'(\|x\|)\|x\|\sum_{i=1}^{Q}\big\|\frac{\partial f_i}{\partial\nu}(x)\big\|^2\big)dx$$

$$-\int_U\big(\,\big|\,Df(x)\,\big|^2\big(m\chi(\|x\|+\|x\|\chi'(\|x\|))\big)dx = 0,$$

that is,

$$(2-m)\int_U\chi(\|x\|)\,\big|\,Df(x)\,\big|^2\,dx + 2\int_U\chi'(\|x\|)\|x\|\sum_{i=1}^{Q}\big\|\frac{\partial f_i}{\partial\nu}(x)\big\|^2\,dx$$

$$= \int_U\chi'(\|x\|)\|x\|\,\big|\,Df(x)\,\big|^2\,dx.$$

Now we fix $r \in (0, \mathrm{dist}(0,\partial U))$ and we let $\{\chi_j\}_{j=1}^{\infty}$ approach $\mathbb{1}_{[0,r]}$ so that

$$(2-m)\int_{B(0,r)}\big|\,Df\,\big|^2 - 2r\int_{\partial B(0,r)}\sum_{i=1}^{Q}\big\|\frac{\partial f_i}{\partial\nu}\big\|^2 d\mathscr{H}^{m-1} = -r\int_{\partial B(0,r)}\big|\,Df\,\big|^2,$$

and one finally computes that for a.e $r \in (0, \mathrm{dist}(0,\partial U))$,

$$\Theta'(r) = (2-m)r^{2-m-1}\int_{B(0,r)}\big|\,Df\,\big|^2 + r^{2-m}\int_{\partial B(0,r)}\big|\,Df\,\big|^2$$

$$= 2r^{2-m}\int_{\partial B(0,r)}\sum_{i=1}^{Q}\big\|\frac{\partial f_i}{\partial\nu}\big\|^2 d\mathscr{H}^{m-1}\geqslant 0.$$

$\square$

6. Squash variations

Here we consider vertical variations above x whose amplitude depend on $x \in U$. Let

$$Y : U \times \ell_2 \to \ell_2$$

be a C^1 map such that $U \cap \{x : Y(x, \cdot) \neq 0\}$ is relatively compact in U. For $f : U \to \mathscr{Q}_Q(\ell_2)$ we define

$$(Y \square f)_t(x) := \bigoplus_{i=1}^{Q} [\![f_i(x) + tY(x, f_i(x))]\!].$$

We say that f is squash stationary if for every such Y one has

$$\frac{d}{dt}\Big|_{t=0} \int_U |\, D(Y \square f)_t \,|^2 = 0.$$

6.1. Proposition. — *A map $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$ is squash stationary if and only if for all Y,*

$$\sum_{i=1}^{Q} \int_U \langle Df_i(x), D_x Y(x, f_i(x)) \rangle + \sum_{i=1}^{Q} \int_U \langle Df_i(x), D_y Y(x, f_i(x)) \circ Df_i(x) \rangle dx = 0. \tag{70}$$

Proof. The derivation of this Euler-Lagrange equation is much simpler than for squeeze variations. One computes

$$|\, D(Y \square f)_t \,|^2 = \sum_{i=1}^{Q} \|D(f_i + tY(\cdot, f_i(\cdot))\|_{\mathrm{HS}}^2$$

$$= |\, Df \,|^2 + 2t \sum_{i=1}^{Q} \langle Df_i, D_x Y(\cdot, f_i(\cdot)) + D_y Y(\cdot, f_i(\cdot)) \circ Df_i \rangle + o(t).$$

Integrating over U and then differentiating at $t = 0$ gives equation (70). $\qquad\square$

6.2. Corollary. — *If $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$ is squash stationary and $B(a, r) \subseteq U$ then*

$$\int_{B(a,r)} |\, Df \,|^2 = \int_{\partial B(a,r)} \sum_{i=1}^{Q} \langle \frac{\partial f_i}{\partial \nu}, f_i \rangle d\mathscr{H}^{m-1}. \tag{71}$$

Proof. Assume for simplicity that $a = 0$. Let $\chi \in C_c^\infty(\mathbb{R})$ be a function which is constant in a neighborhood of 0. We will plug

$$Y(x, y) := \chi(\|x\|)y$$

into equation (70). First we compute

$$D_x Y(x, y) = \chi'(\|x\|) \frac{x}{\|x\|} \otimes y, \qquad D_y Y(x, y) = \chi(\|x\|) \mathrm{id}_{\mathbb{R}^m}.$$

Thus we have

$$\sum_{i=1}^{Q} \int_U \chi'(\|x\|) \langle \frac{\partial f_i}{\partial \nu}(x), f_i(x) \rangle + \int_U \chi(\|x\|) |\, Df(x) \,|^2 dx = 0.$$

Now, we fix $r \in (0, \mathrm{dist}(0, \partial U))$ and we let $\{\chi_j\}_{j=1}^{\infty}$ be an approximation of $\mathbb{1}_{[0,r]}$, so that

$$-\sum_{i=1}^{Q} \int_{\partial B(0,r)} \langle \frac{\partial f_i}{\partial \nu}, f_i \rangle d\mathscr{H}^{m-1} + \int_{B(0,r)} |\, Df \,|^2 = 0.$$

$\qquad\square$

7. Frequency function

Let $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$. For $a \in U \subseteq \mathbb{R}^m$ and $r \in (0, \text{dist}(a, \partial U))$, we define the following quantities

$$D_a(r) := \int_{B(a,r)} |\, Df \,|^2, \qquad H_a(r) := \int_{\partial B(a,r)} |f|^2, \qquad N_a(r) := \frac{r D_a(r)}{H_a(r)},$$

where the last quantity, $N_a(r)$, is defined provided that $H_a(r) \neq 0$. $N_a(r)$ is called the *frequency function* at the point a. When $a = 0$, we simply denote $D(r)$, $H(r)$, $N(r)$ for $D_0(r)$, $H_0(r)$, $N_0(r)$ respectively.

The following lemma ensures that if f is squash stationary and non zero in a neighborhood of a, then $N_a(r)$ is defined for small values of r.

7.1. LEMMA. — *If $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$ is squash stationary and if $H_a(r_0) = 0$ for some $a \in U$ and $r_0 \in (0, \text{dist}(a, \partial U))$, then $f \equiv Q[\![0]\!]$ on $B(a, r_0)$.*

Proof. This is a consequence of (71): if $f \equiv 0$ a.e on $\partial B(a, r)$ then $\mathscr{E}_2(f, B(a, r)) = 0$, thus f vanishes on $B(a, r)$. $\qquad \square$

7.2. LEMMA. — *For $a \in U$, $H_a(r)$ is absolutely continuous in r, for any $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$.*

Proof. For simplicity, assume $a = 0 \in U$. Then

$$\begin{aligned}
H(r) &= \frac{d}{dr} \int_{B(0,r)} |f(x)|^2 dx \\
&= \frac{d}{dr} \left(r^m \int_B |f(rx)|^2 dx \right) \\
&= m r^{m-1} \int_B |f(rx)|^2 dx + 2 r^m \int_B \sum_{i=1}^Q \left\langle f_i(rx), \frac{\partial f_i}{\partial \nu}(rx) \right\rangle dx \\
&= \frac{m}{r} \int_{B(0,r)} |f|^2 + 2 \sum_{i=1}^Q \int_{B(0,r)} \left\langle f_i, \frac{\partial f_i}{\partial \nu} \right\rangle,
\end{aligned}$$

and the last expression is easily seen to be absolutely continuous. $\qquad \square$

7.3. LEMMA. — *If f is squash stationary then for any $a \in U$ and a.e. $r \in (0, \text{dist}(a, \partial U))$,*

$$H_a'(r) = \frac{m-1}{r} H_a(r) + 2 D_a(r). \tag{72}$$

Proof. For simplicity, assume $a = 0 \in U$. Then we have

$$\begin{aligned}
H'(r) &= \frac{d}{dr} \int_{\partial B} r^{m-1} |f(rx)|^2 d\mathscr{H}^{m-1} x \\
&= \frac{m-1}{r} \int_{\partial B(0,1)} r^{m-1} |f(rx)|^2 d\mathscr{H}^{m-1} x \\
&\quad + 2 r^{m-1} \int_{\partial B(0,r)} \sum_{i=1}^Q \left\langle f_i(rx), \frac{\partial f_i}{\partial \nu}(rx) \right\rangle d\mathscr{H}^{m-1} x \\
&= \frac{m-1}{r} \int_{\partial B(0,r)} |f|^2 d\mathscr{H}^{m-1} + 2 \int_{\partial B(0,r)} \sum_{i=1}^Q \left\langle f_i, \frac{\partial f_i}{\partial \nu} \right\rangle d\mathscr{H}^{m-1}.
\end{aligned}$$

Now using (71) we prove (72). $\qquad \square$

7.4. THEOREM. — *If f is squeeze and squash stationary and for $a \in U$, $H_a(r_0) \neq 0$ for some $r_0 \in (0, \text{dist}(a, \partial U))$, then $N_a'(r) \geqslant 0$ for a.e. $r \leqslant r_0$.*

Proof. For simplicity, assume $a = 0 \in U$. Select $r \in (0, r_0]$ at which D and H are differentiable. Then, according to (72), (68) and (71),

$$
\begin{aligned}
N'(r) &= \frac{D(r)}{H(r)} + r\frac{D'(r)}{H(r)} - rD(r)\frac{H'(r)}{H^2(r)} \\
&= \frac{D(r)}{H(r)} + r\frac{D'(r)}{H(r)} - (m-1)\frac{D(r)}{H(r)} - 2r\frac{D^2(r)}{H^2(r)} \\
&= \frac{(2-m)D(r) + rD'(r)}{H(r)} - 2r\frac{D^2(r)}{H^2(r)} \\
&= \frac{r^{m-1}\Theta'(r)}{H(r)} - 2r\frac{D^2(r)}{H^2(r)} \\
&= \frac{2r}{H(r)} \int_{\partial B(0,r)} \sum_{i=1}^{Q} \|\frac{\partial f_i}{\partial \nu}\|^2 - \frac{2r}{H(r)^2}\Big(\int_{\partial B(0,r)} \sum_{i=1}^{Q} \langle f_i, \frac{\partial f_i}{\partial \nu}\rangle \Big)^2.
\end{aligned}
$$

Thus,

$$
N'(r) = \frac{2r}{H^2(r)}\Big[\int_{\partial B(0,r)} \sum_{i=1}^{Q} \|f_i\|^2 \int_{\partial B(0,r)} \sum_{i=1}^{Q} \|\frac{\partial f_i}{\partial \nu}\|^2 - \Big(\int_{\partial B(0,r)} \sum_{i=1}^{Q} \langle f_i, \frac{\partial f_i}{\partial \nu}\rangle \Big)^2 \Big]
$$

is nonnegative by Cauchy-Schwartz inequality. $\qquad\square$

7.5. COROLLARY. — *Assume that $f \in W_2^1(U, \mathcal{Q}_Q(\ell_2))$ is squeeze and squash stationary and $r_0 \in (0, \operatorname{dist}(a, \partial U))$ for $a \in U$. Then, for a.e. $r \leqslant r_0$, we have*

$$
\frac{d}{dr} \ln \frac{H_a(r)}{r^{m-1}} = \frac{2N_a(r)}{r}, \tag{73}
$$

$$
\Big(\frac{r}{r_0}\Big)^{2N_a(r_0)} \frac{H_a(r_0)}{r_0^{m-1}} \leqslant \frac{H_a(r)}{r^{m-1}} \leqslant \Big(\frac{r}{r_0}\Big)^{2N_a(r)} \frac{H_a(r_0)}{r_0^{m-1}}, \tag{74}
$$

$$
\frac{D_a(r)}{r^{m-2}} \leqslant \Big(\frac{r}{r_0}\Big)^{2N_a(r)} \frac{D_a(r_0)}{r^{m-2}} \frac{N_a(r)}{N_a(r_0)}. \tag{75}
$$

Proof. For simplicity, assume $a = 0 \in U$. First we prove (73). By equation (72), we have

$$
\frac{d}{dr}\frac{H(r)}{r^{m-1}} = \frac{H'(r)}{r^{m-1}} - (m-1)\frac{H(r)}{r^m} = \frac{2D(r)}{r^{m-1}}.
$$

Consequently,

$$
\frac{d}{dr} \ln \Big(\frac{H(r)}{r^{m-1}}\Big) = \frac{2D(r)}{r^{m-1}}\frac{r^{m-1}}{H(r)} = \frac{2N(r)}{r}.
$$

We integrate (73) from r to r_0:

$$
\begin{aligned}
\ln \frac{H(r_0)}{r_0^{m-1}} - \ln \frac{H(r)}{r^{m-1}} &= \int_r^{r_0} \frac{d}{d\rho} \ln \frac{H(\rho)}{\rho^{m-1}} d\rho \\
&= \int_r^{r_0} \frac{2N(\rho)}{\rho} d\rho \\
&\geqslant 2N(r) \int_r^{r_0} \frac{d\rho}{\rho}. \tag{76}
\end{aligned}
$$

Therefore

$$
\ln \Big(\frac{H(r_0)}{r_0^{m-1}}\frac{r^{m-1}}{H(r)}\Big) \geqslant \ln \Big(\frac{r_0}{r}\Big)^{2N(r)},
$$

which yields the first inequality in (74). Similarly, by bounding above $N(\rho)$ by $N(r_0)$ in (76), one proves the reverse inequality.

By definition of $N(r)$,
$$\frac{D(r)}{r^{m-2}} = \frac{H(r)}{r^{m-1}} N(r).$$
Therefore, using (74), one has
$$\frac{D(r)}{r^{m-2}} \leqslant \left(\frac{r}{r_0}\right)^{2N(r)} \frac{H(r_0)}{r_0^{m-1}} N(r) = \left(\frac{r}{r_0}\right)^{2N(r)} \frac{D(r_0)}{r_0^{m-2}} \frac{N(r)}{N(r_0)},$$
which is (75). □

8. Regularity of stationary maps

8.1. PROPOSITION. — *If $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$ is squeeze and squash stationary and let $a \in U$ and $B(a, 3r_0) \subseteq U$, then f is locally essentially bounded on $B(a, r_0)$. Specifically,*
$$\|f\|_{L^\infty(B(a,r_0))}^2 \leqslant \frac{C}{r_0^m} \int_{B(a,3r_0)} |f|^2. \tag{77}$$

Proof. Assume $a = 0 \in U$. Let $x_0 \in B(0, r_0)$. Then we have
$$\int_{r_0}^{2r_0} d\rho \int_{\partial B(x_0,\rho)} |f|^2 d\mathscr{H}^{m-1} = \int_{B(x_0,2r_0) \setminus B(x_0,r_0)} |f|^2$$
$$\leqslant \int_{B(0,3r_0)} |f|^2.$$
Therefore there exists some $r_1 \in (r_0, 2r_0)$ such that
$$\int_{\partial B(x_0,r_1)} |f|^2 d\mathscr{H}^{m-1} \leqslant \frac{2}{r_0} \int_{B(0,3r_0)} |f|^2.$$
Moreover one has, according to (74),
$$\int_{B(x_0,r)} |f|^2 = \int_0^r \int_{\partial B(x_0,\rho)} |f|^2$$
$$= \int_0^r H_{x_0}(\rho) d\rho$$
$$\leqslant \int_0^r \left(\frac{\rho}{r_0}\right)^{m-1} H_{x_0}(r_0) d\rho$$
$$= \frac{r^m}{m} \frac{H_{x_0}(r_0)}{r_0^{m-1}}$$
$$\leqslant \frac{r^m}{m} \frac{H_{x_0}(r_1)}{r_1^{m-1}}$$
$$\leqslant \frac{Cr^m}{r_0^m} \int_{B(0,3r_0)} |f|^2.$$
Therefore
$$\frac{1}{\mathscr{L}^m(B(x_0,r))} \int_{B(x_0,r)} |f|^2 \leqslant \frac{C}{r_0^m} \int_{B(0,3r_0)} |f|^2. \tag{78}$$
Whenever x_0 is a Lebesgue density point of $|f|^2 \in L_1(U)$, by letting r tend to 0, (78) gives the desired estimate. □

8.2. REMARK. — If a is a Lebesgue density total branch point of a squeeze and squash stationary map $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$, *i.e.* if there exists $y \in \ell_2$ such that
$$\lim_{r \to 0} \frac{1}{\mathscr{L}^m(B(a,r))} \int_{B(a,r)} \mathscr{G}(f(x), Q[\![y]\!])^2 = 0,$$

then f is continuous at a. Indeed, apply Proposition 8.1 to the squeeze and squash stationary map $\tau_{Q[\![y]\!]}(f)$, one has that

$$\lim_{r \to 0} \sup_{x \in B(a,r)} \mathscr{G}(f(x), f(a))^2 = \lim_{r \to 0} \frac{C}{r^m} \int_{B(a,3r)} |\tau_{Q[\![y]\!]}(f)|^2 = 0.$$

Now, for $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$ and $B(a,r) \subseteq U$, we will denote by $\overline{f}_{a,r}$ a mean of f in the ball $B(a,r)$.

8.3. PROPOSITION. — *If $f \in W_2^1(U, \mathscr{Q}_Q(\ell_2))$ is squeeze and squash stationary, then*

(1) *for any a in U,*

$$\lim_{r \to 0} \Theta_a(r) = \lim_{r \to 0} \frac{1}{r^{m-2}} \int_{B(a,r)} |\, Df \,|^2 = 0.$$

(2) *in case $U = B(0,1)$, f is VMO in the following sense: there is a function $\omega : [0, 2^{-1}] \to \mathbb{R}_+$ satisfying $\lim_{r \to 0} \omega(r) = 0$ such that for any $a \in B(0, 2^{-1})$, $r \in [0, 2^{-1}]$,*

$$\frac{1}{\mathscr{L}^m(B(a,r))} \int_{B(a,r)} \mathscr{G}^2(f, \overline{f}_{a,r}) \leqslant \omega(r).$$

Proof. As in (69), we introduce

$$\mu(A) := \int_A |\, Df \,|^2.$$

We consider two cases.

Case 1. $\lim_{r \to 0} N_a(r) = \alpha > 0$.

We then use the monotonicity of frequency (Theorem 7.4) and (75) to infer that, for some $r_0 < \operatorname{dist}(a, \partial U)$ and $r \in (0, r_0]$,

$$\frac{D_a(r)}{r^{m-2}} \leqslant \left(\frac{r}{r_0} \right)^{2\alpha} \frac{D_a(r_0)}{r_0^{m-2}},$$

which converges to 0 as $r \to 0$.

Case 2. $\lim_{r \to 0} N_a(r) = 0$.

Then, by definition of $N_a(r)$ and by Proposition 8.1,

$$\frac{D_a(r)}{r^{m-2}} = N_a(r) \frac{H_a(r)}{r^{m-1}} \leqslant C N_a(r) \|f\|_{L^\infty(U)},$$

which converges to 0 as well. This establishes (1).

We now come to (2). The functions $\Theta_a : [0, 2^{-1}] \to \mathbb{R}_+$, $a \in B(0, 2^{-1})$, are monotone nondecreasing with respect of r by Proposition 5.2, continuous and satisfy $\Theta_a(0) = 0$ by (1). Let us introduce

$$\omega(r) := \sup_{a \in B(0, 2^{-1})} \frac{\mu(B(a,r))}{r^{m-2}}. \tag{79}$$

By the classical Dini theorem, $\lim_{r \to 0} \omega(r) = 0$. Then (2) follows from Poincaré's inequality. $\qquad \square$

8.4. PROPOSITION (Logarithmic decay of normalized energy). — *Let f be a squeeze and squash stationary map in $W_2^1(U, \mathscr{Q}_Q(\ell_2))$, $B(0,3) \subseteq U$. Then there exist C (depending on $\|f\|_{L^\infty(B)}$ and $\alpha \in (0,1)$) such that for every $a \in B(0, 2^{-1})$ and $r \in (0, 2^{-1}]$ one has*

$$\frac{1}{\mathscr{L}^m(B(a,r))} \int_{B(a,r)} \mathscr{G}^2(f, \overline{f}_{a,r}) \leqslant C \left(\frac{1}{|\ln r|} \right)^\alpha.$$

In fact,

$$\frac{1}{r^{m-2}} \int_{B(a,r)} |Df|^2 \leqslant C\big(\frac{1}{|\ln r|}\big)^\alpha.$$

Proof. We start with $\rho_0 = \frac{1}{2}$ and define a sequence $\{\rho_j\}_j$ inductively by $\rho_{j+1} = \rho_j^2$. Thus $\rho_j = \rho_0^{2^j}$.

Note that, using Proposition 8.1, one has

$$\Theta_a(\rho_{j+1}) = \frac{D_a(\rho_{j+1})}{\rho_{j+1}^{m-2}} = N_a(\rho_{j+1}) \frac{H_a(\rho_{j+1})}{\rho_{j+1}^{m-1}} \leqslant C N_a(\rho_{j+1}), \qquad (80)$$

and by (75)

$$\Theta_a(\rho_{j+1}) = \frac{D_a(\rho_{j+1})}{\rho_{j+1}^{m-2}} \leqslant \left(\frac{\rho_{j+1}}{\rho_j}\right)^{2N_a(\rho_j)} \frac{D_a(\rho_j)}{\rho_j^{m-2}} = \left(\frac{\rho_{j+1}}{\rho_j}\right)^{2N_a(\rho_j)} \Theta_a(\rho_j). \quad (81)$$

First suppose that $\rho_j^{2N_a(\rho_j)} \leqslant 2^{-1}$. Then by (81),

$$\Theta_a(\rho_{j+1}) \leqslant \left(\frac{\rho_{j+1}}{\rho_j}\right)^{2N_a(\rho_j)} \Theta_a(\rho_j) = \rho_j^{2N_a(\rho_j)} \Theta_a(\rho_j) \leqslant \frac{1}{2} \Theta_a(\rho_j).$$

On the other hand, if $\rho_j^{2N_a(\rho_j)} > 2^{-1}$, then

$$2N_a(\rho_j) \ln \rho_j > \ln\big(\frac{1}{2}\big).$$

As $\rho_j = 2^{-2^j}$, we infer that $N_a(\rho_j) < 2^{-j-1}$. In that case, (80) yields

$$\Theta_a(\rho_{j+1}) \leqslant C 2^{-j-1}.$$

Recalling (79), we prove that

$$\omega(\rho_{j+1}) \leqslant \max\big\{C 2^{-j-1}, \frac{1}{2}\omega(\rho_j)\big\}.$$

It is now standard that for some $\alpha \in (0,1)$,

$$\omega(\rho) \leqslant C\big(\frac{1}{|\ln \rho|}\big)^\alpha.$$

We conclude the proof by applying Proposition 8.3(2). □

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