DERIVATION OF THE ARONSSON EQUATION FOR C^1 HAMILTONIANS

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Abstract. It is proved herein that any absolute minimizer u for a suitable Hamiltonian $H \in C^1(\mathbb{R}^n \times \mathbb{R} \times U)$ is a viscosity solution of the Aronsson equation:

$$H_p(Du, u, x) \cdot (H(Du, u, x))_x = 0 \quad \text{in } U.$$

The primary advance is to weaken the assumption that $H \in C^2$, used by previous authors, to the natural condition that $H \in C^1$.

1. Introduction

Let U be an open subset of $\mathbb{R}^n$ and $H(p, z, x) \in C(\mathbb{R}^n \times \mathbb{R} \times U)$. A function $u : U \to \mathbb{R}$ is said to be an *absolute minimizer* for H in U if the following two conditions hold:

(i) u is locally Lipschitz continuous in U;

(ii) whenever V is an bounded open subset of U, $\bar{V} \subset U$, $v \in C(\bar{V})$ is locally Lipschitz continuous in V and $u|_{\partial V} = v|_{\partial V}$, we have

$$\operatorname{ess\,sup}_{x \in V} H(Du(x), u(x), x) \leq \operatorname{ess\,sup}_{x \in V} H(Dv(x), v(x), x).$$

Here and later, $Du = (u_{x_1}, \ldots, u_{x_n})$ denotes the spatial gradient of u.

The study of absolute minimizers was initiated by G. Aronsson in [1], [2], [4] in the case $n = 1$, and in [3] in the case $H(p, z, x) = |p|$ (equivalently, $H(p) = |p|^2$), although in [3] he primarily used the Lipschitz constant in place of the L^∞ functionals indicated above. The initial study of absolutely minimizing functions in the full generality above was provided by Jensen, Barron and Wang in [7]. In particular, they showed, in some generality, that any absolute minimizer for H is a viscosity solution of the *Aronsson equation:*

$$H_p(Du(x), u(x), x) \cdot (H(Du(x), u(x), x))_x = 0 \quad \text{in } U, \tag{1.1}$$

where $H \in C^2(\mathbb{R}^n \times \mathbb{R} \times U)$, H_p is the gradient of $H(p, z, x)$ in p, $(H(Du(x), u(x), x))_x$ is the (formal) gradient of $x \mapsto H(Du(x), u(x), x)$ and the "dot" denotes the Euclidean inner-product.

Subsequently a simpler derivation of this result under somewhat weaker hypotheses was given in [9], wherein the essential assumptions were that H is C^2 and quasiconvex in p (see Section 2). The hypothesis that $H \in C^2$ is unnatural in the sense that the equation (1.1) makes perfect sense if $H \in C^1$. In fact, if H has the simple form $H(p, z, x) = \|p\|$ and $\|\cdot\|$

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is *any* norm on $\mathbb{R}^n$, the appropriate variant of (1.1) is derived by Aronsson, Crandall and Juutinen, [5].

In this paper we give a positive answer to the interesting question, explicitly posed in [9] and left open in both [7] and [9], of whether or not the Aronsson equation is satisfied by absolutely minimizing functions if H is merely C^1 in two cases. More precisely, if $H = H(p, x)$ is independent of z, quasiconvex in p, and C^1, then any absolute minimizer for H indeed solves (1.1). The case where $H = H(p, z, x)$ also depends on z is more complex; in this case we obtain the same result under the assumption that H is convex in p. The only issue left unsettled as regards the satisfaction of the Aronsson equation when $H \in C^1$, is whether or not it is still satisfied if the convexity assumption in the z-dependent case is relaxed to quasiconvexity. We believe that the answer is yes, and remain interested in the question.

Let us briefly describe the role of the convexity assumption. In [9], a change of variables was used to reduce to the case in which H is monotonic in the z variable. This monotonicity is needed to deal with the z-dependent case. The proofs in [9] and the current paper both use the idea of "comparion with cones" from Crandall, Evans and Gariepy, [12]. Cones are solutions of the Hamilton-Jacobi equation $H =$ constant in appropriate sense. The cones used in [9] and the current paper are different. Since H is assumed to be C^2 in [9], by solving the Hamiton-Jacobi equation via the charateristic method, it was possible to choose smooth cones which are C^1 perturbations of the test funtion. In that case, quasiconvexity is enough to implement the strategy of changing variables. In our situation, where H is only C^1, the cones we use are more direct generalization of the cones in [12]; they appear already in Champion and De Pascale [8] and Fathi and Siconolfi [15] with somewhat different technicalities. These cones are viscosity solutions of the Hamilton-Jacobi equation and not C^1 perturbations of the test function. We use the convexity assumption to implement a successful change of variables.

Our results are new even if $H = H(p)$ depends only on p. The proofs of [7] and [9] are inadequate to establish these results, and we will combine a variety of techniques, some of which are motivated by proofs in [5], [10]. We note that in the generality of current paper (or [7] and [9]), the Aronsson equation does not characterize absolute minimizing functions; that is, a viscosity solution of the Aronsson equation might not be an absolute minimizer. Two simple counterexamples are given in Yu [20]. It is an interesting problem to delineate conditions guaranteeing that solutions of the Aronsson equation are absolutely minimizing. Some progress has been made in this direction. For example, a viscosity solution of the Aronsson equation is an absolute minimizer if H satisfies one of the following: (a) $H = H(p, x) \in C^2$ is independent of z, convex and coercive in p (see [20]) and (b) $H = H(p) \in C^2$ only depends on p and is quasiconvex and coercive in p (see Yu [21], Gariepy, Wang and Yu [16]). The proofs in the papers just cited also used versions of "comparison with cones."

Currently, perhaps the main use of the Aronsson equation is to prove the uniqueness of absolute minimizers. See, for instance, Jensen [17], Juutinen[18], Crandall, Gunnarson and

Wang [14], and Jensen, Wang and Yu [19], etc. However, uniqueness fails for solutions of the Dirichlet problem for the Aronsson equation and for absolutely minimizing functions subject to Dirichlet conditions, except in some special situations. A simple example of nonuniqueness is given in [20].

The implications of our results, although we regard them as very interesting, are limited by the negative facts cited above. Of course, they do provide necessary conditions; when our results apply, absolutely minimizing functions must satisfy the Aronsson equation.

2. Preliminaries and the Main Results

We will use $|x|$ to denote the Euclidean norm of $x \in \mathbb{R}^n$ and $x \cdot y$ to denote the Euclidean inner-product of $x, y \in \mathbb{R}^n$.

Balls are denoted as follows:

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}, \overline{B}_r(x) = \{y \in \mathbb{R}^n : |y - x| \le r\}$$

The notation $A := B$ means that A is defined to be B.

Throughout this paper, U is an open subset of $\mathbb{R}^n$, $\overline{U}$ is its closure, ∂U is its boundary, and

$$H \in C(\mathbb{R}^n \times \mathbb{R} \times U). \tag{2.1}$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *quasiconvex* if

$$\{p \in \mathbb{R}^n : \ f(p) \le \lambda\} \quad \text{is convex for any } \lambda \in \mathbb{R}.$$

It is equivalent to require that

$$f(tp + (1-t)q) \le f(p) \vee f(q) \quad \text{for any } p, q \in \mathbb{R}^n \text{ and } t \in [0,1] \ .$$

For example, if $f(p) = g(h(p))$, where $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and h is a convex function, then f is a quasiconvex function which is not convex in general.

Throughout this paper, $H(p, z, x)$ is at least quasiconvex in p. H_p denotes the gradient of H in p, H_x is the gradient of H in x, and H_z is the partial derivative of H with respect to z. Gradients are regarded as row vectors. The formal expression

$$\begin{aligned}(H(Du(x), u(x), x))_x =& H_p(Du(x), u(x), x) D^2 u(x) + \\ & H_z(Du(x), u(x), x) Du(x) + H_x(Du(x), u(x), x)\end{aligned}$$

is used to interpret the equation

$$\mathcal{A}[u] := H_p(Du(x), u(x), x) \cdot (H(Du(x), u(x), x))_x = 0 \tag{2.2}$$

in the viscosity sense as used in Crandall, Ishii and Lions, [13].

We recall that $u \in C(U)$ is a *viscosity subsolution* of $\mathcal{A}[u] = 0$ provided that whenever $\varphi \in C^2(U)$ and $x_0 \in U$ is a local maximum of $u - \varphi$ with $u(x_0) - \varphi(x_0) = 0$, then

$$\mathcal{A}[\varphi](x_0) \ge 0. \tag{2.3}$$

This inequality is appropriate as $\mathcal{A}[u]$ is formally nondecreasing in $D^2u(x)$ for fixed $Du(x), u(x), x$. Likewise, u is a *viscosity supersolution* if $\mathcal{A}[\varphi](x_0) \le 0$ whenever $u(x_0) - \varphi(x_0) = 0$ and x_0 is a local minimum of $u - \varphi$. Finally, u is a *viscosity solution* of $\mathcal{A}[u] = 0$ if it is both a viscosity subsolution and a viscosity supersolution.

Our first main result concerns the case in which H is independent of z.

Theorem 2.1. *If $H \in C^1(\mathbb{R}^n \times U)$ and $p \mapsto H(p, x)$ is quasiconvex for each $x \in U$, then any absolute minimizer u for H is a viscosity solution of $\mathcal{A}[u] = 0$.*

In the general case in which H does depend on z, we need to replace the quasiconvexity assumption on H by convexity.

Theorem 2.2. *$H \in C^1(\mathbb{R}^n \times \mathbb{R} \times U)$ and $p \mapsto H(p, z, x)$ is convex for each $(z, x) \in \mathbb{R} \times U$, then any absolute minimizer u for H is a viscosity solution of $\mathcal{A}[u] = 0$.*

Assuming that u is absolutely minimizing in U for H, it will suffice to prove that u is a subsolution of the Aronsson equation (2.2). The proof that u is a supersolution is then obtained by either applying this result to the Hamiltonian $H(-p, -z, x)$ (for which $-u$ is absolutely minimizing), or by rerunning the previous proof with obvious modifications.

Thus, if $x_0 \in U$, $\varphi \in C^2(U)$ and

$$\begin{aligned} &\text{(i) } u(x) - \varphi(x) \le u(x_0) - \varphi(x_0) = 0, \text{ equivalently,} \\ &\text{(ii) } u(x) - u(x_0) \le \varphi(x) - \varphi(x_0) \text{ and } u(x_0) = \varphi(x_0), \end{aligned} \tag{2.4}$$

for x near x_0, we need to show that $\mathcal{A}[\varphi](x_0) \ge 0$.

In a standard way, replacing $\varphi(x)$ by $\varphi(x) + |x - x_0|^4$, we may assume

$$\begin{aligned} &\text{(i) } u(x) - \varphi(x) < u(x_0) - \varphi(x_0) = 0, \text{ equivalently,} \\ &\text{(ii) } u(x) - u(x_0) < \varphi(x) - \varphi(x_0) \text{ and } u(x_0) = \varphi(x_0), \end{aligned} \tag{2.5}$$

for $x \in U \setminus \{x_0\}$ near x_0. Finally, we may also assume that

$$H_p(D\varphi(x_0), \varphi(x_0), x_0) \ne 0, \tag{2.6}$$

for, otherwise, $\mathcal{A}[\varphi](x_0) = 0$.

We prepare a simple proposition which is used in the proofs.

Proposition 2.3. *Let $H \in C(\mathbb{R}^n \times \mathbb{R} \times U)$ be quasiconvex in p. Let V be a bounded open subset of U with $\bar{V} \subset U$.*
(i) Suppose $x_0 \in V$, $\varphi \in C^1(V)$, and f is Lipschitz continuous in V. If

$$f(x) - \varphi(x) \le f(x_0) - \varphi(x_0) = 0 \quad \textit{in } V, \tag{2.7}$$

then

$$H(D\varphi(x_0), \varphi(x_0), x_0) \le \lim_{r \downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(Df, f, x). \tag{2.8}$$

(ii) Let u be an absolute minimizer for H in U. Assume that $x_0 \in V$ and f is a Lipschitz continuous function in V satisfying

$$u(x) - f(x) \le u(x_0) - f(x_0) = 0 \quad \text{for } x \in \partial V. \tag{2.9}$$

Then

$$\lim_{r\downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(Du, u, x) \le \operatorname{ess\,sup}_V H(Df, f, x). \tag{2.10}$$

Proof. To prove (i), first note that, for reasons of continuity,

$$\lim_{r\downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(Df, f, x) = \lim_{r\downarrow 0} \operatorname{ess\,sup}_{B_r(x_0)} H(Df, f(x_0), x_0) \tag{2.11}$$

Without loss of generality (see (2.5)), we assume that

$$f(x) - \varphi(x) < f(x_0) - \varphi(x_0) = 0 \quad \text{for } x \in \overline{B}_r(x_0) \setminus \{x_0\} \tag{2.12}$$

for small $0 < r$. Let f_ε be a standard mollification of f and $x_\varepsilon \in \overline{B}_r(x_0)$ satisfy

$$f_\varepsilon(x_\varepsilon) - \varphi(x_\varepsilon) = \max_{\bar{B}_r(x_0)} (f_\varepsilon - \varphi).$$

In view of (2.12), $x_\varepsilon \to x_0$ as $\varepsilon \downarrow 0$. Hence, for small ε,

$$\begin{aligned} H(D\varphi(x_\varepsilon), \varphi(x_\varepsilon), x_\varepsilon) &= H(Df_\varepsilon(x_\varepsilon), \varphi(x_\varepsilon), x_\varepsilon) \\ &\le \operatorname{ess\,sup}_{x\in B_r(x_0)} H(Df(x), \varphi(x_\varepsilon), x_\varepsilon) \\ &= \operatorname{ess\,sup}_{x\in B_r(x_0)} H(Df(x), f(x_0), x_0) + \mathrm{o}(1) \end{aligned}$$

as $\varepsilon \downarrow 0$. The inequality above is due to the quasiconvexity of H in the p variable (see the form of Jensen's inequality in [7]), while the equality is from $x_\varepsilon \to x_0$, $\varphi(x_0) = f(x_0)$, and uniform continuity of H on compact sets. Sending $\varepsilon \downarrow 0$, then $r \downarrow 0$, the result follows (recall (2.11)).

We turn to (ii). Set

$$f_{\varepsilon,\delta}(x) = f(x) + \varepsilon |x - x_0|^2 - \delta. \tag{2.13}$$

Then

$$u(x_0) - f_{\varepsilon,\delta}(x_0) = \delta > 0,$$

while, on ∂V,

$$u(x) - f_{\varepsilon,\delta}(x) \le u(x) - f(x) - \varepsilon \min_{\partial V} |x - x_0|^2 + \delta \le -\varepsilon \min_{\partial V} |x - x_0|^2 + \delta.$$

It follows that if

$$-\varepsilon \min_{\partial V} |x - x_0|^2 + \delta < 0, \tag{2.14}$$

then there is a nonempty connected component V' of $\{x \in V : u(x) - f_{\varepsilon,\delta}(x) > 0\}$ which contains x_0 and is compactly contained in V. Then $u = f_{\varepsilon,\delta}$ on $\partial V'$; in consequence, since u is absolutely minimizing for H, we have, for $B_r(x_0) \subset V'$,

$$\begin{aligned} \operatorname{ess\,sup}_{B_r(x_0)} H(Du, u, x) &\le \operatorname{ess\,sup}_{V'} H(Du, u, x) \\ &\le \operatorname{ess\,sup}_{V'} H(Df_{\varepsilon,\delta}, f_{\varepsilon,\delta}, x) \le \operatorname{ess\,sup}_V H(Df_{\varepsilon,\delta}, f_{\varepsilon,\delta}, x). \end{aligned}$$

The relation (2.10) follows upon sending $r \downarrow 0$ and then $\varepsilon, \delta \downarrow 0$, subject to (2.14).

The final preliminary observation of this section is that we may assume, without loss of generality, that

$$\lim_{|p|\to\infty} H(p,z,x) = +\infty \text{ uniformly for } \ (z,x) \in \mathbb{R} \times \bar{U}. \tag{2.15}$$

This will simplify technicalities below. All of our conclusions are local, so we may assume that $\bar{U}$ is compact, $H \in C^1(\mathbb{R}^n \times \mathbb{R} \times \bar{U})$, $u \in C(\bar{U})$ and Du is bounded. To reduce to the case in which (2.15) holds, let u be the absolutely minimizing function under consideration and put

$$\begin{aligned} &R := \|Du\|_{L^\infty(U)} + \max_{\bar{U}} |u(x)| + 1, \\ &M := \min\left\{H(p,z,x) : |p|, |z| \le R, x \in \bar{U}\right\}, \end{aligned} \tag{2.16}$$

and let P_R be the radial retraction of $\mathbb{R}^n$ on $\overline{B}_R(0)$, as given by

$$P_R(p) = \begin{cases} p, \ |p| \le R, \\ R\dfrac{p}{|p|}, \ |p| \ge R. \end{cases}$$

Now define

$$\hat{H}(p,z,x) = \max(H(p,z,x), |p - P_R p| + M). \tag{2.17}$$

Since the maximum of quasiconvex functions is quasiconvex, $\hat{H}$ is quasiconvex in p. Moreover, by the construction,

$$\begin{aligned} &\hat{H}(Du(x), u(x), x) = H(Du(x), u(x), x) \text{ for } x \in \bar{U}, \\ &H \le \hat{H} \text{ and } \hat{H} \text{ satisfies (2.15) in place of } H. \end{aligned} \tag{2.18}$$

Thus u is absolutely minimizing for $\hat{H}$. Finally, if (2.4) holds, then $|D\varphi(x_0)| < R$ (this is well-known and also a consequence of Proposition 2.3 (i) with $H(p,z,x) = |p|$), so the derivatives of H required to compute $\mathcal{A}[\varphi](x_0)$ exist and are the same for H and $\hat{H}$.

3. Proofs of Theorems 2.1 and 2.2

3.1. **The Main Ideas in a Simple Case.** Here we take $H = H(p) = |p|^2$ to illustrate the main new idea in a simple case. The proof resembles that in [5], as modified in [10], in that "comparison with cones" is used to derive the Aronsson equation in the viscosity sense quite directly. However, it uses a new twist which permits generalizations not otherwise easily obtained. Barron and Jensen [6] also used a related argument, in a technically more complex way and setting.

Assume that $u \in C(U)$ is absolutely minimizing for H and let (2.5) hold for $\varphi, x_0, x \in \overline{B}_r(x_0) \subset U$. Let

$$k_r = \frac{\max_{x\in\partial B_r(x_0)} u(x) - u(x_0)}{r} = \frac{u(x_r) - u(x_0)}{r}, \tag{3.1}$$

where $x_r \in \partial B_r(x_0)$. Here k_r is the least number such that

$$u(x) \le u(x_0) + k_r|x - x_0| \text{ for } x \in \partial B_r(x_0). \tag{3.2}$$

Assuming that (2.4) holds, we establish the claim:

$$|D\varphi(x_0)| \le k_r. \tag{3.3}$$

Taking $f(x) = u(x_0) + k_r|x - x_0|$ and $V = B_r(x_0)$ in Proposition 2.3 (ii) yields

$$\lim_{\tau \downarrow 0} \operatorname{ess\,sup}_{B_\tau(x_0)} |Du| \le |k_r| = k_r, \tag{3.4}$$

while Proposition 2.3 (i) with $f = u$ yields

$$|D\varphi(x_0)| \le \lim_{\tau \downarrow 0} \operatorname{ess\,sup}_{B_\tau(x_0)} |Du|. \tag{3.5}$$

The estimate (3.3) follows from (3.4) coupled with (3.5).

In the last equality of (3.4), we took as known that $0 \le k_r$, which corresponds to the simple fact that if u is absolutely minimizing and $u \le c$ on ∂V, then $u \le c$ in V. In all, the argument is a variant, which easily generalizes, of well-established reasonings.

From (2.5) and (3.3), we find

$$\begin{aligned} |D\varphi(x_0)| \le k_r =& \frac{u(x_r) - u(x_0)}{r} \le \frac{\varphi(x_r) - \varphi(x_0)}{r} \\ &= \int_0^1 D\varphi(x_0 + t(x_r - x_0)) \cdot \left(\frac{x_r - x_0}{r} \right) dt. \end{aligned} \tag{3.6}$$

We deduce several things from this. First, since $(x_r - x_0)/r$ is a unit vector, if it has an accumulation point ω as $r \downarrow 0$, then

$$|D\varphi(x_0)| \le \liminf_{r \downarrow 0} k_r \le D\varphi(x_0) \cdot \omega.$$

Hence, if, as we are assuming (see (2.6)), $D\varphi(x_0) \neq 0$, then

$$\omega = \frac{D\varphi(x_0)}{|D\varphi(x_0)|}, \text{ which implies that } \lim_{r \downarrow 0} \frac{x_r - x_0}{r} = \frac{D\varphi(x_0)}{|D\varphi(x_0)|}. \tag{3.7}$$

Next, again since $(x_r - x_0)/r$ is a unit vector, (3.6) implies that there must exist $0 < t_r < 1$ such that for

$$x_{t_r} = x_0 + t_r(x_r - x_0)$$

we have

$$|D\varphi(x_0)| \le |D\varphi(x_{t_r})|. \tag{3.8}$$

By Taylor approximation,

$$\begin{aligned} &|D\varphi(x_{t_r})|^2 - |D\varphi(x_0)|^2 = \\ &\qquad 2\left((D\varphi(x_0) D^2\varphi(x_0)) \cdot (x_{t_r} - x_0) \right) + \mathrm{o}(|x_{t_r} - x_0)|. \end{aligned} \tag{3.9}$$

Using (3.7), (3.8) and the above, we find

$$0 \le \lim_{r\downarrow 0} \frac{|D\varphi(x_{t_r})|^2 - |D\varphi(x_0)|^2}{|x_{t_r} - x_0|}$$
$$= \frac{2}{|D\varphi(x_0)|}\left((D\varphi(x_0)D^2\varphi(x_0)) \cdot D\varphi(x_0)\right).$$

3.2. **The General Strategy.** We explain the basic ideas, motivated by the simple case, to find the proofs given below. However, we will have to modify these ideas a bit to actually make it all work.

- Step I: use the idea of comparison with cones to find proper cone functions C_r to generalize the role of $k_r|x - x_0|$ in (3.2) in the form

$$u(x) \le u(x_0) + C_r(x, x_0) \text{ on } \partial B_r(x_0). \tag{3.10}$$

- Step II: Find a point $x_{r'}$ in $B_r(x_0) \setminus \{x_0\}$ such that

$$H(D\varphi(x_0), \varphi(x_0), x_0) \le H(D\varphi(x_{r'}), \varphi(x_{r'}), x_{r'})$$

  and (something close to)

$$C_r(x_{r'}, x_0) \le \varphi(x_{r'}) - \varphi(x_0). \tag{3.11}$$

  From (the precise variant of) (3.11), derive that

$$\lim_{r\downarrow 0} \frac{x_{r'} - x_0}{|x_{r'} - x_0|} = \lambda H_p(D\varphi(x_0), \varphi(x_0), x_0)$$

  for some $\lambda > 0$.
- Step III: Derive $\mathcal{A}[\varphi](x_0) \ge 0$ (see (2.2)) using Step II and

$$\lim_{r\downarrow 0} \frac{H(D\varphi(x_{r'}), \varphi(x_{r'}), x_{r'}) - H(D\varphi(x_0), \varphi(x_0), x_0)}{|x_{r'} - x_0|} \ge 0.$$

For the case $H = H(p, z, x)$, i.e, H has z dependence, we also use the idea of changing of variables as in [9] to make $H_z \ge 0$ in some suitable domain.

3.3. **The Proof Theorem 2.1: The Case $H = H(p, x)$.** We always assume in this section that

$$H = H(p, x) \in C(\mathbb{R}^n \times \bar{U})$$

is quasiconvex in p and independent of z. Since the result we seek to prove is local, we hereafter replace U by

$$B_R(x_0) \text{ where } R > 0 \text{ and } \overline{B}_R(x_0) \subset U. \tag{3.12}$$

Moreover, taking R sufficiently small, we may also assume that

$$
\begin{aligned}
&\text{(i) } u\in C(\overline{B}_R(x_0)) \text{ is absolutely minimizing for } H \text{ in } B_R(x_0).\\
&\text{(ii) } \varphi\in C^2(\overline{B}_R(x_0)),\ x_0\in U \text{ and (2.5) holds in the form}\\
&\qquad u(x)-u(x_0)<\varphi(x)-\varphi(x_0) \text{ for } x\in \overline{B}_R(x_0)\setminus\{x_0\}\,,\text{ and } u(x_0)=\varphi(x_0).\\
&\text{(iii) (2.15) holds.}
\end{aligned}
\tag{3.13}
$$

The appropriate "cone functions" are found in [8] and [15]. To begin, for $k\in\mathbb{R}$, $x\in\overline{B}_R(x_0)$ and $p\in\mathbb{R}^n$ one defines

$$
L(p,x,k)=\max_{\{q\in\mathbb{R}^n: H(q,x)\le k\}} q\cdot p. \tag{3.14}
$$

We will always assume that L is well-defined and finite on arguments which appear by asking that $k\ge k_0(r)$ when we are working in $\overline{B}_r(x_0)$, where

$$
k_0(r) \text{ is the least number } k \text{ for which } \left\{p: \max_{x\in\overline{B}_r(x_0)} H(p,x)\le k\right\} \text{ is nonempty.}
$$

There are other ways to display $k_0(r)$. We have

$$
k_0(r)=\min_{p\in\mathbb{R}^n}\max_{x\in\overline{B}_r(x_0)} H(p,x), \tag{3.15}
$$

which is attained and finite by (2.15). We will also use that $k_0(r)=\bar{H}(p_0)$ where

$$
\bar{H}(p)=\max_{x\in\overline{B}_r(x_0)} H(p,x), \bar{H}(p_0)=\min_{\mathbb{R}^n}\bar{H}(p). \tag{3.16}
$$

Here $\bar{H}$ and p_0 depend on r, but we leave this dependence implicit.

Remark 3.1. If $H(p,x)\ge H(0,x)=0$ for all p,x, then $k_0(r)=0$ for all r.

In view of (3.14) and (2.15), L has the following properties - all "obvious" - as a function of $x\in\overline{B}_r(x_0)$, $p\in\mathbb{R}^n$ and $k_0(r)\le k$:

$$
\begin{aligned}
&\text{(i) } x\mapsto L(p,x,k) \text{ is upper-semicontinuous,}\\
&\text{(ii) } p\mapsto L(p,x,k) \text{ is Lipschitz continuous with a constant depending only on } k,\\
&\text{(iii) } p\mapsto L(p,x,k) \text{ is convex, positively 1-homogeneous, and } L(0,x,k)=0,\\
&\text{(iv) If } 0<M, \text{ then there is a } k_M \text{ such that } L(p,x,k)\ge M|p| \text{ for } k_M\le k,\\
&\text{(v) } k\mapsto L(p,x,k) \text{ is nondecreasing and continuous from the right.}
\end{aligned}
\tag{3.17}
$$

Let

$$
L(p,x,k-)=\lim_{\hat{k}\uparrow k} L(p,x,\hat{k}),\ \ L(p,x,k+)=\lim_{\hat{k}\downarrow k} L(p,x,\hat{k}). \tag{3.18}
$$

While $L(p,x,k+)=L(p,x,k)$ ((3.17) (v)), in general $L(p,x,k-)<L(p,x,k)$. The condition which rules this out is

$$
\partial\left\{q: H(q,x)<k\right\}=\left\{q: H(q,x)=k\right\}. \tag{3.19}
$$

Clearly (3.19) holds if H is convex in p and $k>\min_{q\in\mathbb{R}^n} H(q,x)$.

Let $r \le R$ and $x \in \overline{B}_r(x_0)$. A *path* from x_0 to x in $\overline{B}_r(x_0)$ is an absolutely continuous mapping $\xi : [0,T] \to \overline{B}_r(x_0)$ such that $\xi(0) = x_0$ and $\xi(T) = x$ where $0 \le T$. The set of such paths is denoted by $\text{path}\,(x,r)$:

$$\text{path}\,(x,r) = \left\{ \text{paths } \xi \text{ from } x_0 \text{ to } x \text{ in } \overline{B}_r(x_0) \right\}. \tag{3.20}$$

In the discourse, if ξ, T occur together, then it is assumed that $[0,T]$ is the domain of ξ.

We note right away that if p_0 is from (3.16) and $\xi \in \text{path}\,(x,r)$, then for $y \in \overline{B}_r(x_0)$ and $k \ge k_0(r)$, we have

$$H(p_0, y) \le k_0(r) \implies L(p, y, k) \ge p_0 \cdot p. \tag{3.21}$$

Hence, if $\xi \in \text{path}\,(x,r)$,

$$p_0 \cdot (x - x_0) = \int_0^T p_0 \cdot \dot{\xi}(t)\, dt \le \int_0^T L(\dot{\xi}(t), \xi(t), k)\, dt. \tag{3.22}$$

Here $\dot{\xi}$ is the derivative of ξ.

It follows that for $k_0(r) \le k$, and $x \in \overline{B}_r(x_0)$ the quantity

$$C_{k,r}(x, x_0) := \inf \left\{ \int_0^T L(\dot{\xi}(t), \xi(t), k)\, dt : \xi \in \text{path}\,(x,r) \right\}. \tag{3.23}$$

is well-defined and finite. The $C_{k,r}$ will provide our "cone functions." By (3.17) (v), $k \mapsto C_{k,r}$ is nondecreasing. We set

$$C_{k-,r}(x, x_0) = \lim_{\hat{k} \uparrow k} C_{\hat{k},r}(x, x_0), \;\; C_{k+,r}(x, x_0) = \lim_{\hat{k} \downarrow k} C_{\hat{k},r}(x, x_0). \tag{3.24}$$

Since $L(0, x, k) = 0$, $C_{k,r}(x_0, x_0) \le 0$. It follows from this and (3.22) that

$$C_{k,r}(x_0, x_0) = 0. \tag{3.25}$$

Next, note that if ξ is a path from x_0 to $x \in \overline{B}_r(x_0)$, and $y \in \overline{B}_r(x_0)$, then $\eta(t) = \xi(t)$, $0 \le t \le T$, $\eta(t) = x + (t - T)(y - x)$ for $T \le t \le T + 1$ is a path from x_0 to y in $\overline{B}_r(x_0)$. Hence

$$C_{k,r}(y, x_0) \le \int_0^T L(\dot{\xi}(t), \xi(t), k)\, dt + \int_0^1 L(y - x, x + t(y - x), k)\, dt,$$

which implies

$$C_{k,r}(y, x_0) \le C_{k,r}(x, x_0) + K|y - x| \tag{3.26}$$

where

$$K = \max_{w \in \overline{B}_r(x_0)} \max_{H(q,w) \le k} |q|$$

is finite, again owing to (2.15). That is, $x \mapsto C_{k,r}(x, x_0)$ is Lipschitz continuous in $\overline{B}_r(x_0)$. In particular, the gradient $DC_{k,r}(x, x_0)$ exists for almost all x by Rademacher's Theorem. We have recalled the proof of:

Lemma 3.2. *Let (2.15) hold. Then $x \mapsto C_{k,r}(x, x_0)$ is Lipschitz continuous on $\overline{B}_r(x_0)$, uniformly for bounded k, $k_0(r) \le k$.*

Further properties of $C_{k,r}$ are established in the Appendix. According to Fathi-Siconolfi [15], the following lemma holds.

Lemma 3.3. *Suppose (2.15) holds and $k_0(r) \le k$. Then $C_{k,r}$ is a viscosity solution of*

$$H(DC_{k,r}(x,x_0),x) = k \quad \textit{in } B_r(x_0)\backslash\{x_0\}.$$

In particular, $H(DC_{k,r}(x,x_0),x) = k$ a.e.

Assuming (3.13) and $0 < r \le R$, define k_r as follows

$$k_r = \inf\left\{k : k_0(r) \le k \text{ and } u(x) \le u(x_0) + C_{k,r}(x,x_0) \text{ for } x \in \partial B_r(x_0)\right\}. \tag{3.27}$$

We recall that $k_0(r)$ is defined in (3.15) (and apologize for the distracting simultaneous use of $k_0(r)$ and k_r.)

The quantity k_r is well-defined due to (3.17) (iv), which implies that for any $M > 0$ we have

$$C_{k,r}(x,x_0) \ge Mr \text{ for } x \in \partial\overline{B}_r(x_0)$$

provided k is sufficiently large. In fact, by

Several lemmas provide the core of the proof of Theorem 2.1.

Lemma 3.4. *Let (3.13) hold and $0 < r \le R$. Then $H(D\varphi(x_0),x_0) \le k_r$.*

Proof. First observe that if $k_r < k$, then, via the definition of k_r and the fact that $k \mapsto C_k(x,x_0)$ is nondecreasing, we have

$$u(x) \le u(x_0) + C_{k,r}(x,x_0) \text{ for } x \in \partial B_r(x_0). \tag{3.28}$$

We claim that

$$H(D\varphi(x_0),x_0) \le \lim_{s\downarrow 0} \operatorname{ess\,sup}_{B_s(x_0)} H(Du,x) \le \operatorname{ess\,sup}_{B_r(x_0)} H(DC_k,x) = k.$$

for $k_r < k$, whence the result. The first inequality is from Proposition 2.3 (i) with $f = u$, the second from 3.28 and Proposition 2.3 (ii) with $f(x) = C_{k,r}(x,x_0) + u(x_0)$, and the equality is from Lemma 3.3. □

Lemma 3.5. *Let $H \in C^1(\mathbb{R}^n \times B_R(x_0))$, $\varphi \in C^1(\overline{B}_R(x_0))$, $H_p(D\varphi(x_0),x_0) \ne 0$ and*

$$h_0 = H(D\varphi(x_0),x_0). \tag{3.29}$$

Then $k_0(r) < h_0$ for sufficiently small r.

Proof. Since $H_p(D\varphi(x_0),x_0) \ne 0$, there is a p such that $H(p,x_0) < h_0$, and then an $r > 0$ such that $H(p,x) < h_0$ for $x \in \overline{B}_r(x_0)$. But this implies that $k_0(r) < h_0$. □

Lemma 3.6. *Let (3.13) hold and $H \in C^1(\mathbb{R}^n \times \overline{B}_R(x_0))$ and $H_p(D\varphi(x_0),x_0) \ne 0$. Then for sufficiently small $0 < r$, there exists a path $\xi : [0,T_r] \to \overline{B}_r(x_0)$, such that $\xi(T_r) \ne x_0$ and*

$$\begin{aligned} &\int_0^{T_r} L(\dot\xi,\xi,h_0-)\,dt < \varphi(\xi(T_r)) - \varphi(x_0) \textit{ and} \\ &h_0 = H(D\varphi(x_0),x_0) \le H(D\varphi(\xi(T_r)),\xi(T_r)). \end{aligned} \tag{3.30}$$

Proof. By the definition of k_r and $h_0 \le k_r$, for $k < h_0$ there is an $x_k \in \partial B_r(x_0)$ such that

$$C_{k,r}(x_k, x_0) \le u(x_k) - u(x_0).$$

Let $k \uparrow h_0$ along a sequence such that $x_k \to y_r \in \partial B_r(x_0)$. This yields

$$C_{h_0-,r}(y_r, x_0) \le u(y_r) - u(x_0).$$

Let $\xi \in \text{path}\,(y_r, r)$ be provided by Lemma 4.3, that is $\xi \in \text{path}\,(y_r, r)$ and

$$\int_0^T L(\dot\xi, \xi, h_0-)\,dt = C_{h_0-,r}(y_r, x_0) = C_{h_0-,r}(\xi(T), x_0).$$

Combining the two relations above with $u(y_r) - u(x_0) < \varphi(y_r) - \varphi(x_0)$ yields

$$\int_0^T L(\dot\xi, \xi, h_0-)\,dt < \varphi(\xi(T)) - \varphi(x_0) = \int_0^T \frac{d}{dt}\varphi(\xi(t))\,dt = \int_0^T D\varphi(\xi(t)) \cdot \dot\xi(t)\,dt. \tag{3.31}$$

Thus there are positive values of t such that $\dot\xi(t)$ exists and

$$L(\dot\xi(t), \xi(t), h_0-) < D\varphi(\xi(t)) \cdot \dot\xi(t).$$

By the definition of L, this implies that $H(D\varphi(\xi(t)), \xi(t)) \ge h_0$. Let $t_r \in [0, T]$ be the largest value of t for which $H(D\varphi(\xi(t)), \xi(t)) \ge h_0$. Then $H(D\varphi(\xi(t)), \xi(t)) < h_0$ on $(t_r, T]$ implies

$$\varphi(\xi(T)) - \varphi(\xi(t_r)) = \int_{t_r}^T D\varphi(\xi(t)) \cdot \dot\xi(t)\,dt \le \int_{t_r}^T L(\dot\xi, \xi, h_0-)\,dt,$$

and so, using (3.31),

$$\int_0^{t_r} L(\dot\xi, \xi, h_0-)\,dt + \varphi(\xi(T)) - \varphi(\xi(t_r)) < \varphi(\xi(T)) - \varphi(x_0)$$

or, using also the definition of t_r,

$$\int_0^{t_r} L(\dot\xi, \xi, h_0-)\,dt < \varphi(\xi(t_r)) - \varphi(x_0) \text{ and } H(D\varphi(x_0), x_0) = h_0 \le H(\varphi(\xi(t_r)), \xi(t_r)). \tag{3.32}$$

It remains to remark that $\xi(t_r) \ne x_0$. Indeed, if it were the case that $\xi(t_r) = x_0$, then the integral on the left of (3.32) is nonnegative by (3.22), in contradiction to the strict inequality. The assertions of the lemma thus hold if we put $T_r = t_r$ and replace ξ by its restriction to $[0, T_r]$. □

Remark 3.7. The conditions (3.13) were assumed in Lemma 3.6. However, all that was used in the proof was $k_0(r) < h_0 \le k_r$ and (3.13) (ii), with C^1 in place of C^2. The inequality $k_0(r) < h_0$ was a trivial consequence of $H \in C^1$ and $H_p(D\varphi(x_0), \varphi(x_0)) \ne 0$ (Lemma 3.5), while Lemma 3.4 supplied $h_0 \le k_r$.

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Take $r = 1/m$ and m sufficiently large so that the assertions of Lemma 3.6 hold. Let $\xi, T_m = T_{1/m}$ (ξ varies with m, but we have enough subscripts) be provided by the lemma and put $x_m = \xi(T_m)$. We have

$$0 < |x_m - x_0| \leq \frac{1}{m},\ \xi \in \text{path}\left(x_m, \frac{1}{m}\right),$$
$$\int_0^{T_m} L(\dot{\xi}, \xi, h_0-)\, dt < \varphi(x_m) - \varphi(x_0),$$
$$h_0 = H(D\varphi(x_0), x_0) \leq H(D\varphi(x_m), x_m).$$

Passing to a subsequence, we can assume that

$$\lim_{m\to\infty} \frac{x_m - x_0}{|x_m - x_0|} = Q \in \partial B_1(0). \tag{3.33}$$

Put

$$\bar{H}_m(p) = \max_{x \in \overline{B}_{\frac{1}{m}}(x_0)} H(p, x). \tag{3.34}$$

For $\delta > 0$ we have

$$\begin{aligned} \max_{\bar{H}_m(q) \leq h_0 - \delta} (q \cdot (x_m - x_0)) &\leq \int_0^{T_m} \max_{\bar{H}_m(q) \leq h_0 - \delta} \left(q \cdot \dot{\xi}(t)\right)\, dt \\ &\leq \int_0^{T_m} L(\dot{\xi}(t), \xi(t), h_0-)\, dt \\ &< \varphi(x_m) - \varphi(x_0). \end{aligned} \tag{3.35}$$

Dividing both of the extremes above by $|x_m - x_0|$ and sending $m \to \infty$ yields

$$q \cdot Q \leq D\varphi(x_0) \cdot Q \text{ if } H(q, x_0) = \lim_{m\to\infty} \bar{H}_m(q) < h_0 - \delta. \tag{3.36}$$

Thus

$$q \cdot Q \leq D\varphi(x_0) \cdot Q \text{ if } H(q, x_0) < h_0 = H(D\varphi(x_0), x_0). \tag{3.37}$$

This inequality remains true for $q \in C$, where C is the convex set

$$C = \overline{\{q : H(q, x_0) < H(D\varphi(x_0), x_0)\}}. \tag{3.38}$$

Since $H_p(D\varphi(x_0), x_0) \neq 0$, $D\varphi(x_0) \in C$. Thus (3.37) implies

$$D\varphi(x_0) \cdot Q = \max_{q \in C} (q \cdot Q); \tag{3.39}$$

in particular, Q is an exterior normal to C at $D\varphi(x_0)$. As the unique outward normal direction is that of $H_p(D\varphi(x_0), x_0)$, there exists a $\lambda > 0$ such that

$$Q = \lambda H_p(D\varphi(x_0), x_0). \tag{3.40}$$

Finally, by calculus,

$$
\begin{aligned}
0 &\leq \lim_{r\to 0} \frac{H(D\varphi(x_m), x_m) - H(D\varphi(x_0), x_0)}{|x_m - x_0|} \\
&= \left(H(D\varphi(x), x)_x \Big|_{x=x_0} \right) \cdot Q \qquad (3.41)\\
&= \lambda \left(H(D\varphi(x), x)_x \Big|_{x=x_0} \right) \cdot H_p(D\varphi(x_0), x_0).
\end{aligned}
$$

□

3.4. **The Proof Theorem 2.2: The Case $H = H(p, z, x)$.**

We assume throughout this section that

$$H = H(p, z, x) \in C(\mathbb{R}^n \times \mathbb{R} \times \overline{B}_R(x_0))$$

is quasiconvex in p. When necessary, we strengthen this to requiring convexity in p. As in Section 3.3, we will also refer to the conditions (3.13). In addition, we will assume, when (3.13) holds, that for some $\varepsilon > 0$

$$0 \leq H_z(p, z, x) \tag{3.42}$$

if $|z - \varphi(x_0)| + |x - x_0| < \varepsilon$ and $|H(D\varphi(x_0), \varphi(x_0), x_0) - H(p, z, x)| \leq \varepsilon$. This condition might be satisfied simply because H is nondecreasing in z; alternatively, if H is C^1 and convex in p, we may attain it via a change of variables. This is established at the end of this section. Note, however, that if H is nondecreasing in z, then $H(-p, -z, x)$ is not, unless H is independent of z. Thus there is no "two sided" condition of this kind as regards showing that u is both a sub and supersolution of the Aronsson equation.

Put

$$H^*(p, x) := H(p, u(x), x) \tag{3.43}$$

and let $k_0^*(r)$, L^* be computed from H^* as $k_0(r)$, L were from H in Section 3.3. Let $C^*_{k,r}$ be computed from L^* as $C_{k,r}$ was from L and k^*_r be computed from the $C^*_{k,r}$ as k_r was from the $C_{k,r}$ in Section 3.3. For example,

$$L^*(x, p, k) = \max_{H^*(q,x)\leq k} q \cdot p = \max_{H(q,u(x),x)\leq k} q \cdot p,$$

and

$$C^*_{k,r}(x, x_0) = \inf \left\{ \int_0^T L^*(\xi, \dot{\xi}, k)\, dt : \xi \in \text{path}\,(x, r) \right\}.$$

According to Lemma 3.3

$$H(DC^*_{k,r}(x), u(x), x) = k \text{ a.e. in } B_r(x_0). \tag{3.44}$$

Lemma 3.8. *Let (3.13) and (3.42) hold. Then, when r is small enough,*

$$h_0 := H(D\varphi(x_0), \varphi(x_0), x_0) \leq k^*_r. \tag{3.45}$$

Proof. If $h_0 \le k_0^*(r)$, there is nothing to prove, as $k_0^*(r) \le k_r^*$ by definition. Hence assume that $k_0^*(r) < h_0$. We argue by contradiction. If (3.45) does not hold, then there exists $0 < \delta$ such that $k_0^*(r), k_r^* < h_0 - \delta$. Then Lemma 4.4 implies that

$$u(x) < u(x_0) + C^*_{h_0-\delta,r}(x,x_0) \quad \text{on } \partial B_r(x_0).$$

Choose $\kappa > 0$ such that

$$u(x) < u(x_0) + C^*_{h_0-\delta,r}(x,x_0) - \kappa \quad \text{on } \partial B_r(x_0).$$

Let

$$V = \{x \in B_r(x_0) : \ u(x) > u(x_0) + C^*_{h_0-\delta,r}(x,x_0) - \kappa\}.$$

Note that $x_0 \in V$ since $C^*_{k,r}(x_0,x_0) = 0$ ((3.25)). We have that

$$u = u(x_0) + C^*_{h_0-\delta,r}(x,x_0) - \kappa \text{ on } \partial V.$$

Since $x_0 \in V$, according to Proposition 2.3 (i), the absolutely minimizing property of u implies

$$\begin{aligned} h_0 &\le \operatorname{ess\,sup}_V H(Du,u,x) \\ &\le \operatorname{ess\,sup}_V H(DC^*_{h_0-\delta}(x,x_0), u(x_0) + C^*_{h_0-\delta,r}(x,x_0) - \kappa, x). \end{aligned} \tag{3.46}$$

Now, by continuity of H, u, $u(x_0) = \varphi(x_0)$, and $C^*_{h_0-\delta}(x_0,x_0) = 0$,

$$H(DC^*_{h_0-\delta}(x,x_0), u(x_0) + C^*_{h_0-\delta,r}(x,x_0) - \kappa, x) - H(DC^*_{h_0-\delta}(x,x_0), u(x), x)$$

can be made as small as we like for $x \in \overline{B}_r(x_0)$ by choosing r, κ sufficiently small. Moreover,

$$H(DC^*_{h_0-\delta}(x,x_0), u(x), x) = h_0 - \delta = H(D\varphi(x_0), \varphi(x_0), x_0) - \delta.$$

Thus, by (3.42), (3.46) and the definition of V, if r, κ, δ are sufficiently small,

$$\begin{aligned} &\operatorname{ess\,sup}_V H(DC^*_{h_0-\delta,r}(x,x_0), u(x_0) + C^*_{h_0-\delta,r}(x) - \kappa, x) \\ &\qquad \le \operatorname{ess\,sup}_V H(DC^*_{h_0-\delta,r}(x,x_0), u(x), x) = h_0 - \delta. \end{aligned}$$

Hence $h_0 \le h_0 - \delta$. This is a contradiction. □

Lemma 3.9. *Let (3.13) hold, $H \in C^1(\mathbb{R}^n \times \mathbb{R} \times \overline{B}_R(x_0))$ and $H_p(D\varphi(x_0), \varphi(x_0), x_0) \neq 0$. Then for sufficiently small $0 < r$, there exists a path $\xi : [0, T_r] \to \overline{B}_r(x_0)$, such that $\xi(T_r) \neq x_0$ and*

$$\begin{aligned} &\int_0^{T_r} L^*(\xi, \dot{\xi}, h_0-)\, dt < \varphi(\xi(T_r)) - \varphi(x_0) \ \text{and} \\ &\qquad h_0 = H(D\varphi(x_0), \varphi(x_0), x_0) \le H(D\varphi(\xi(T_r)), \varphi(\xi(T_r)), \xi(T_r)). \end{aligned} \tag{3.47}$$

Proof. We may directly apply Lemma 3.6 and Remark 3.7 to the $C^*_{k,r}$, as we have $h_0 \le k_r^*$ from Lemma 3.8 and $k_0^*(r) < h_0$ by the proof of Lemma 3.5. The result is (3.47), but with

$$h_0 \le H^*(D\varphi(\xi(T_r)), \xi(T_r)) = H(D\varphi(\xi(T_r)), u(\xi(T_r)), \xi(T_r))$$

on the right of the final inequality of (3.47). However, by (3.42), if r is sufficiently small, we may use $u(\xi(T_r)) \le \varphi(\xi(T_r))$ to make the replacement which results in (3.47). □

<u>*Proof of Theorem 2.2*</u>. In view of Lemma 3.9, the proceedings (3.33)-(3.40) remain valid with H, L, replaced with H^*, L^*. In particular, (3.40) becomes

$$Q = \lambda H_p(D\varphi(x_0), \varphi(x_0), x_0).$$

Using (3.47), (3.41) becomes

$$\begin{aligned} 0 &\le \lim_{r\to 0} \frac{H(D\varphi(x_m), \varphi(x_m), x_m) - H(D\varphi(x_0), \varphi(x_0), x_0)}{|x_m - x_0|} \\ &= \left(H(D\varphi(x), \varphi(x), x)_x\big|_{x=x_0} \right) \cdot Q \qquad (3.48) \\ &= \lambda \left(H(D\varphi(x), \varphi(x), x)_x\big|_{x=x_0} \right) \cdot H_p(D\varphi(x_0), \varphi(x_0), x_0). \end{aligned}$$

□

We conclude this section with the demonstration that (3.42) may be attained via a change of variables if H is C^1 and convex in p, even if the "original H" is not nondecreasing in z. The demonstration borrows from one in [9]. To this end, we make some reductions. Assuming that (3.13) holds, let

$$H(p_0, \varphi(x_0), x_0) = \min_{\mathbb{R}^n} H(p, \varphi(x_0), x_0)$$

and put

$$\begin{aligned} &\tilde{H}(p, z, x) = H(p + p_0, z + \varphi(x_0) + p_0 \cdot (x - x_0), x), \\ &\tilde{u}(x) = u(x) - u(x_0) - p_0 \cdot (x - x_0),\ \ \tilde{\varphi}(x) = \varphi(x) - \varphi(x_0) - p_0 \cdot (x - x_0). \end{aligned}$$

Then a direct check shows that $\tilde{u}$ is absolutely minimizing for $\tilde{H}$ iff u is absolutely minimizing for H. Moreover, if $\tilde{\mathcal{A}}$ is the Aronsson operator for $\tilde{H}$, then $\tilde{\mathcal{A}}[\tilde{\varphi}](x_0) = \mathcal{A}[\varphi]$. That is, without loss of generality, we can simply assume that

$$u(x_0) = \varphi(x_0) = 0 \ \text{ and } \ H(0, \varphi(x_0), x_0) = \min_{\mathbb{R}^n} H(p, \varphi(x_0), x_0).$$

Since $H_p(D\varphi(x_0), \varphi(x_0), x_0) = H_p(D\varphi(x_0), 0, x_0) \neq 0$, we have

$$H(D\varphi(x_0), 0, x_0) > H(0, 0, x_0). \tag{3.49}$$

Next, it follows from (3.49) that there exists $\delta, \varepsilon' > 0$ such that if

$$|z| + |x - x_0| \le \varepsilon' \text{ and } |H(D\varphi(x_0), 0, x_0) - H(p, z, x)| \le \varepsilon', \tag{3.50}$$

we then have

$$H(p, z, x) - H(0, z, x) \ge \delta. \tag{3.51}$$

Therefore, owing to convexity in the p variable, when (3.50) holds

$$p \cdot H_p(p, z, x) \ge H(p, z, x) - H(0, z, x) \ge \delta. \tag{3.52}$$

Now define w, ψ by

$$u = G(w),\ \ \varphi = G(\psi) \text{ and } \hat{H}(p, z, x) = H(G'(z)p, G(z), x) \tag{3.53}$$

where

$$G(s) = s + \frac{\beta}{2}s^2$$

and $\beta \geq 1$ is to be determined later.

The functions w, ψ are well-defined if

$$-\frac{1}{2\beta} < u, \varphi \tag{3.54}$$

and we ask that $-1/\beta < w, \psi$, as G is a diffeomorphism of $(-1/\beta, \infty)$ onto $(-1/(2\beta), \infty)$. The condition (3.54) is guaranteed by

$$L_0|x - x_0| < \frac{1}{2\beta} \Longleftrightarrow \beta|x - x_0| < \frac{1}{2L_0} \tag{3.55}$$

where L_0 is a Lipschitz common constant for u, φ, as $u(x_0) = \varphi(x_0) = 0$. Moreover, then $\hat{H}(Dw, w, x) = H(Du, u, x)$, etc, and w is absolutely minimizing for $\hat{H}$ in $B_r(x_0)$.

Given $\varepsilon' > 0$ such that (3.50) holds, we may choose $\varepsilon > 0$ such that

$$\begin{aligned} &\beta(|z| + |x - x_0|) \leq \varepsilon \text{ and} \\ &|\hat{H}(D\psi(x_0), 0, x_0) - \hat{H}(p, z, x)| = |H(D\varphi(x_0), 0, x_0) - H(G'(z)p, G(z), x)| \leq \varepsilon \end{aligned} \tag{3.56}$$

implies (3.50) and (3.55). Recall that $\beta \geq 1$, so $\beta|z|$ controls the size of z, as well as the size of the perturbations βz of $G'(z)$ from 1 and $\beta z^2/2$ of $G(z)$ from z. Note that ε is independent of β. Hence

$$\begin{aligned} \hat{H}_z(p, z, x) &= \beta H_p(G'(z)p, G(z), x) \cdot p + (1 + \beta z)H_z(G'(z)p, G(z), x) \\ &= \frac{\beta}{1 + \beta z} H_p(G'(z)p, G(z), x) \cdot G'(z)p + (1 + \beta z)H_z(G'(z)p, G(z), x) \\ &\geq \frac{\delta\beta}{1 + \varepsilon} + (1 + \beta z)H_z(G'(z)p, G(z), x). \end{aligned}$$

Note that (3.56) provides a bound on p. Thus if β is sufficiently large and ε is sufficiently small, we have

$$\hat{H}_z(p, z, x) \geq 0$$

when

$$|z| + |x - x_0| \leq \frac{\varepsilon}{\beta} \text{ and } |H(D\varphi(x_0), 0, x_0) - H(G'(z)p, G(z), x)| \leq \varepsilon. \tag{3.57}$$

Finally, if $\hat{\mathcal{A}}$ is the Aronsson operator for $\hat{H}$, then a calculation shows that

$$\hat{\mathcal{A}}[\psi](x_0) = G'(\varphi(x_0))\mathcal{A}[\varphi](x_0) = G'(0)\mathcal{A}[\varphi](x_0) = \mathcal{A}[\varphi](x_0),$$

and we are done.

4. Appendix

We establish a few properties of the $C_{k,r}$, beginning with a basic lower-semcontinuity and compactness result.

Working with $C_{k,r}$ is simplified if we recall that we may assume $|\dot{\xi}(t)| = 1$ almost everywhere in computing it. This is attained by noting that if $\xi \in \text{path}\,(x, r)$, then

$$\eta : \left[0, \int_0^T |\dot{\xi}(\tau)|\, d\tau\right] \to \overline{B}_r(x_0)$$

is well-defined by

$$\eta\left(\int_0^t |\dot{\xi}(\tau)|\, d\tau\right) = \xi(t). \tag{4.1}$$

Moreover, η has 1 as a Lipschitz constant and $|\dot{\eta}| = 1$ a.e. The substitution $s = \int_0^t |\dot{\xi}(\tau)|\, d\tau$ in

$$\int_0^{T'} L\left(\eta(s), \frac{d}{ds}\eta(s), k\right) ds, \; T' = \int_0^T |\dot{\xi}(\tau)|\, d\tau,$$

yields

$$\int_0^{T'} L\left(\eta(s), \frac{d}{ds}\eta(s), k\right) ds = \int_0^T L(\dot{\xi}(t), \xi(t), k)\, dt \tag{4.2}$$

because L is positive homogeneous of degree 1 in p. Thus

$$\begin{gathered} C_{k,r}(x, x_0) := \inf\left\{\int_0^T L(\dot{\xi}(t), \xi(t), k)\, dt : \xi \in \text{upath}\,(x, r)\right\} \\ \text{where upath}\,(x, r) = \left\{\xi \in \text{path}\,(x, r) : |\dot{\xi}(t)| = 1 \text{ a.e.}\right\} \end{gathered} \tag{4.3}$$

The term "upath" is a mnemonic for "unit speed path."

Lemma 4.1. *Let* $0 \le T_m,\ \xi_m : [0, T_m] \to \overline{B}_r(x_0), m = 1, 2, \ldots$ *be a sequence of Lipschitz continuous paths satisfying* $|\dot{\xi}_m| \le 1$, *a.e.*
(a) Assume that $\lim_{m\to\infty} T_m = T$ *and*

$$\lim_{m\to\infty} \max_{0\le t\le \min(T_m,T)} |\xi_m(t) - \xi(t)| = 0. \tag{4.4}$$

Then for $k_0(r) < k$,

$$\liminf_{m\to\infty} \int_0^{T_m} L(\dot{\xi}_m, \xi_m, k)\, dt \ge \int_0^T L(\dot{\xi}, \xi, k-)\, dt. \tag{4.5}$$

(b) Let $\varepsilon > 0$. *Let* $\xi_m : [0, T_m] \to \overline{B}_r(x_0)$, $m = 1, 2, \ldots,$ *be a sequence of Lipschitz continuous paths satisfying* $|\dot{\xi}_m| = 1$ *ae and for which* $\int_0^{T_m} L(\dot{\xi}_m, \xi_m, k_0(r) + \varepsilon)\, dt$ *is bounded above. Then* ξ_m *has a subsequence satisfying the assumptions of (a) for some* $0 \le T$ *and* $\xi : [0, T] \to \overline{B}_r(x_0)$.

Proof. Part (b) of the Lemma is immediate from standard considerations, once we notice that the T_m are bounded. Let $\bar{H}(p) = \max_{x\in\overline{B}_r(x_0)} H(x,p)$ and p_0 be a minimum point for $\bar{H}$; that is, $\bar{H}(p_0) = k_0(r)$. Then there exists $\delta > 0$ such that

$$B_\delta(p_0) \subset \left\{q : \bar{H}(q) < \bar{H}(p_0) + \varepsilon\right\} \subset \{q : H(x,q) < k_0(r) + \varepsilon\} \text{ for } x \in \overline{B}_r(x_0).$$

Hence

$$\begin{aligned} L(p,x,k_0(r)+\varepsilon) &= \max_{H(q,x)\le k_0(r)+\varepsilon} q\cdot p \\ &= \max_{H(q,x)\le k_0(r)+\varepsilon} (q-p_0)\cdot p + p_0\cdot p \ge \delta|p| + p_0\cdot p. \end{aligned} \tag{4.6}$$

Then $|\dot{\xi}_m| = 1$ ae implies that

$$\delta T_m + p_0\cdot(\xi_m(T_m) - \xi_m(0)) \le \int_0^{T_m} L(\dot{\xi}_m(s), \xi_m(s), k_0(r)+\varepsilon)\, ds.$$

It now follows from the assumptions that T_m is bounded. Passing to a subsequence along which the T_m converge and then another along which the ξ_m converge suitably via Arzela-Ascoli yields the desired T and ξ.

To prove the assertions of part (a), we first note that the integrands in (4.5) are uniformly bounded because $|\dot{\xi}_m|, |\dot{\xi}| \le 1$; that is

$$|L(\dot{\xi}_m, \xi_m, k)|, |L(\dot{\xi}, \xi, k)| \le C \tag{4.7}$$

for some C. If we show that that for $0 < \delta < T$ and $\hat{k} < k$,

$$\liminf_{m\to\infty} \int_0^{T-\delta} L(\dot{\xi}_m, \xi_m, k)\, dt \ge \int_0^{T-\delta} L(\dot{\xi}, \xi, \hat{k})\, dt, \tag{4.8}$$

then it follows from (4.7) and $T_m \to T$ that

$$\liminf_{m\to\infty} \int_0^{T_m} L(\dot{\xi}_m, \xi_m, k)\, dt = \lim_{\delta\downarrow 0} \liminf_{m\to\infty} \int_0^{T-\delta} L(\dot{\xi}_m, \xi_m, k)\, dt \ge \int_0^{T} L(\dot{\xi}, \xi, \hat{k})\, dt.$$

Then (4.5) is obtained, via the monotone convergence theorem, upon letting $\hat{k} \uparrow k$.

Now

$$\left\{q : H(q, \xi(t)) \le \hat{k}\right\} \subset \{q : H(q, \xi_m(t)) \le k\} \text{ for } 0 \le t \le T-\delta$$

as soon as m is sufficiently large by continuity of H and (4.4). Thus, for large m,

$$L(\dot{\xi}_m(t), \xi_m(t), k) \ge L(\dot{\xi}(t), \xi_m(t), \hat{k})$$

and so

$$\int_0^{T-\delta} L(\dot{\xi}_m(t), \xi_m(t), k)\, dt \ge \int_0^{T-\delta} L(\dot{\xi}(t), \xi_m(t), \hat{k})\, dt. \tag{4.9}$$

Pass now to a subsequence along which the lim inf is attained and extract a further subsequence along which $\dot{\xi}_m$ converges weakly in $L^2(0, T-\delta)$. It must be that the weak limit is $\dot{\xi}$, and the integrand on the right of (4.9) is convex in its second argument, hence the integral is lower semicontinuous with respect to weak convergence. The result follows. □

The next result is an important tool for us. It is the variant of the existence of a minimizing path valid in our situation.

Proposition 4.2. *Let $0 < r \le R$ and $k_0(r) < k$. Then for $x \in \overline{B}_r(x_0)$ there exists $\xi \in$ path (x, r) such that*

$$\int_0^T L(\dot{\xi}, \xi, k-)\, dt = C_{k-,r}(x, x_0). \tag{4.10}$$

Proof. Let $x \in \overline{B}_r(x_0)$. For each pair of positive integers $l \le m$, there is a $\xi_m \in$ upath (x,r) such that

$$\int_0^{T_m} L\left(\xi_m, \dot{\xi}_m, k - \frac{1}{l}\right) dt - \frac{1}{m} \le \int_0^{T_m} L\left(\xi_m, \dot{\xi}_m, k - \frac{1}{m}\right) dt - \frac{1}{m} \le C_{k-\frac{1}{m},r}(x, x_0). \tag{4.11}$$

Applying Lemma 4.1 (b), pass to a subsequence ξ_{m_j} of the ξ_m satisfying the assumptions of Lemma 4.1 (a) and let $\xi : [0, T] \to \overline{B}_r(x_0)$, $T = \lim_{j\to\infty} T_{m_j}$, be the limit of the ξ_{m_j}. Then use Lemma 4.1 (a) to pass to the limit inferior as $j \to \infty$ in (4.11) with m replaced by m_j. This results in

$$\int_0^T L\left(\xi, \dot{\xi}, \left(k - \frac{1}{l}\right) -\right) dt \le C_{k-,r}(x, x_0).$$

Now let $l \to \infty$ to establish

$$\int_0^T L(\dot{\xi}, \xi, k-)\, dt \le C_{k-,r}(x, x_0).$$

Since

$$C_{k-\varepsilon,r}(x, x_0) \le \int_0^T L(\dot{\xi}, \xi, k-)\, dt$$

for every $\varepsilon > 0$, the opposite inequality also holds. □

We record the following associated properties of C_k as a function of k.

Lemma 4.3. *Let $0 < r < R$ and $k_0(r) < k$. Then for $x \in \overline{B}_r(x_0)$*

$$C_{k+,r}(x, x_0) = C_{k,r}(x, x_0), \tag{4.12}$$

and

$$C_{k-,r}(x, x_0) = \min\left\{\int_0^T L(\dot{\xi}, \xi, k-)\, dt : \xi \in \text{path}\,(x, r)\right\}. \tag{4.13}$$

Proof. For $\xi \in$ path (x, r) the monotone convergence theorem and the continuity from the right of $k \mapsto L_k$ imply

$$\lim_{\hat{k}\downarrow k} \int_0^T L(\dot{\xi}, \xi, \hat{k})\, dt = \int_0^T L(\dot{\xi}, \xi, k)\, dt.$$

As the infimum of upper-semicontinuous functions is upper-semicontinuous, $k \mapsto C_k(x, x_0)$ is upper-semicontinuous (which is equivalent to continuity from the right for nondecreasing functions). Thus (4.12) holds.

To prove (4.13), we first make the obvious remark that for $\hat{k} < k$

$$C_{\hat{k},r}(x,x_0) \le \inf \left\{ \int_0^T L(\dot{\xi},\xi,k-)\,dt : \xi \in \text{path}\,(x,r) \right\}$$

which establishes (4.13) with "≤" in place of "=" and "inf" in place of "min." Proposition 4.2 shows the inf is a min, and is attained with equality.

□

The next lemma is more elementary, and was not needed in Section 3.3.

Lemma 4.4. *Let $k_0(r) \le k < \hat{k}$. There is a $\delta > 0$ such that for any $x \in \overline{B}_r(x_0)$*

$$C_{k,r}(x,x_0) + \delta|x-x_0| \le C_{\hat{k},r}(x,x_0) \tag{4.14}$$

for $x \in \overline{B}_r(x_0)$.

Proof. Let $k_0(r) \le k < \hat{k}$. Obvious arguments show that there is a $\delta > 0$ such that

$$Q \in \{q : H(q,y) \le k\} \implies B_\delta(Q) \subset \left\{q : H(q,y) \le \hat{k}\right\}. \tag{4.15}$$

It follows from (4.15) that

$$L(p,x,k) + \delta|p| \le L(p,x,\hat{k}), \tag{4.16}$$

and from this that

$$\int_0^T L(\dot{\xi},\xi,k)\,dt + \delta \int_0^T |\dot{\xi}|\,dt \le \int_0^T L(\dot{\xi},\xi,\hat{k}). \tag{4.17}$$

The estimate (4.14) follows at once. □

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