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Regularity of Dirac-Harmonic Maps

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For any n-dimensional compact spin Riemannian manifold M with a given spin structure and a spinor bundle ΣM, and any compact Riemannian manifold N, we show an ϵ-regularity theorem for weakly Dirac-harmonic maps $(\phi, \psi) : M \otimes \Sigma M \to N \otimes \phi^* TN$. As a consequence, any weakly Dirac-harmonic map is proven to be smooth when $n = 2$. A weak convergence theorem for approximate Dirac-harmonic maps is established when $n = 2$. For $n \geq 3$, we introduce the notation of stationary Dirac-harmonic maps and obtain a Liouville theorem for stationary Dirac-harmonic maps in $\mathbb{R}^n$. If, in addition, $\psi \in W^{1,p}$ for some $p > \frac{2n}{3}$, then we obtain an energy monotonicity formula and prove a partial regularity theorem for any such a stationary Dirac-harmonic map.

1 Introduction

The notation of Dirac-harmonic maps is inspired by the supersymmetric nonlinear sigma model from the quantum field theory [8], and is a very natural and interesting extension of harmonic maps. In a series of papers [4, 5], Chen–Jost–Li–Wang recently introduced the subject of Dirac-harmonic maps and studied some analytic aspects of Dirac-harmonic maps from a spin Riemann surface into another Riemannian manifold. In order to review

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some of the main theorems of [4, 5] and to motivate the aim of this paper, let us briefly describe the mathematical framework given by [4, 5].

For $n \geq 2$, let (M, g) be a compact n-dimensional spin Riemannian manifold with a given spin structure and an associated spinor bundle $\Sigma (= \Sigma M)$, and (N, h) be a compact k-dimensional Riemannian manifold without boundary. By Nash's theorem, we may assume that (N, h) is isometrically embedded into an Euclidean space $\mathbb{R}^K$ for a sufficiently large K. Let ∇^M and ∇^N be the Levi-Civita connection on M and N, respectively. Let $\langle \cdot, \cdot \rangle$ be a Hermitian metric on Σ (a complex vector bundle of complex dimension n), and ∇^Σ be the Levi-Civita connection on Σ compatible with the metrics $\langle \cdot, \cdot \rangle$ and g. For a map $\phi : M \to N$, let $\phi^* TN$ denote the pullback bundle of TN by ϕ that is equipped with the pullback metric $\phi^* h$ and the connection $\nabla^{\phi^* TN}$. On the twist bundle $\Sigma \otimes \phi^* TN$, there is a metric, still denoted as $\langle \cdot, \cdot \rangle$, induced from the $\langle \cdot, \cdot \rangle$ and $\phi^* h$. There is also a Levi-Civita connection $\overline{\nabla}$ on $\Sigma \otimes \phi^* TN$ induced from ∇^Σ and $\nabla^{\phi^* TN}$. The Dirac operator $\mathcal{D}$ along ϕ is defined as follows. For any section $\psi \in \Gamma(\Sigma \otimes \phi^* TN)$,

$$D\psi = f_\alpha \circ \overline{\nabla}_{f_\alpha} \psi, \tag{1}$$

where $\{f_\alpha\}_{\alpha=1}^n$ is a local orthonormal frame on M, and $\circ : TM \otimes_{\mathbb{C}} \Sigma \to \Sigma$ is the Clifford multiplication. More precisely, if we write ψ in the local coordinate as $\psi = \psi^i \otimes \frac{\partial}{\partial y_i}(\phi)$, where $\psi^i \in \Gamma\Sigma$ is a section of Σ for $1 \leq i \leq k$ and $\{\frac{\partial}{\partial y_i}\}_{i=1}^k$ is a local coordinate frame on N, then

$$D\psi = \partial \psi^i \otimes \frac{\partial}{\partial y_i}(\phi) + \left(f_\alpha \circ \psi^i\right) \otimes \nabla^{\phi^* TN}_{f_\alpha} \left(\frac{\partial}{\partial y_i}(\phi) \right), \tag{2}$$

where $\partial = f_\alpha \circ \nabla^\Sigma_{f_\alpha}$ is the standard Dirac operator on the spin bundle Σ.

The Dirac-harmonic energy functional was first introduced by Chen–Jost–Li–Wang in [4, 5]:

$$L(\phi, \psi) = \int_M [|d\phi|^2 + \langle \psi, D\psi \rangle] dv_g = \int_M \left[g^{\alpha\beta} h_{ij}(\phi) \frac{\partial \phi^i}{\partial x_\alpha} \frac{\partial \phi^j}{\partial x_\beta} + \langle \psi, D\psi \rangle \right] \sqrt{g}\, dx. \tag{3}$$

Critical points of $L(\phi, \psi)$ are called Dirac-harmonic maps, which are natural extensions of harmonic maps and harmonic spinors. In fact, when $\psi = 0$, $L(\phi, 0) = \int_M |d\phi|^2 \, dv_g$ is the Dirichlet energy functional of $\phi : M \to N$, and its critical points are harmonic maps that have been extensively studied (see Lin–Wang [20] for relevant references). On the other hand, when $\phi = \text{constant} : M \to N$ is a constant map, $L(\text{constant}, \psi) = \int_M \langle \psi, D\psi \rangle \, dv_g$ is the Dirac functional of $\psi \in (\Gamma\Sigma)^k$, and its critical points are harmonic spinors $\partial \psi = 0$ that have also been well studied (see Lawson–Michelsohn [19]).

Studying the regularity of weakly Dirac-harmonic maps is one of our main interests. For this purpose, we introduce the natural Sobolev space in which the functional $L(\cdot,\cdot)$ is well defined. Recall the Sobolev space $H^1(M,N)$ is defined by $H^1(M,N)=\{u\in H^1(M,\mathbb{R}^K)\,:\,u(x)\in N\ \text{ a.e. } x\in M\}$.

Definition 1.1. For $\phi\in H^1(M,N)$, the set of sections $\psi\in\Gamma(\Sigma\otimes\phi^*TN)$ is defined to be all $\psi=(\psi^1,\ldots,\psi^K)\in(\Gamma\Sigma)^K$ such that

$$\sum_{i=1}^K \nu_i\psi^i(x)=0\ \text{ a.e. } x\in M,\ \forall\nu=(\nu_1,\ldots,\nu_K)\in(T_{\phi(x)}N)^\perp.$$

We say that $\psi=(\psi^1,\ldots,\psi^K)\in S^{1,\frac{4}{3}}(\Gamma(\Sigma\otimes\phi^*TN))$ if $d\psi^i\in L^{\frac{4}{3}}(M)$ and $\psi^i\in L^4(M)$ for all $1\le i\le K$.

Definition 1.2. A pair of maps $(\phi,\psi)\in H^1(M,N)\times S^{1,\frac{4}{3}}(\Gamma(\Sigma\otimes\phi^*TN))$ is called a weakly Dirac-harmonic map, if it is a critical point of $L(\cdot,\cdot)$ over the Sobolev space $H^1(M,N)\times S^{1,\frac{4}{3}}(\Gamma(\Sigma\otimes\phi^*TN))$.

Remark 1.3. First, the Hölder inequality implies that if $(\phi,\psi)\in H^1(M,N)\times S^{1,\frac{4}{3}}(\Gamma(\Sigma\otimes\phi^*TN))$, then

$$\left|\int_M\langle\psi,D\psi\rangle\,dv_g\right|\le C\|\psi\|_{L^4(M)}\left[\|d\psi\|_{L^{\frac{4}{3}}(M)}+\|d\phi\|_{L^2(M)}\|\psi\|_{L^4(M)}\right]<+\infty.$$

Hence, $L(\phi,\psi)$ is well defined for any $(\phi,\psi)\in H^1(M,N)\times S^{1,\frac{4}{3}}(\Gamma(\Sigma\otimes\phi^*TN))$. Second, it is useful to note that

$$\int_M\langle\psi,D\psi\rangle\,dv_g=\int_M\mathrm{Re}\langle\psi,D\psi\rangle\,dv_g,\tag{4}$$

where $\mathrm{Re}(z)$ denotes the real part for $z\in\mathbb{C}$. In fact, since D is self-adjoint, i.e.

$$\int_M\langle\psi,D\xi\rangle\,dv_g=\int_M\langle D\psi,\xi\rangle\,dv_g,\ \forall\ \psi,\ \xi\in S^{1,\frac{4}{3}}(\Gamma(\Sigma\otimes\phi^*TN)),$$

we have

$$\int_M\overline{\langle\psi,D\psi\rangle}\,dv_g=\int_M\langle D\psi,\psi\rangle\,dv_g=\int_M\langle\psi,D\psi\rangle\,dv_g.$$

This yields (4).

The Euler–Lagrange equation of a Dirac-harmonic map $(\phi, \psi) \in H^1(M, N) \times S^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ is (see [4])

$$\tau(\phi) = \mathcal{R}^N(\phi, \psi), \tag{5}$$

$$D\psi = 0, \tag{6}$$

where $\tau(\phi)$ is the tension field of ϕ given by

$$\tau(\phi) = \mathrm{tr}(\nabla^{M\otimes\phi^* TN} d\phi) = \left(\Delta\phi^i + g^{\alpha\beta}\Gamma^i_{jl}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\frac{\partial\phi^l}{\partial x_\beta}\right)\frac{\partial}{\partial y_i}(\phi),$$

and $\mathcal{R}^N(\phi, \psi) \in \Gamma(\phi^* TN)$ is defined by

$$\mathcal{R}^N(\phi, \psi) = \frac{1}{2}\sum R^m_{lij}(\phi)\langle\psi^i, \nabla\phi^l \circ \psi^j\rangle\frac{\partial}{\partial y_m}(\phi).$$

Here $\Gamma^i_{jl}(\phi)$ is the Christoffel symbol of the Levi-Civita connection of N, $\nabla\phi^l \circ \psi^j$ denotes the Clifford multiplication of the vector field $\nabla\phi^l$ with the spinor ψ^j, and R^m_{lij} is a component of the Riemannian curvature tensor of (N, h).

Among other things, Chen–Jost–Li–Wang proved in their Theorems 2.2 and 2.3 in [5] that *if* (M^2, g) *is a spin Riemann surface and* $N = S^{K-1} \subset \mathbb{R}^K$ *is the standard sphere, then any weakly Dirac-harmonic map* (ϕ, ψ) *is in* $C^\infty(M^2, S^{K-1}) \times C^\infty(\Gamma(\Sigma \times \phi^*(TS^{K-1})))$, which was extended to any compact hypersurface in $\mathbb{R}^K$ by Zhu [27]. The crucial observation in [5] is that the nonlinearity in (5) is of Jacobian determinant structure. This theorem is an extension of that on harmonic maps by Hélein [14]. Our motivation in this paper is (1) to extend the above theorem to all Riemannian manifold $N \subset \mathbb{R}^K$, and (2) to study the regularity problem of stationary Dirac-harmonic maps in higher dimensions $n \geq 3$.

Denote by $i_M > 0$ the injectivity radius of M. For $0 < r < i_M$ and $x \in M$, denote by $B_r(x)$ the geodesic ball in M with center x and radius r. Our first result is an ϵ-regularity theorem.

Theorem 1.4. For $n \geq 2$, there exists $\varepsilon_0 > 0$ depending only on (M, g) and (N, h) such that if $(\phi, \psi) \in H^1(M, N) \times S^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ is a weakly Dirac-harmonic map satisfying, for some $x_0 \in M$ and $0 < r_0 \leq \frac{1}{2}i_M$,

$$\sup_{x\in B_{r_0}(x_0), 0<r\leq r_0}\left\{\frac{1}{r^{n-2}}\int_{B_r(x)}(|d\phi|^2 + |\psi|^4)dv_g\right\} < \varepsilon_0^2, \tag{7}$$

then (ϕ, ψ) is smooth in $B_{r_0}(x_0)$.

Since $\int_M (|d\phi|^2 + |\psi|^4)$ is conformally invariant when $n(=\dim\ M) = 2$ (see [4], Lemma 3.1), it is not hard to see that there exists $0 < r_0 = r_0(M, \phi, \psi) \le i_M$ such that

$$\sup_{x\in M}\int_{B_{r_0}(x)} (|d\phi|^2 + |\psi|^4) \le \epsilon_0^2,$$

where $\epsilon_0 > 0$ is the same constant as in Theorem 1.4. Hence, the following theorem is an immediate consequence of Theorem 1.4.

Theorem 1.5. For $n = 2$, assume $(\phi, \psi) \in H^1(M, N) \times S^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ is a weakly Dirac-harmonic map. Then $(\phi, \psi) \in C^\infty(M, N) \times C^\infty(\Gamma(\Sigma \otimes \phi^* TN))$.

We would remark that Theorem 1.5 was proved by Chen–Jost–Li–Wang [5] for $N = S^{K-1} \subset \mathbb{R}^K$ and by Zhu [27] for hypersurfaces $N \subset \mathbb{R}^K$. While we circulated this preprint, we learnt from Chen that in a forthcoming article [7], Chen–Jost–Wang–Zhu also independently obtain Theorem 1.5 through a different method.

For $n \ge 3$, it is well known in the context of harmonic maps that in order for a harmonic map to enjoy partial regularity, we need to pose the stationarity condition (see, e.g. Evans [9], Bethuel [2], and Rivieré [21]). For the same purpose, we also introduce the notion of stationary Dirac-harmonic maps.

Definition 1.6. We call a weakly Dirac-harmonic map

$$(\phi, \psi) \in H^1(M, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$$

to be a stationary Dirac-harmonic map, if, in addition, it is a critical point of $L(\phi, \psi)$ with respect to the domain variations, i.e. for any family of diffeomorphisms $F_t(x) := F(t, x) \in C^1((-1, 1) \times M, M)$ with $F_0(x) = x$ for $x \in M$, and $F_t(x) = x$ for any $x \in \partial M$ and $t \in (-1, 1)$ when $\partial M \ne \emptyset$, then we have

$$\frac{d}{dt}\bigg|_{t=0} \left[\int_M \big(|d\phi_t|^2 + \langle \psi_t, D\psi_t\rangle\big) dv_g\right] = 0, \tag{8}$$

where $\phi_t(x) = \phi(F_t(x))$ and $\psi_t = \psi(F_t(x))$.

Motivated by [4], we define *stress-energy tensor* $\mathcal{S}$ for a stationary Dirac-harmonic map (ϕ, ψ) by

$$\mathcal{S}_{\alpha\beta} = \left\langle \frac{\partial \phi}{\partial x_\alpha}, \frac{\partial \phi}{\partial x_\beta}\right\rangle - \frac{1}{2}|d\phi|^2\delta_{\alpha\beta} + \frac{1}{2}\mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha} \circ \nabla_{\frac{\partial}{\partial x_\beta}} \psi\right\rangle,\ 1 \le \alpha, \beta \le n, \tag{9}$$

where $\{\frac{\partial}{\partial x_\alpha}\}$ is a local coordinate frame on M.

It turns out that the stationarity property is equivalent to that the stress-energy tensor $\mathcal{S}$ is divergence free (see Lemma 4.2):

$$\sum_{\alpha,\beta} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta} \mathcal{S}_{\beta\gamma}\right) = 0,\ 1 \le \gamma \le n, \tag{10}$$

in the sense of distributions.

An immediate consequence of (10), which we prove in Section 4, is the following Liouville property of stationary Dirac-harmonic maps.

Theorem 1.7. For $n \ge 3$, let $(M, g) = (\mathbb{R}^n, g_0)$ be the n-dimensional Euclidean space associated with the spinor bundle Σ. If $(\phi, \psi) \in H^1(\mathbb{R}^n, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ is a stationary Dirac-harmonic map, then $\phi \equiv$ constant and $\psi \equiv 0$.

In dimensions $n \ge 3$, the stationarity property is a necessary condition for smoothness of weakly Dirac-harmonic maps. In fact, Chen–Jost–Li–Wang [4] proved that any smooth Dirac-harmonic map $(\phi, \psi) \in C^\infty(M, N) \times C^\infty(\Gamma(\Sigma \times \phi^* TN))$ has its stress-energy tensor divergence free and hence is a stationary Dirac-harmonic map. Hence, Theorem 4.3 extends a corresponding Liouville theorem on smooth Dirac-harmonic maps by Chen–Jost–Wang [6].

An important implication of (10) is the following monotonicity inequality (see Section 4): *there exist* $0 < r_0 < i_M$ *and* $C_0 > 0$ *depending only on* (M, g) *such that if* $(\phi, \psi) \in H^1(M, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ *is a stationary Dirac-harmonic map, then for any* $x_0 \in M$ *and* $0 < r \le r_0$*, it holds*

$$\begin{aligned} \frac{d}{dr}\left(e^{C_0 r} r^{2-n} \int_{B_r(x)} |d\phi|^2 dv_g\right) &\ge e^{C_0 r} r^{2-n} \int_{\partial B_r(x)} 2\left|\frac{\partial \phi}{\partial r}\right|^2 dH^{n-1} \\ &\quad + e^{C_0 r} r^{2-n} \int_{\partial B_r(x_0)} \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle dH^{n-1}. \end{aligned} \tag{11}$$

However, we point out that (11) does not yield that the renormalized energy $e^{C_0 r} r^{2-n} \int_{B_r(x)} |d\phi|^2 dv_g$ is monotone increasing with respect to r, since the second term of the right-hand side of (11)

$$e^{C_0 r} r^{2-n} \int_{\partial B_r(x)} \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle dH^{n-1}$$

may change signs. In order to utilize (11) to control $r^{2-n} \int_{B_r(x)} (|d\phi|^2 + |\psi|^4)\, dv_g$, we need to assume $d\psi \in L^p$ for some $p > \frac{2n}{3}$. In fact, we have the following theorem.

Theorem 1.8. For $n \geq 3$, let $(\phi, \psi) \in H^1(M, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ be a stationary Dirac-harmonic map. If, in addition, $d\psi \in L^p(M)$ for some $p > \frac{2n}{3}$, then there exists a closed subset $\mathcal{S}(\phi) \subset M$, with $H^{n-2}(\mathcal{S}(\phi)) = 0$, such that $(\phi, \psi) \in C^\infty(M \setminus \mathcal{S}(\phi))$.

Now let us outline the main ingredients to prove Theorem 1.4 as follows.

(1) We observe that the Dirac-harmonic property is invariant under totally geodesic, isometric embedding. More precisely, let $\Phi : (N, h) \to (\tilde{N}, \tilde{h})$ be a totally geodesic, isometric embedding map. If, for $\phi : M \to N$ and $\psi \in \Gamma(\Sigma \otimes \phi^* TN)$, (ϕ, ψ) is a weakly Dirac-harmonic map, then for $\tilde{\phi} = \Phi(\phi) : M \to \tilde{N}$ and $\tilde{\psi} = \Phi_*(\psi) = \psi^i \otimes \frac{\partial}{\partial z_i}(\tilde{\phi}) \in \Gamma(\Sigma \otimes \tilde{\phi}^* T\tilde{N})$, $(\tilde{\phi}, \tilde{\psi})$ is a weakly Dirac-harmonic map.

(2) By employing the enlargement argument of Hélein [15, 16] in the context of harmonic maps, we can assume that $TN\big|_{\phi(M)}$ is trivial so that there exists an orthonormal tangent frame $\{e_i\}_{i=1}^k$ on $\phi^* TN$.

(3) We use this moving frame to rewrite the Dirac-harmonic map equation (5) into the form

$$d^*(\langle d\phi, e_i \rangle) = \sum_j \Theta_{ij} \langle d\phi, e_j \rangle, \tag{12}$$

where $\Theta = (\Theta_{ij}) \in L^2(B_{r_0}(x_0), so(n) \otimes \wedge^1(\mathbb{R}^n))$ satisfies $|\Theta| \leq C(|d\phi| + |\psi|^2)$.

(4) The smallness condition (7) guarantees that we can apply the Coulomb gauge construction, due to Revieré [22] ($n = 2$) and Rivieré–Struwe [23] ($n \geq 3$), to further rewrite (12) into an equation in which the nonlinearity has the Jacobian determinant structure similar to that of harmonic maps.

(5) We utilize the duality between the Hardy space and Bounded Mean Oscillation (BMO) space to obtain a decay estimate in the Morrey space, which yields the Hölder continuity of ϕ.

(6) By adapting the hole-filling technique developed by Giaquinta–Hildebrandt [13] in the context of harmonic maps, we establish the higher order regularity of (ϕ, ψ). We point out that in dimension two, a different proof of higher order regularity of Dirac-harmonic maps has been provided by Chen–Jost–Li–Wang [5].

As a byproduct of the rewriting of Dirac-harmonic maps under the above Coulomb gauge frame, we also obtain a convergence theorem of weakly convergent sequences of approximate Dirac-harmonic maps in dimension two, which extends a corresponding convergence of approximate harmonic maps from surfaces due to Bethuel [3] (see also

Freire–Müller–Struwe [11], Wang [24], and Rivieré [22]). More precisely, we have the following theorem.

Theorem 1.9. For $n = 2$, let $(\phi_p, \psi_p) \in H^1(M, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ be a sequence of weak solutions to the approximate Dirac-harmonic map equation

$$\tau(\phi_p) = \mathcal{R}^N(\phi_p, \psi_p) + u_p, \tag{13}$$

$$D\psi_p = v_p. \tag{14}$$

Assume that $u_p \to 0$ strongly in $H^{-1}(M)$ and $v_p \rightharpoonup 0$ weakly in $L^{\frac{4}{3}}(M)$. If $\phi_p \rightharpoonup \phi$ in $H^1(M, N)$ and $\psi_p \rightharpoonup \psi$ in $\mathcal{S}^{1,\frac{4}{3}}$, then $(\phi, \psi) \in H^1(M, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ is a weakly Dirac-harmonic map.

The paper is organized as follows. In Section 2, we rewrite the equation of Dirac-harmonic maps via moving frames. In Section 3, we use the Coulomb gauge construction, duality between Hardy space and BMO space, and a decay estimate in Morrey space to first prove the Hölder continuity part of Theorem 1.4 and then adopt the hole-filling technique by Giaquinta–Hildebrandt [13] to prove the higher order regularity part of Theorem 1.4. In Section 4, we discuss various properties of stationary Dirac-harmonic maps and prove Theorems 4.3 and 1.8. In Section 5, we prove the convergence Theorem 1.9.

2 Dirac-Harmonic Maps via Moving Frames

In this section, we first show that a Dirac harmonic map (ϕ, ψ) is invariant under a totally geodesic, isometric embedding so that Hélein's enlargement argument (see [15, 16]) guarantees that we can assume there is an orthonormal frame $\{e_i\}_{i=1}^k$ of $\phi^* TN$. Then employing this orthonormal frame, we write the equation of Dirac-harmonic maps into the form (12).

We begin with the following proposition.

Proposition 2.1. Let $(\widetilde{N}, \widetilde{h})$ be another compact Riemannian manifold without boundary and $f : (N, h) \to (\widetilde{N}, \widetilde{h})$ be a totally geodesic, isometric embedding. Let $(\phi, \psi) \in H^1(M, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma M \otimes \phi^* TN))$ be a weakly Dirac-harmonic map. Define $\widetilde{u} = f(u) \in H^1(M, \widetilde{N})$ and $\widetilde{\psi} = f_*(\psi) = \psi^i \otimes \frac{\partial}{\partial z_i}(\widetilde{\phi}) \in \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes (\widetilde{\phi})^* T\widetilde{N}))$. Then $(\widetilde{\phi}, \widetilde{\psi})$ is also a weakly Dirac-harmonic map.

Proof. By the chain rule formula of tension fields (see Jost [18]), we have

$$\tau(\widetilde{\phi}) = \mathrm{tr}\big[\nabla^{f*T\widetilde{N}} df(d\phi, d\phi)\big] + f_*(\tau(\phi)) = f_*(\tau(\phi)) = f_*(\mathcal{R}^N(\phi, \psi)),$$

where we have used the fact that f is totally geodesic, i.e. $\nabla^{f^*T\widetilde{N}} df = 0$, and the Dirac-harmonic map equation (5).

Set $\widehat{N} = f(N)$. Then $(\hat{N}, \widetilde{h})$ is a totally geodesic, submanifold of $(\widetilde{N}, \widetilde{h})$. Moreover, if $y = (y_1, \dots, y_k)$ is a local coordinate system on N, then $z = (z_1, \dots, z_k) = f(y)$ is a local coordinate system on $\widehat{N}$ and $\frac{\partial}{\partial z_i} = f_*(\frac{\partial}{\partial y_i}), 1 \le i \le k$ is a local coordinate frame on $\widehat{N}$. Since $f : (N, h) \to (\widehat{N}, \widetilde{h})$ is an isometry, we have

$$\begin{aligned} f_*(\mathcal{R}^N(\phi, \psi)) &= f_*\left(\frac{1}{2}(R^N)^m_{lij}(\phi)\langle\psi^i, \nabla\phi^l \circ \psi^j\rangle \frac{\partial}{\partial y_m}(\phi)\right) \\ &= \frac{1}{2}(R^{\widehat{N}})^m_{lij}(\widetilde{\phi})\langle\psi^i, \nabla\widetilde{\phi}^l \circ \psi^j\rangle \frac{\partial}{\partial z_m}(\widetilde{\phi}) \\ &= \mathcal{R}^{\widehat{N}}(\widetilde{\phi}, \widetilde{\psi}) = \mathcal{R}^{\widetilde{N}}(\widetilde{\phi}, \widetilde{\psi}), \end{aligned}$$

where we have used the fact that $(R^{\widehat{N}})^m_{lij}(\widetilde{\phi}) = (R^{\widetilde{N}})^m_{lij}(\widetilde{\phi})$ in the last two steps, which follows from the Gauss–Codazzi equation since $\widehat{N} \subseteq \widetilde{N}$ is a totally geodesic submanifold.

To see that $\widetilde{\psi}$ satisfies (6), denote $\widetilde{D}$ as the Dirac operator along the map $\widetilde{\phi}$. Then it follows from (2.6) in [4] that

$$\widetilde{D}\widetilde{\psi} = f_*(D\psi) + (\nabla\phi^i \circ \psi^j) \otimes \nabla^{f^*T\widetilde{N}} df\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = 0,$$

where we have used the fact that both $D\psi = 0$ and $\nabla^{f^*T\widetilde{N}} df = 0$. ■

With the help of Proposition 2.1, we can now adapt the same enlargement argument as that by Hélein [15, 16] and assume that (N, h) is parallelized. Hence, there exists a global orthonormal frame $\{\hat{e}_i\}_{i=1}^k$ on (N, h). Set $e_i(x) = \hat{e}_i(\phi(x))$, $1 \le i \le k$. Then $\{e_i\}$ is an orthonormal frame along ϕ^*TN. Using this frame, we can write the spinor field ψ along map ϕ as

$$\psi = \sum_{i=1}^k \psi^i \otimes e_i,\ \ \psi^i \in \Gamma(\Sigma),\ \ 1 \le i \le k.$$

Let $\{\frac{\partial}{\partial x_\alpha}\}_{\alpha=1}^n$ be a local coordinate frame on M. Recall the tension field of ϕ is defined by (see Jost [18])

$$\tau(\phi) = g^{\alpha\beta}\nabla^{\phi^*TN}_{\frac{\partial}{\partial x_\alpha}}\left(\frac{\partial\phi}{\partial x_\beta}\right).$$

Denote the components of $\tau(\phi)$ and $\mathcal{D}\psi$ with respect to the frame $\{e_i\}$ by

$$\tau^i(\phi) = \langle \tau(\phi), e_i\rangle_{\phi^* h},\ 1 \le i \le k,$$

$$(D\psi)^i = \langle D\psi, e_i\rangle_{\phi^* h},\ 1 \le i \le k.$$

Under these notations, we have the following lemma.

Lemma 2.2. If $(\phi, \psi) \in H^1(M,N) \times \mathcal{S}^{1,\frac{4}{3}}(\Gamma(\Sigma \otimes \phi^* TN))$ is a weakly Dirac-harmonic map, then it holds, for $1 \le i \le n$,

$$(D\psi)^i = 0, \tag{15}$$

$$\tau^i(\phi) = R^N(\phi)(e_i, e_j, e_l, e_m)\langle \phi_*(\xi_\alpha),\ e_j\rangle_{\phi^* h}\langle \psi^m,\ \xi_\alpha \circ \psi^l\rangle, \tag{16}$$

where $\{\xi_\alpha\}_{\alpha=1}^n$ is a local orthonormal frame on M.

Proof. It suffices to prove (16). To do this, let $\{\phi_t\}$ be a variation of ϕ such that $\frac{\partial \phi_t}{\partial t}|_{t=0} = \eta = \eta^i e_i$ for $(\eta^1, \ldots, \eta^k) \in C_0^\infty(M, \mathbb{R}^k)$. Then we have $\psi_t = \psi^i \otimes e_i(\phi_t)$. Then we have

$$\begin{aligned}\frac{\partial}{\partial t} D\psi_t &= \partial\psi^i \otimes \nabla_{\frac{\partial}{\partial t}} e_i(\phi_t) + (\xi_\alpha \circ \psi^i) \otimes \nabla_{\frac{\partial}{\partial t}} \nabla_{\xi_\alpha} e_i(\phi_t) \\ &= \partial\psi^i \otimes \nabla_{\frac{\partial}{\partial t}} e_i(\phi_t) + (\xi_\alpha \circ \psi^i) \otimes \nabla_{\xi_\alpha} \nabla_{\frac{\partial}{\partial t}} e_i(\phi_t) + (\xi_\alpha \circ \psi^i) \otimes R^{\phi_t^* TN}(\phi_t)\left(\frac{\partial}{\partial t}, \xi_\alpha\right) e_i(\phi_t) \\ &= D\left(\psi^i \otimes \nabla_{\frac{\partial}{\partial t}} e_i(\phi_t)\right) + (\xi_\alpha \circ \psi^i) \otimes R^{\phi_t^* TN}(\phi_t)\left(\frac{\partial}{\partial t}, \xi_\alpha\right) e_i(\phi_t).\end{aligned}$$

This, combined with the fact that $D\psi = 0$ and D is self-adjoint, implies

$$\begin{aligned}\frac{d}{dt}|_{t=0} \int_M \langle \psi_t, D\psi_t\rangle\, dv_g &= \int_M \left\langle \frac{\partial}{\partial t}|_{t=0}\psi_t, D\psi\right\rangle + \int_M \left\langle \psi, \frac{\partial}{\partial t}|_{t=0} D\psi_t\right\rangle \\ &= \int_M \left\langle \psi^i,\ \xi_\alpha \circ \psi^j\right\rangle \left\langle e_i,\ R^{\phi^* TN}(\phi)(\eta, \phi_*(\xi_\alpha)) e_j\right\rangle dv_g \\ &= \int_M \eta^l \langle \phi_*(\xi_\alpha),\ e_m\rangle \langle \psi^i,\ \xi_\alpha \circ \psi^j\rangle \langle e_i,\ R^N(e_l, e_m) e_j\rangle\, dv_g,\end{aligned}$$

On the other hand, it is well known that

$$\frac{d}{dt}\Big|_{t=0} \int_M |d\phi_t|^2\, dv_g = 2\int_M \langle \tau(\phi), e_l\rangle \eta^l\, dv_g = 2\int_M \tau^l(\phi)\eta^l\, dv_g.$$

Hence, combining these formulae together, we obtain (16). ∎

3 The ϵ-Decay Estimate and Regularity Theorem

In this section, we utilize the skew symmetry of the nonlinearity in the right-hand side of the Dirac-harmonic map equation (16) and adapt the Coulomb gauge construction technique developed by Rivieré [22] ($n = 2$) and Rivieré–Struwe [23] ($n \geq 3$) to establish an energy decay estimate for Dirac-harmonic maps in Morrey spaces under the smallness condition. As a consequence, we prove the Hölder continuity part of Theorems 1.4 and 1.5.

Since the regularity issue is a local result, we assume, for simplicity, that for $x_0 \in M$, the geodesic ball $B_{i_M}(x_0) \subset M$ with the metric g is identified by (B_2, g_0). Here B_2 is the ball centered at 0 and radius 2 in $\mathbb{R}^n$, and g_0 is the Euclidean metric on $\mathbb{R}^n$. We also assume that the spin bundle Σ restricted in B_2 is given by $\Sigma|_{B_2} \equiv B_2 \times \mathbb{C}^L$, with $L = \mathrm{rank}_{\mathbb{C}} \Sigma$.

Let $(\phi, \psi) \in H^1(B_2, N) \times \mathcal{S}^{1,\frac{4}{3}}(B_2, \mathbb{C}^L \otimes \phi^* TN)$ be a weakly Dirac-harmonic map and $\{e_i\}_{i=1}^k$ be an orthonormal frame of $\phi^* TN$ as given in Section 2. Write $\psi = \psi^i \otimes e_i$ for some $\psi^i \in \mathbb{C}^L$, $1 \leq i \leq k$.

Now we define Ω, the $k \times k$ matrix whose entries are 1-forms, by

$$\Omega_{ij} = \sum_{\alpha=1}^{n} \left[\sum_{l,m=1}^{k} R^N(\phi)(e_i, e_j, e_l, e_m) \left\langle \psi^m, \frac{\partial}{\partial x_\alpha} \circ \psi^l \right\rangle \right] dx_\alpha, \quad \text{for } 1 \leq i, j \leq k. \tag{17}$$

Then we have the following simple fact.

Proposition 3.1. Let Ω be given by (17). Then Ω_{ij} is real valued for any $1 \leq i, j \leq k$, and Ω is skew symmetric, i.e.

$$\Omega_{ij} = -\Omega_{ji}, \ 1 \leq i, j \leq k.$$

Proof. First observe that the skew symmetry of Clifford multiplication $\circ$ and the properties of Hermitian metric $\langle \cdot, \cdot \rangle$ give

$$\overline{\left\langle \psi^m, \frac{\partial}{\partial x_\alpha} \circ \psi^l \right\rangle} = \left\langle \frac{\partial}{\partial x_\alpha} \circ \psi^l, \psi^m \right\rangle = -\left\langle \psi^l, \frac{\partial}{\partial x_\alpha} \circ \psi^m \right\rangle.$$

On the other hand, the curvature operator $R^N(\phi)(\cdot, \cdot, \cdot, \cdot)$ is skew symmetric in its last two components:

$$R^N(\phi)(\cdot, \cdot, e_l, e_m) = -R^N(\phi)(\cdot, \cdot, e_m, e_l).$$

Thus, we conclude that

$$\overline{\Omega_{ij}} = \Omega_{ij},$$

so that Ω is real valued. Here $\Omega_{ij} = -\Omega_{ji}$ follows from skew symmetry of $R^N(\phi)(\cdot,\cdot,\cdot,\cdot)$ with respect to its first two components. ■

In terms of Ω, the Dirac-harmonic map equation (16) can be written as

$$\tau^i(\phi) = \sum_{j=1}^{k} \Omega_{ij} \cdot \langle d\phi, e_j\rangle, \quad 1 \le i \le k, \tag{18}$$

where $\cdot$ denotes the inner product of 1-forms, and $\langle d\phi, e_j\rangle = \sum_{\alpha=1}^{n} \langle \frac{\partial \phi}{\partial x_\alpha}, e_j\rangle \, dx_\alpha$.

Denote by d^* the conjugate operator of d. Then we have, for $1 \le i \le k$,

$$d^*(\langle d\phi, e_i\rangle) = \langle \tau(\phi), e_i\rangle + \langle d\phi, de_i\rangle = \tau^i(\phi) + \langle de_i, e_j\rangle \cdot \langle d\phi, e_j\rangle.$$

Hence, we have

$$d^*(\langle d\phi, e_i\rangle) = \sum_{l=1}^{k} \Theta_{il} \cdot \langle d\phi, e_l\rangle; \ \Theta_{ij} \equiv \Omega_{ij} + \langle de_i, e_j\rangle, \ \forall 1 \le i, j \le k. \tag{19}$$

Before proving Theorem 1.4, we recall the definition of Morrey spaces.

Definition 3.2. For $1 \le p \le n$, $0 < \lambda \le n$, and a domain $U \subseteq \mathbb{R}^n$, the Morrey space $M^{p,\lambda}(U)$ is defined by

$$M^{p,\lambda}(U) := \left\{ f \in L^p_{\text{loc}}(U) : \|f\|_{M^{p,\lambda}(U)} < +\infty \right\},$$

where

$$\|f\|^p_{M^{p,\lambda}(U)} = \sup \left\{ r^{\lambda-n} \int_{B_r} |f|^p : B_r \subseteq U \right\}.$$

It is easy to see that for $1 \le p \le n$, $M^{p,n}(U) = L^p(U)$ and $M^{p,p}(U)$ behaves like $L^n(U)$ from the view of scalings.

Now we recall the Coulomb gauge construction theorem in Morrey spaces with small Morrey norms, due to Rivieré [22] for $n = 2$ and Rivieré–Struwe [23] for $n \ge 3$, which plays a critical role in our proof here.

Lemma 3.3. There exist $\epsilon(n) > 0$ and $C(n) > 0$ such that if $R \in L^2(B_1, \mathrm{so(k)} \otimes \wedge^1\mathbb{R}^n)$ satisfies

$$\|R\|_{M^{2,2}(B_1)} \le \epsilon(n), \tag{20}$$

then there exist $P \in H^1(B_1, \mathrm{SO(k)})$ and $\xi \in H^1(B_1, \mathrm{so(k)} \otimes \wedge^2\mathbb{R}^n)$ such that

$$P^{-1}RP + P^{-1}dP = d^*\xi \text{ in } B_1, \tag{21}$$

$$d\xi = 0 \text{ in } B_1, \quad \xi = 0 \text{ on } \partial B_1. \tag{22}$$

Moreover, ∇P and $\nabla\xi$ belong to $M^{2,2}(B_1)$ with

$$\|\nabla P\|_{M^{2,2}(B_1)} + \|\nabla\xi\|_{M^{2,2}(B_1)} \le C(n)\|R\|_{M^{2,2}(B_1)} \le C(n)\epsilon(n). \tag{23}$$

Here so(k) denotes a Lie algebra of SO(k).

The crucial step to prove Theorem 1.4 is the following lemma.

Lemma 3.4. There exist $\epsilon_0 > 0$ such that if $(\phi, \psi) \in H^1(B_2, N) \times \mathcal{S}^{1,\frac{4}{3}}(B_2, \mathbb{C}^L \otimes \phi^*TN)$ is a weakly Dirac-harmonic map satisfying

$$\|\nabla\phi\|^2_{M^{2,2}(B_2)} + \|\psi\|^4_{M^{4,2}(B_2)} \le \epsilon_0^2, \tag{24}$$

then for any $\alpha \in (0,1)$, $\phi \in C^\alpha(B_1, N)$. Moreover,

$$[\phi]_{C^\alpha(B_1)} \le C\|\nabla\phi\|_{M^{2,2}(B_2)}. \tag{25}$$

Proof. By Proposition 3.1 and (19), we have $\Theta = (\Theta_{ij}) = (\Omega_{ij} - \langle e_i, de_j\rangle) \in L^2(B_1, \mathrm{so(k)} \otimes \wedge^1\mathbb{R}^n)$. Moreover, (20) implies

$$\|\Theta\|_{M^{2,2}(B_1)} \le C(N)\big[\||\psi|^2\|_{M^{2,2}(B_1)} + \|\nabla\phi\|_{M^{2,2}(B_1)}\big] \le C(N)\epsilon_0 \le \epsilon(n),$$

provided $\epsilon_0 > 0$ is chosen to be sufficiently small, where $\epsilon(n) > 0$ is the same constant as in Lemma 3.3. Hence, applying Lemma 3.3 with R replaced by $-\Theta$, we conclude that there are $P \in H^1(B_1, \mathrm{SO(k)})$ and $\xi \in H^1(B_1, \mathrm{so(k)} \otimes \wedge^1\mathbb{R}^n)$ such that

$$P^{-1}dP - P^{-1}\Theta P = d^*\xi,\ d\xi = 0 \text{ in } B_1,\ \xi = 0 \text{ on } \partial B_1, \tag{26}$$

and

$$\|\nabla P\|_{M^{2,2}(B_1)} + \|\nabla\xi\|_{M^{2,2}(B_1)} \le C(n)\|\Theta\|_{M^{2,2}(B_1)} \le C(n)\epsilon_0. \tag{27}$$

Write $P = (P_{ij})$, $P^{-1} = (P_{ji})$, and $\xi = (\xi_{ij})$. Since $P^{-1}P = I_k$, we have $dP^{-1} = -P^{-1}dPP^{-1}$. Multiplying P^{-1} to (19) and applying (26), we obtain

$$d^*\left[P^{-1}\begin{pmatrix}\langle d\phi, e_1\rangle\\ \vdots \\ \langle d\phi, e_k\rangle\end{pmatrix}\right] = [dP^{-1}P + P^{-1}\Theta P]\cdot P^{-1}\begin{pmatrix}\langle d\phi, e_1\rangle\\ \vdots \\ \langle d\phi, e_k\rangle\end{pmatrix}$$
$$= -d^*\xi\cdot P^{-1}\begin{pmatrix}\langle d\phi, e_1\rangle\\ \vdots \\ \langle d\phi, e_k\rangle\end{pmatrix}. \tag{28}$$

The components of (28) can be written as

$$-d^*(P_{ji}\langle d\phi, e_j\rangle) = d^*\xi_{il}\cdot(P_{ml}\langle d\phi, e_m\rangle), \ 1\le i\le k, \ \text{ in } B_1. \tag{29}$$

To proceed, recall the definition of BMO spaces. For any domain $U\subseteq\mathbb{R}^n$, BMO(U) is defined to be the set of functions $f\in L^1_{\text{loc}}(U)$ such that

$$[f]_{\text{BMO}(U)} \equiv \sup\left\{\frac{1}{|B_r|}\int_{B_r}|f-\bar f_r|\,dx : B_r\subseteq U\right\} < +\infty,$$

where $\bar f_r = \frac{1}{|B_r|}\int_{B_r} f$ is the average of f over B_r. By Poincaré inequality, it follows that

$$[f]_{\text{BMO}(U)} \le C\|\nabla f\|_{M^{p,p}(U)}, \ \forall 1\le p\le n. \tag{30}$$

For any $0 < R\le\frac12$, let $B_R\subset B_1$ be an arbitrary ball of radius R and $\eta\in C_0^\infty(B_1)$ be such that $0\le\eta\le1$, $\eta\equiv1$ in B_R, $\eta\equiv0$ outside B_{2R}, and $|\nabla\eta|\le\frac4R$. For $1\le i\le k$, let

$$\sum_{j=1}^k P_{ji}\langle d((\phi-\bar\phi_r)\eta), e_j\rangle = df_i + d^*g_i \quad\text{in } \mathbb{R}^n \tag{31}$$

be Hodge decomposition of $\sum_{j=1}^k P_{ji}\langle d((\phi-\bar\phi_r)\eta), e_j\rangle$ on $\mathbb{R}^n$, where $f_i\in H^1(\mathbb{R}^n)$, $g_i\in H^1(\mathbb{R}^n,\wedge^2\mathbb{R}^n)$ is a closed 2-form, i.e. $dg_i = 0$ in $\mathbb{R}^n$ (see Iwaniec–Martin [17] for more details). Moreover, we have the estimate

$$\|\nabla f_i\|_{L^2(\mathbb{R}^n} + \|\nabla g_i\|_{L^2(\mathbb{R}^n)} \le C\|d\phi\|_{L^2(B_{2R})}. \tag{32}$$

Taking d^* of both sides of (31) and applying (29), we have that for $1\le i\le k$,

$$-\Delta f_i = d^*\xi_{il}\cdot(P_{ml}\langle d\phi, e_m\rangle) \ \text{ in } B_R, \tag{33}$$
$$\Delta g_i = dP_{ji}\wedge\langle d\phi, e_j\rangle + P_{ji}d\phi\wedge de_j \ \text{ in } B_R. \tag{34}$$

Now we define two auxiliaries $f_i^2 \in H^1(B_R)$ and $g_i^2 \in H^1(B_R, \wedge^2\mathbb{R}^n)$ on B_R by

$$\Delta f_i^2 = 0 \ \text{ in } B_R, \ f_i^2 = f_i \ \text{ on } \partial B_R, \tag{35}$$

$$\Delta g_i^2 = 0 \ \text{ in } B_R, \ g_i^2 = g_i \ \text{ on } \partial B_R. \tag{36}$$

Set $f_i^1 = f_i - f_i^2$ and $g_i^1 = g_i - g_i^2$. Then f_i^1 and g_i^1 belong to $H_0^1(B_R)$. For $1 < p < \frac{n}{n-1}$, let $p' = \frac{p}{p-1}$ be its Hölder conjugate. Recall the duality characterization of $\|\nabla u\|_{L^p(B_R)}$ for $u \in W_0^{1,p}(B_R)$:

$$\|\nabla u\|_{L^p(B_R)} \le C \sup\left\{\int_{B_R} \nabla u \cdot \nabla v \, dx : \ v \in W_0^{1,p'}(B_R), \ \|\nabla v\|_{L^{p'}(B_R)} \le 1\right\}. \tag{37}$$

Since $p' > n$, the Sobolev embedding theorem implies that $W_0^{1,p'}(B_R) \hookrightarrow C^{1-\frac{n}{p'}}(B_R)$ and for $v \in W_0^{1,p'}(B_R)$, with $\|\nabla v\|_{L^{p'}(B_R)} \le 1$, there holds

$$\|v\|_{L^\infty(B_R)} \le C R^{1-\frac{n}{p'}}, \ \|\nabla v\|_{L^2(B_R)} \le C R^{\frac{n}{2}-\frac{n}{p'}}. \tag{38}$$

For any such v, we can employ (33), upon integration by parts, use the duality between the Hardy space $\mathcal{H}^1$ and the BMO space to estimate f_i^1, similar to Bethuel [2] and Rivieré–Struwe [23], as follows:

$$\begin{aligned}
\int_{B_R} \nabla f_i^1 \cdot \nabla v &= -\int_{B_R} \Delta f_i \cdot v = \int_{B_R} d^*\xi_{il} \cdot (P_{ml}\langle d\phi, e_m\rangle)\, v \\
&= -\int_{B_R} d^*\xi_{il} \cdot d(P_{lm} e_m v)(\phi - \bar{\phi}_R) \\
&\le C \|d^*\xi_{il} \cdot d(P_{lm} e_m v)\|_{\mathcal{H}^1(\mathbb{R}^n)} [\phi]_{\mathrm{BMO}(B_R)} \\
&\le C\|\nabla \xi\|_{L^2(B_R)} \left(\|\nabla P\|_{L^2(B_R)} + \|\nabla\phi\|_{L^2(B_R)}\right) \|v\|_{L^\infty(B_R)} [\phi]_{\mathrm{BMO}(B_R)} \\
&\quad + C\|\nabla\xi\|_{L^2(B_R)} \|\nabla v\|_{L^2(B_R)} [\phi]_{\mathrm{BMO}(B_R)} \\
&\le C\epsilon_0 R^{\frac{n-2}{2}} \left[R^{\frac{n-2}{2}} \|v\|_{L^\infty(B_R)} + \|\nabla v\|_{L^2(B_R)}\right] [\phi]_{\mathrm{BMO}(B_R)} \\
&\le C\epsilon_0 R^{\frac{n-2}{2}} \left[R^{1-\frac{n}{p'}+\frac{n-2}{2}} + R^{\frac{n}{2}-\frac{n}{p'}}\right] [\phi]_{\mathrm{BMO}(B_R)} \\
&\le C\epsilon_0 R^{\frac{n}{p}-1} [\phi]_{\mathrm{BMO}(B_R)},
\end{aligned}$$

where we have used that $\|\nabla e_m\|_{L^2(B_R)} \le C\|\nabla\phi\|_{L^2(B_R)}$, (47), (27), and (38) in the derivation of these inequalities. Taking supremum over all such v's and using (37), we obtain

$$\left(R^{p-n}\int_{B_R} \left|\nabla f_i^1\right|^p\right)^{\frac{1}{p}} \le C\epsilon_0 [\phi]_{\mathrm{BMO}(B_R)}. \tag{39}$$

The estimation of g_i^1 can be achieved in a way similar to that of f_i^1. In fact, for any $v \in W_0^{1,p'}(B_R)$ satisfying (38), we have

$$\begin{aligned}
\int_{B_R} \nabla g_i^1 \cdot \nabla v &= -\int_{B_R} \Delta g_i^1 \cdot v = -\int_{B_R} \Delta g_i \cdot v \\
&= -\int_{B_R} \left[d P_{ji} \wedge \langle d\phi, e_j\rangle + P_{ji} d\phi \wedge de_j\right] v \\
&= \int_{B_R} [d P_{ji} \wedge d(ve_j) + d(P_{ji} v) \wedge de_j](\phi - \bar{\phi}_R) \\
&\le C[\|d P_{ji} \wedge d(ve_j)\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|d(P_{ji} v) \wedge de_j\|_{\mathcal{H}^1(\mathbb{R}^n)}]\, [\phi]_{\mathrm{BMO}(B_R)} \\
&\le C\|\nabla P\|_{L^2(B_R)} \left(\|\nabla v\|_{L^2(B_R)} + \|\nabla\phi\|_{L^2(B_R)}\|v\|_{L^\infty(B_R)}\right) [\phi]_{\mathrm{BMO}(B_R)} \\
&\quad + C\|\nabla\phi\|_{L^2(B_R)} \left(\|\nabla v\|_{L^2(B_R)} + \|\nabla P\|_{L^2(B_R)}\|v\|_{L^\infty(B_R)}\right) [\phi]_{\mathrm{BMO}(B_R)} \\
&\le C\epsilon_0 R^{\frac{n}{p}-1}\, [\phi]_{\mathrm{BMO}(B_R)}\,.
\end{aligned}$$

Taking supremum over all such v's and using (37) yield

$$\left(R^{p-n}\int_{B_R} \left|\nabla g_i^1\right|^p\right)^{\frac{1}{p}} \le C\epsilon_0\, [\phi]_{\mathrm{BMO}(B_R)}. \tag{40}$$

Now we want to estimate f_i^2 and g_i^2. Since both f_i^2 and g_i^2 are harmonic, by the classical Campanato estimates for harmonic functions (see, e.g. Giaquinta [12]), (39), and (40), we have that for any $0 \le r \le R$, it holds

$$\begin{aligned}
r^{p-n}\int_{B_r} \left[\left|\nabla f_i^2\right|^p + \left|\nabla g_i^2\right|^p\right] &\le C\left(\frac{r}{R}\right)^p \left\{R^{p-n}\int_{B_R} \left[\left|\nabla f_i^2\right|^p + \left|\nabla g_i^2\right|^p\right]\right\} \\
&\le C\left(\frac{r}{R}\right)^p \left\{R^{p-n}\int_{B_R} \left[\left(|\nabla f_i|^p + |\nabla g_i|^p\right) + \left(|\nabla f_i^1|^p + |\nabla g_i^1|^p\right)\right]\right\} \\
&\le C\left(\frac{r}{R}\right)^p \left\{R^{p-n}\int_{B_R} |\nabla\phi|^p + \epsilon_0^p[\phi]^p_{\mathrm{BMO}(B_R)}\right\}.
\end{aligned} \tag{41}$$

Therefore, using (31), (39)–(41), and

$$|d\phi| \le \max_{i=1}^{k} \left|\sum_{j=1}^{k} P_{ji}\langle d\phi, e_j\rangle\right|,$$

we have

$$\begin{aligned}
r^{p-n}\int_{B_r} |\nabla\phi|^p &\le Cr^{p-n}\int_{B_r} \left[\left|\nabla f_i^2\right|^p + \left|\nabla g_i^2\right|^p\right] + Cr^{p-n}\int_{B_r} \left[|\nabla f_i^1|^p + |\nabla g_i^2|^p\right] \\
&\le C\left(\frac{r}{R}\right)^p \left\{R^{p-n}\int_{B_R} |\nabla\phi|^p + \epsilon_0^p\, [\phi]^p_{\mathrm{BMO}(B_R)}\right\} + Cr^{p-n}\int_{B_R} \left[|\nabla f_i^1|^p + |\nabla g_i^2|^p\right] \\
&\le C\left(\frac{r}{R}\right)^p \left\{R^{p-n}\int_{B_R} |\nabla\phi|^p + \left[1 + \left(\frac{r}{R}\right)^{-n}\epsilon_0^p\right] [\phi]^p_{\mathrm{BMO}(B_R)}\right\}.
\end{aligned} \tag{42}$$

As in [23], we set for $x_0 \in B_1$ and $0 < r \leq 1$,

$$\Phi(x_0, r) = r^{p-n} \int_{B_r(x_0)} |\nabla \phi|^p,$$

and for $0 < R \leq 1$,

$$\Psi(R) = \sup \{\Phi(x_0, r) : \; x_0 \in B_1, \; 0 < r \leq R\}.$$

Then we have

$$\sup_{x_0 \in B_1} [\phi]^p_{\mathrm{BMO}(B_R(x_0))} \leq C\Psi(R).$$

Thus, (42) yields that there is a universal constant $C > 0$ such that for any $x_0 \in B_1$ and $0 < r < R \leq 1$, it holds

$$\Phi(x_0, r) \leq C \left(\frac{r}{R}\right)^p \left[1 + \left(\frac{r}{R}\right)^{-n} \epsilon_0^p\right] \Psi(R). \tag{43}$$

Now for any given $\alpha \in (0, 1)$, choose $\lambda \in (0, 1)$ such that $2C \leq \lambda^{p(\alpha-1)}$, and choose $\epsilon_0 > 0$ such that $\epsilon_0^p = \lambda^n$. Then we have that

$$\Phi(x_0, \lambda R) \leq 2C\lambda^p \Psi(R) \leq \lambda^{p\alpha} \Psi(R) \leq \lambda^{p\alpha} \Psi(R_0) \tag{44}$$

holds for any $x_0 \in B_1$, $0 < R_0 < 1$, and $0 < R \leq R$. Taking supremum with respect to x_0 and $R < R_0$ gives

$$\Psi(\lambda R_0) \leq \lambda^{p\alpha} \Psi(R_0), \; \forall 0 < R_0 < 1. \tag{45}$$

The iteration of (45) yields

$$\Psi(r) \leq \left(\frac{r}{R_0}\right)^{p\alpha} \Psi(R_0), \forall 0 < r \leq R_0 < 1. \tag{46}$$

This, combined with Morrey's decay lemma (see Giaquinta [12]), implies that for any $\alpha \in (0, 1)$, $\phi \in C^\alpha(B_1)$ with $[\phi]_{C^\alpha(B_1)} \leq C\|\nabla \phi\|_{M^{2,2}(B_2)}$. ■

Next we present a proof on the higher order regularity of (ϕ, ψ). The ideas are suitable modifications of the hole-filling-type argument by Giaquinta–Hildebrandt [13] in the context of harmonic maps. More precisely, we have the following $C^{1,\alpha}$-regularity lemma.

Lemma 3.5. There exist $\epsilon_0 > 0$ such that if $(\phi, \psi) \in H^1(B_2, N) \times \mathcal{S}^{1,\frac{4}{3}}(B_2, \mathbb{C}^L \otimes \phi^* TN)$ is a weakly Dirac-harmonic map satisfying

$$\|\nabla \phi\|^2_{M^{2,2}(B_2)} + \|\psi\|^4_{M^{4,2}(B_2)} \leq \epsilon_0^2, \tag{47}$$

then there exists $\mu \in (0,1)$ such that $(\phi,\psi) \in C^{1,\mu}(B_1,N) \times C^{1,\mu}(B_1,\mathbb{C}^L \otimes \phi^* TN)$.

Proof. The proof is divided into several steps.

Step 1. For any $\beta \in (0,1)$, there exists $C > 0$ depending only on ϵ_0 such that

$$r^{2-n}\int_{B_r(x_0)} |\psi|^4 \le C r^{2\beta},\ \forall\ x_0 \in B_1,\ \text{and}\ r \le \frac{1}{2}. \tag{48}$$

To see this, first observe that the equation $D\psi = 0$ can be written as

$$\partial\psi^i = -\Gamma^i_{jl}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\left(\frac{\partial}{\partial x_\alpha}\circ\psi^l\right),\ \forall 1 \le i \le k, \tag{49}$$

where $\Gamma^i_{jl}(\phi)$ is the Christoffel symbol of (N,h) at ϕ. Note that the Lichnerowitz's formula (see [19]) gives

$$\partial^2\psi^i = -\Delta\psi^i + \frac{1}{2}R\psi^i = -\Delta\psi^i,$$

since the scalar curvature $R = 0$ on Ω. Therefore, taking ∂ of (49) gives

$$\Delta\psi^i = \partial\left[\Gamma^i_{jl}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\left(\frac{\partial}{\partial x_\alpha}\circ\psi^l\right)\right]. \tag{50}$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \eta \le 1$, $\eta = 1$ in $B_{\frac{r}{2}}(x_0)$, and $\eta = 0$ outside $B_r(x_0)$. Define

$$\psi_2^i(x) = -\int_{\mathbb{R}^n}\frac{\partial G(x,y)}{\partial y_\beta}\frac{\partial}{\partial y_\beta}\circ\left[\Gamma^i_{jl}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\left(\frac{\partial}{\partial x_\alpha}\circ\psi^l\right)\eta^2\right](y)\,dy, \tag{51}$$

where $G(x,y) = c_n|x-y|^{2-n}$ is the fundamental solution of Δ on $\mathbb{R}^n$. Then it is easy to see

$$\left|\psi_2^i\right|(x) \le C\int_{\mathbb{R}^n}\frac{(|\eta\nabla\phi||\eta\psi|)(y)}{|x-y|^{n-1}}\,dy = C I_1(|\eta\nabla\phi||\eta\psi|)(x), \tag{52}$$

where I_1 is the Riesz potential operator of order 1, that is the operator whose convolution kernel is $|x|^{1-n}$, $x \in \mathbb{R}^n$. Since $\nabla\phi \in M^{2,2}(B_2)$ and $\psi \in M^{4,2}(B_2)$, by the Hölder inequality we have $|\nabla\phi||\psi| \in M^{\frac{4}{3},2}(B_2)$. Hence, $|\eta\nabla\phi||\eta\psi| \in M^{\frac{4}{3},2}(\mathbb{R}^n)$ and

$$\||\eta\nabla\phi||\eta\psi|\|_{M^{\frac{4}{3},2}(\mathbb{R}^n)} \le C\,\||\nabla\phi||\psi|\|_{M^{\frac{4}{3},2}(B_r(x_0))} \le C\|\nabla\phi\|_{M^{2,2}(B_r(x_0))}\|\psi\|_{M^{4,2}(B_r(x_0))}.$$

By Adams' inequality on Morrey spaces (see Adams [1]) $I_1 : M^{\frac{4}{3},2}(\mathbb{R}^n) \to M^{\frac{4}{2}}(\mathbb{R}^n)$, we have

$$\left\|\psi_2^i\right\|_{M^{4,2}(\mathbb{R}^n)} \le C\,\||\eta\nabla\phi||\eta\psi|\|_{M^{\frac{4}{3},2}(\mathbb{R}^n)} \le C\|\nabla\phi\|_{M^{2,2}(B_r(x_0))}\|\psi\|_{M^{4,2}(B_r(x_0))}. \tag{53}$$

By the definition of ψ_2^i, we have

$$\Delta\psi_2^i = \partial\left[\Gamma^i_{jl}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\left(\frac{\partial}{\partial x_\alpha}\circ\psi^l\right)\eta^2\right] = \partial\left[\Gamma^i_{jl}(\phi)\frac{\partial\phi^j}{\partial x_\alpha}\left(\frac{\partial}{\partial x_\alpha}\circ\psi^l\right)\right]\ \text{ on } B_{\frac{r}{2}}(x_0)$$

so that $\Delta(\psi^i - \psi_2^i) = 0$ on $B_{\frac{r}{2}}(x_0)$. Hence, by the standard estimate on harmonic functions and (53), we have that for any $\theta \in (0, \frac{1}{2})$,

$$\left\|\psi^i - \psi_2^i\right\|_{M^{4,2}(B_{\theta r}(x_0))} \le C\theta^{\frac{1}{2}} \left\|\psi^i - \psi_2^i\right\|_{M^{4,2}(B_r(x_0))} \le C\theta^{\frac{1}{2}} \|\psi\|_{M^{4,2}(B_r(x_0))}. \tag{54}$$

Putting (53) and (54) together gives

$$\|\psi\|_{M^{4,2}(B_{\theta r}(x_0))} \le C\left(\theta^{\frac{1}{2}} + \epsilon_0\right)\|\psi\|_{M^{4,2}(B_r(x_0))}. \tag{55}$$

For any $\beta \in (0,1)$, choose $\theta = \theta(\beta) \in (0, \frac{1}{2})$ such that $2C \le \theta^{\frac{\beta-1}{2}}$ and then choose ϵ_0 such that $2C\epsilon_0 \le \theta^{\frac{\beta}{2}}$, we would have

$$\|\psi\|_{M^{4,2}(B_{\theta r}(x_0))} \le \theta^{\frac{\beta}{2}} \|\psi\|_{M^{4,2}(B_r(x_0))}. \tag{56}$$

By iteration, this clearly yields (48).

Step 2. For any $\beta \in (0,1)$, $x_0 \in B_1$, and $0 < r \le \frac{1}{2}$, it holds

$$r^{2-n} \int_{B_r(x_0)} |\nabla\phi|^2 \le C(\epsilon_0) r^{2\beta}. \tag{57}$$

To do it, let $v \in H^1(B_r(x_0), \mathbb{R}^K)$ be such that

$$\Delta v = 0 \ \text{ in } B_r(x_0); \quad v = \phi \ \text{ on } \partial B_r(x_0).$$

Then the maximum principle and (25) of Lemma 3.4 imply that for any $\beta \in (0,1)$

$$\|v - \phi\|_{L^\infty(B_r(x_0))} \le \operatorname{osc}_{B_r(x_0)} \phi \le C r^\beta. \tag{58}$$

Multiplying (5) by $\phi - v$, integrating over $B_r(x_0)$, and using (58) and (48), we have

$$\begin{aligned} \int_{B_r(x_0)} |\nabla(\phi - v)|^2 &\le C\left[\int_{B_r(x_0)} |\nabla\phi|^2 |\phi - v| + \int_{B_r(x_0)} |\nabla\phi| \, |\psi|^2 \, |\phi - v|\right] \\ &\le C r^\beta \left[\int_{B_r(x_0)} |\nabla\phi|^2 + \int_{B_r(x_0)} |\psi|^4\right] \\ &\le C r^\beta \int_{B_r(x_0)} |\nabla\phi|^2 + C r^{n-2+3\beta}. \end{aligned} \tag{59}$$

Hence, by the standard estimate on harmonic functions, we have

$$\begin{aligned} (\theta r)^{2-n} \int_{B_{\theta r}(x_0)} |\nabla\phi|^2 &\le 2\left[(\theta r)^{2-n} \int_{B_{\theta r}(x_0)} |\nabla v|^2 + (\theta r)^{2-n} \int_{B_{\theta r}(x_0)} |\nabla(\phi - v)|^2\right] \\ &\le C\theta^2 r^{2-n} \int_{B_r(x_0)} |\nabla\phi|^2 + 2(\theta r)^{2-n} \int_{B_r(x_0)} |\nabla(\phi - v)|^2 \\ &\le C\left[\left(\theta^2 + \theta^{2-n} r^\beta\right) r^{2-n} \int_{B_r(x_0)} |\nabla\phi|^2 + \theta^{2-n} r^{3\beta}\right]. \end{aligned} \tag{60}$$

It is not hard to see that we can choose small $r_0 > 0$ and $\theta = \theta(r_0, \beta, n) \in (0, \frac{1}{2})$ such that for any $0 < r \leq r_0$

$$(\theta r)^{2-n} \int_{B_{\theta r}(x_0)} |\nabla \phi|^2 \leq \theta^{2\beta} r^{2-n} \int_{B_r(x_0)} |\nabla \phi|^2 + r^{2\beta}. \tag{61}$$

This, combined with the iteration scheme as in Giaquinta [12], implies that for any $\beta \in (0,1)$ and $x_0 \in B_1$, it holds

$$r^{2-n} \int_{B_r(x_0)} |\nabla \phi|^2 \leq C r^{2\beta}, \ \forall 0 < r \leq r_0.$$

This yields (57).

Step 3. There exists $\mu \in (0,1)$ such that $(\phi, \psi) \in C^{1,\mu}(B_1)$. As in Step 2, let $v \in H^1(B_r(x_0), \mathbb{R}^K)$ be a harmonic function with $v = \phi$ on $\partial B_r(x_0)$. Then, as in (59), by using the estimates from Steps 1 and 2, we would have that for any $\beta \in (\frac{2}{3}, 1)$,

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla(\phi - v)|^2 &\leq C \left[\int_{B_r(x_0)} |\nabla \phi|^2 |\phi - v| + \int_{B_r(x_0)} |\nabla \phi| \, |\psi|^2 \, |\phi - v| \right] \\
&\leq C r^{\beta} \left[\int_{B_r(x_0)} |\nabla \phi|^2 + \int_{B_r(x_0)} |\psi|^4 \right] \leq C r^{n-2+3\beta}.
\end{aligned}$$

Hence, using the Campanato estimate for harmonic functions, we have

$$\begin{aligned}
(\theta r)^{-n} \int_{B_{\theta r}(x_0)} |\nabla \phi - \overline{\nabla \phi}_{x_0, \theta r}|^2 &\leq 2 \left[(\theta r)^{-n} \int_{B_{\theta r}(x_0)} |\nabla v - \overline{\nabla v}_{x_0, r}|^2 + (\theta r)^{-n} \int_{B_{\theta r}(x_0)} |\nabla(\phi - v)|^2 \right] \\
&\leq C \left[\theta^2 r^{-n} \int_{B_r(x_0)} |\nabla \phi - \overline{\nabla \phi}_{x_0, r}|^2 + \theta^{-n} r^{3\beta - 2} \right] \\
&\leq \frac{1}{2} r^{-n} \int_{B_r(x_0)} |\nabla \phi - \overline{\nabla \phi}_{x_0, r}|^2 + C r^{2\mu}
\end{aligned} \tag{62}$$

provided that we first choose $\theta \in (0, \frac{1}{2})$ sufficiently small, and then choose r_0 so small that $C\theta^{-n} r^{3\beta - 2} \leq r^{2\mu}$ for $0 < r \leq r_0$, where

$$\overline{\nabla \phi}_{x_0, r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \nabla \phi$$

is the average of $\nabla \phi$ over $B_r(x_0)$. It follows from the same iteration scheme as in [12] that

$$r^{-n} \int_{B_r(x_0)} |\nabla \phi - \overline{\nabla \phi}_{x_0, r}|^2 \leq C r^{2\mu}, \ \forall x_0 \in B_1, \ 0 < r \leq r_0.$$

This, combined with the characterization of C^{μ} by the Campanato space, implies $\nabla \phi \in C^{\mu}(B_1)$. Substituting $\nabla \phi \in C^{\mu}(B_1)$ into (49), one can easily conclude that $\psi \in C^{1,\mu}(B_1)$. ∎

Thus, we complete the proof of Theorem 1.4.

Completion of Proof of Theorem 1.4. Combining Lemmas 3.4 and 3.5, we know that $(\phi, \psi) \in C^{1,\mu}(B_{\frac{r_0}{2}}(x_0))$. The higher order regularity then follows from the standard bootstrap argument for both (5) and (6). We omit the details. □

4 Stationary Dirac-Harmonic Maps

In this section, we introduce the notion of stationary Dirac-harmonic maps, which is a natural extension of stationary harmonic maps. Any smooth Dirac-harmonic map is a stationary Dirac-harmonic map (see [4]). We prove several interesting properties, including a partial regularity theorem, for stationary Dirac-harmonic maps. To simplify the presentation, we assume throughout this section that $(M, g) = (\Omega, g_0)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain and g_0 is the Euclidean metric on $\mathbb{R}^n$. Thus, the spinor bundle Σ associated with M can also be identified by $\Sigma = \Omega \times \mathbb{C}^L$, $L = \mathrm{rank}_{\mathbb{C}}\Sigma$. We remark that one can modify the proofs of Lemmas 4.2 and 4.4 and Proposition 4.5 without difficulties so that (10) and (11) in Section 1 hold for any general Riemannian manifold (M, g), we leave the details to the interested readers.

Definition 4.1. A weakly Dirac-harmonic map $(\phi, \psi) \in H^1(\Omega, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Omega, \mathbb{C}^L \otimes \phi^*TN))$ is called to be a stationary Dirac-harmonic map, if it is also a critical point of $L(\phi, \psi)$ with respect to the domain variations, i.e. for any $Y \in C_0^\infty(\Omega, \mathbb{R}^n)$, it holds

$$\frac{d}{dt}\Big|_{t=0}\left[\int_\Omega \left(|\nabla\phi_t|^2 + \langle\psi_t, D\psi_t\rangle\right)\right] = 0, \tag{63}$$

where $\phi_t(x) = \phi(x + tY(x))$ and $\psi_t(x) = \psi(x + tY(x))$.

We now derive the stationarity identity for stationary Dirac-harmonic maps defined above.

Lemma 4.2. Let $(\phi, \psi) \in H^1(\Omega, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Omega, \mathbb{C}^L \otimes \phi^*TN))$ be a weakly Dirac-harmonic map. Then (ϕ, ψ) is a stationary Dirac-harmonic map iff for any $Y \in C_0^\infty(\Omega, \mathbb{R}^n)$, it holds

$$\int_\Omega \left[\left\langle \frac{\partial\phi}{\partial x_\alpha}, \frac{\partial\phi}{\partial x_\beta}\right\rangle - \frac{1}{2}|\nabla\phi|^2\delta_{\alpha\beta} + \frac{1}{2}\mathrm{Re}\left\langle\psi, \frac{\partial}{\partial x_\alpha}\circ\nabla_{\frac{\partial}{\partial x_\beta}}\psi\right\rangle\right]\frac{\partial Y_\alpha}{\partial x_\beta} = 0. \tag{64}$$

□

Proof. For $t \in \mathbb{R}$ with small $|t|$, denote $y = F_t(x) = x + tY(x) : \Omega \to \Omega$ and $x = F_t^{-1}(y)$. It is a standard calculation (see, e.g. [20]) that

$$\frac{d}{dt}\Big|_{t=0}\int_\Omega |d\phi_t|^2\ dx = \int_\Omega \left[2\left\langle\frac{\partial\phi}{\partial x_\alpha}, \frac{\partial\phi}{\partial x_\beta}\right\rangle\frac{\partial Y_\alpha}{\partial x_\beta} - |\nabla\phi|^2\mathrm{div}(Y)\right]. \tag{65}$$

Now we compute $\frac{d}{dt}|_{t=0}\int_\Omega \langle \psi_t, D\psi_t\rangle$. First, by Remark 1.3, we have

$$\int_\Omega \langle \psi_t, D\psi_t\rangle = \int_\Omega \mathrm{Re}\,\langle \psi_t, D\psi_t\rangle\,.$$

Before taking $\frac{d}{dt}$, we perform a change of variable as follows. For $y = F_t(x)$, since

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial y_\beta}{\partial x_\alpha}\frac{\partial}{\partial y_\beta},$$

we have

$$D\psi_t = \frac{\partial}{\partial x_\alpha}(F_t(x)) \circ \nabla_{\frac{\partial}{\partial x_\alpha}(F_t(x))}\psi = \frac{\partial y_\beta}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}(y) \circ \nabla_{\frac{\partial}{\partial y_\beta}}\psi.$$

Thus

$$\int_\Omega \mathrm{Re}\,\langle \psi_t, D\psi_t\rangle = \int_\Omega \sum_{\alpha,\beta}\frac{\partial y_\beta}{\partial x_\alpha}\mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha}(y)\circ \nabla_{\frac{\partial}{\partial y_\beta}}\psi\right\rangle \mathrm{Jac}F_t^{-1}\,dy.$$

Since

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Jac}F_t^{-1} = -\mathrm{div}(Y) \text{ and } \frac{d}{dt}\Big|_{t=0}\frac{\partial y_\beta}{\partial x_\alpha} = \frac{\partial Y_\beta}{\partial x_\alpha},$$

we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\int_\Omega \mathrm{Re}\,\langle \psi_t, D\psi_t\rangle &= \int_\Omega \mathrm{Re}\,\langle \psi, D\psi\rangle\left[\frac{d}{dt}|_{t=0}\mathrm{Jac}\big(F_t^{-1}\big)\right] + \int_\Omega\sum_{\alpha,\beta}\mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha}\circ\nabla_{\frac{\partial}{\partial x_\beta}}\psi\right\rangle\frac{\partial Y_\beta}{\partial x_\alpha}\\ &= -\int_\Omega \mathrm{Re}\,\langle \psi, D\psi\rangle\,\mathrm{div}(Y) + \int_\Omega\sum_{\alpha,\beta}\mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha}\circ\nabla_{\frac{\partial}{\partial x_\beta}}\psi\right\rangle\frac{\partial Y_\beta}{\partial x_\alpha}\\ &= \int_\Omega \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha}\circ\nabla_{\frac{\partial}{\partial x_\beta}}\psi\right\rangle\frac{\partial Y_\beta}{\partial x_\alpha}, \qquad (66)\end{aligned}$$

where we have used $D\psi = 0$ in the last step. It is clear that (64) follows from (65) and (66). ■

It is well known that any stationary harmonic map $u : \mathbb{R}^n \to N$ with finite Dirichlet energy is a constant map. Here we prove that the same conclusion holds for stationary Dirac-harmonic maps.

Theorem 4.3. For $n \geq 3$, assume that $(\phi, \psi) \in H^1_{\mathrm{loc}}(\mathbb{R}^n, N) \times \mathcal{S}^{1,\frac{4}{3}}_{\mathrm{loc}}(\Gamma(\Sigma \times \phi^* TN))$ is a stationary Dirac-harmonic map satisfying

$$\int_{\mathbb{R}^n}\left(|\nabla\phi|^2 + |\nabla\psi|^{\frac{4}{3}} + |\psi|^4\right) < +\infty. \qquad (67)$$

Then $\phi \equiv$ constant and $\psi \equiv 0$.

Proof. For any large $R > 0$, let $\eta \in C_0^\infty(\mathbb{R})$ be such that $\eta \equiv 1$ for $|t| \le R$, $\eta \equiv 0$ for $|t| \ge 2R$, and $|\eta'(t)| \le \frac{4}{R}$. Let $Y(x) = x\eta(|x|) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Note that

$$\frac{\partial Y_\beta}{\partial x_\alpha} = \eta(|x|)\delta_{\alpha\beta} + \frac{x_\alpha x_\beta}{|x|}\eta'(|x|).$$

Substituting Y into the stationarity identity (64) yields

$$\begin{aligned}
&\left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 \eta(|x|) - \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha} \circ \nabla_{\frac{\partial}{\partial x_\alpha}} \psi \right\rangle \eta(|x|) \\
&\quad = \int_{\mathbb{R}^n} \left[\left\langle \frac{\partial\phi}{\partial x_\alpha}, \frac{\partial\phi}{\partial x_\beta} \right\rangle - \frac{1}{2}|\nabla\phi|^2 \delta_{\alpha\beta} + \frac{1}{2}\mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha} \circ \nabla_{\frac{\partial}{\partial x_\beta}} \psi \right\rangle\right] \eta'(|x|)\frac{x_\alpha x_\beta}{|x|} \\
&\quad = \int_{\mathbb{R}^n} \left[\left(\left|\frac{\partial\phi}{\partial r}\right|^2 - \frac{1}{2}|\nabla\phi|^2\right) + \frac{1}{2}\mathrm{Re}\left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle\right] |x|\eta'(|x|).
\end{aligned}$$

Since $D\psi = 0$, we have

$$\frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha} \circ \nabla_{\frac{\partial}{\partial x_\alpha}} \psi \right\rangle \eta(|x|) = \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Re}\left\langle \psi, D\psi \right\rangle \eta(|x|) = 0.$$

The right-hand side can be estimated by

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \left[\left(\left|\frac{\partial\phi}{\partial r}\right|^2 - \frac{1}{2}|\nabla\phi|^2\right) + \frac{1}{2}\mathrm{Re}\left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle\right] |x|\eta'(|x|) \right| &\le C\int_{B_{2R}\setminus B_R} \left[|\nabla\phi|^2 + |\psi||\nabla\psi|\right] \\
&\le C\int_{B_{2R}\setminus B_R} \left[|\nabla\phi|^2 + |\psi|^4 + |\nabla\psi|^{\frac{4}{3}}\right].
\end{aligned} \tag{68}$$

On the other hand, the left-hand side is bounded below by

$$\left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 \eta(|x|) \ge \left(\frac{n}{2} - 1\right) \int_{B_R} |\nabla\phi|^2. \tag{69}$$

Since $n \ge 3$, (68) and (69) imply

$$\int_{B_R} |\nabla\phi|^2 \le C\int_{B_{2R}\setminus B_R} \left[|\nabla\phi|^2 + |\psi|^4 + |\nabla\psi|^{\frac{4}{3}}\right], \tag{70}$$

this and (67) imply, after sending R to ∞,

$$\int_{\mathbb{R}^n} |\nabla\phi|^2 = 0,$$

i.e. $\phi \equiv$ constant. Substituting this ϕ into the equation of ψ, we have $\partial\psi = 0$ in $\mathbb{R}^n$. Since $\psi \in L^4(\mathbb{R}^n)$, it follows easily that $\psi \equiv 0$. ■

Now we derive an identity for stationary Dirac-harmonic maps, which is similar to the monotonicity identity for stationary harmonic maps.

Lemma 4.4. Assume that $(\phi, \psi) \in H^1(\Omega, N) \times \mathcal{S}^{1,\frac{4}{3}}(\Omega, \mathbb{C}^L \otimes \phi^* TN)$ is a stationary Dirac-harmonic map. Then for any $x_0 \in \Omega$ and $0 < R_1 \le R_2 < \mathrm{dist}(x_0, \partial\Omega)$, it holds

$$R_2^{2-n} \int_{B_{R_2}(x_0)} |\nabla\phi|^2 \, dx - R_1^{2-n} \int_{B_{R_1}(x_0)} |\nabla\phi|^2 \, dx = \int_{R_1}^{R_2} r^{2-n} \left(\int_{\partial B_r(x_0)} \left[2 \left| \frac{\partial \phi}{\partial r} \right|^2 + \mathrm{Re} \left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle \right] dH^{n-1} \right) dr, \tag{71}$$

where $\frac{\partial}{\partial r} = \frac{\partial}{\partial |x - x_0|}$.

Proof. For simplicity, assume $x_0 = 0 \in \Omega$. The argument is similar to that of Theorem 4.3. For completeness, we outline it again. For $\epsilon > 0$ and $0 < r < \mathrm{dist}(0, \partial\Omega)$, let $\eta_\epsilon(x) = \eta_\epsilon(|x|) \in C_0^\infty(B_r)$ be such that $0 \le \eta_\epsilon \le 1$, $\eta_\epsilon = 1$ for $|x| \le r(1-\epsilon)$. Choose $Y(x) = x\eta_\varepsilon(|x|)$. Note

$$\frac{\partial Y_\beta}{\partial x_\alpha} = \delta_{\alpha\beta}\eta_\epsilon(|x|) + \frac{x_\alpha x_\beta}{|x|}\eta_\epsilon'(|x|).$$

Substituting Y into (64), we have

$$\begin{aligned}
&(2-n)\int_{B_r} |\nabla\phi(x)|^2 \eta_\epsilon(|x|) + \int_{B_r} \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha} \circ \nabla_{\frac{\partial}{\partial x_\alpha}} \psi \right\rangle \eta_\epsilon(|x|) \\
&= -\int_{B_r} \left[2\left\langle \frac{\partial\phi}{\partial x_\alpha}, \frac{\partial\phi}{\partial x_\beta} \right\rangle - |\nabla\phi|^2 \delta_{\alpha\beta} + \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial x_\alpha} \circ \nabla_{\frac{\partial}{\partial x_\beta}} \psi \right\rangle \right] \eta_\epsilon'(|x|) \frac{x_\alpha x_\beta}{|x|} \\
&= -\int_{B_r} \left[\left(2\left| \frac{\partial\phi}{\partial r} \right|^2 - |\nabla\phi|^2 \right) + \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle \right] |x| \eta_\epsilon'(|x|).
\end{aligned}$$

Using the equation $D\psi = \frac{\partial}{\partial x_\alpha} \circ \nabla_{\frac{\partial}{\partial x_\alpha}} \psi = 0$ and sending ϵ to 0 yield

$$(2-n)\int_{B_r} |\nabla\phi|^2 \, dx + r\int_{\partial B_r} |\nabla\phi|^2 \, dH^{n-1} = 2r\int_{\partial B_r} \left| \frac{\partial\phi}{\partial r} \right|^2 dH^{n-1} + r\int_{\partial B_r} \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle dH^{n-1}, \tag{72}$$

or equivalently,

$$\frac{d}{dr}\left(r^{2-n} \int_{B_r} |\nabla\phi|^2 \, dx \right) = r^{2-n} \int_{\partial B_r} \left[2\left| \frac{\partial\phi}{\partial r} \right|^2 + \mathrm{Re}\left\langle \psi, \frac{\partial}{\partial r} \circ \nabla_{\frac{\partial}{\partial r}} \psi \right\rangle \right] dH^{n-1}.$$

Integrating r from R_1 to R_2 yields (4.4). ∎

In contrast with stationary harmonic maps, (71) does not imply that the renormalized energy

$$R^{2-n}\int_{B_R(x_0)}|\nabla\phi|^2$$

is monotone increasing with respect to R yet. In order to have such a monotonicity property, we need to assume that $\nabla\psi$ has higher integrability. More precisely, we have the following proposition.

Proposition 4.5. Assume that $(\phi,\psi)\in H^1(\Omega,N)\times\mathcal{S}^{1,\frac{4}{3}}(\Omega,\mathbb{C}^L\otimes\phi^*TN)$ is a stationary Dirac-harmonic map. If, in addition, $\nabla\psi\in L^p(\Omega)$ for some $\frac{2n}{3}<p\le n$, then there exists $C_0>0$ depending only on $\|\nabla\psi\|_{L^p(\Omega)}$ such that for any $x_0\in\Omega$ and $0<R_1<R_2<\mathrm{dist}(x_0,\partial\Omega)$, it holds

$$R_1^{2-n}\int_{B_{R_1}(x_0)}|\nabla\phi|^2\le R_2^{2-n}\int_{B_{R_2}(x_0)}|\nabla\phi|^2+C_0R_2^{3-\frac{2n}{p}}. \tag{73}$$

Proof. For simplicity, assume $x_0=0$. For $x\in\Omega$, denote

$$f(x)=\left|\mathrm{Re}\left\langle\psi,\frac{\partial}{\partial r}\circ\nabla_{\frac{\partial}{\partial r}}\psi\right\rangle(x)\right|.$$

Since $\nabla\psi\in L^p(\Omega)$, by the Sobolev embedding theorem we have $\psi\in L^{\frac{np}{n-p}}(\Omega)$. Since $f(x)\le C|\psi||\nabla\psi|$, the Hölder inequality implies that $f\in L^q(\Omega)$ with $q=\frac{np}{2n-p}$. Since $p>\frac{2n}{3}$, it is easy to see that $q>\frac{n}{2}$. It then follows that for any $R<R_0=\mathrm{dist}(0,\partial\Omega)$,

$$\begin{aligned}\int_0^R r^{1-n}\int_{B_r}f(x)\,dx&\le\left(\int_0^R r^{1-\frac{n}{q}}\,dr\right)\|f\|_{L^q(B_{R_0})}\\&=\left(\frac{q}{2q-n}\right)R^{2-\frac{n}{q}}\|f\|_{L^q(B_{R_0})}<+\infty,\end{aligned} \tag{74}$$

and

$$R^{2-n}\int_{B_R}f\le R^{2-\frac{n}{q}}\|f\|_{L^q(B_{R_0})}. \tag{75}$$

For any $0<R_1\le r\le R_2<R_0$, set

$$g(r)=\int_{\partial B_r}f(x)\,dH^{n-1}.$$

Then, by integration by parts, we have

$$\begin{aligned}
&\int_{R_1}^{R_2} r^{2-n}\int_{\partial B_r}\left|\mathrm{Re}\left\langle\psi,\frac{\partial}{\partial r}\circ\nabla_{\frac{\partial}{\partial r}}\psi\right\rangle\right| dH^{n-1}\\
&\quad=\int_{R_1}^{R_2} r^{2-n}g(r)\,dr=\int_{R_1}^{R_2} r^{2-n}\,d\left(\int_{B_r} f(x)\,dx\right)\\
&\quad=R_2^{2-n}\int_{B_{R_2}} f(x)\,dx-R_1^{2-n}\int_{B_{R_1}} f(x)\,dx+(n-2)\int_{R_1}^{R_2} r^{1-n}\left(\int_{B_r} f(x)\,dx\right).
\end{aligned}$$

This, combined with (71), implies

$$\begin{aligned}
&R_2^{2-n}\int_{B_{R_2}}|\nabla\phi|^2+R_2^{2-n}\int_{B_{R_2}} f+(n-2)\int_0^{R_2} r^{1-n}\int_{B_r} f\\
&\geq R_1^{2-n}\int_{B_{R_1}}|\nabla\phi|^2+R_1^{2-n}\int_{B_{R_1}} f+(n-2)\int_0^{R_1} r^{1-n}\int_{B_r} f+\int_{R_1}^{R_2} r^{2-n}\int_{\partial B_r}\left|\frac{\partial\phi}{\partial r}\right|^2 dH^{n-1}.
\end{aligned}$$

It is easy to see that this inequality, (74), and (75) imply (73). ■

With the help of Proposition 4.5 and Theorem 1.4, we can prove Theorem 1.8.

Proof of Theorem 1.8. For simplicity, assume $M=\Omega\subseteq\mathbb{R}^n$ and $g=g_0$ is the Euclidean metric on $\mathbb{R}^n$. Since $\nabla\psi\in L^p(\Omega)$ for some $p>\frac{2n}{3}$, we have by Sobolev's embedding theorem that $\psi\in L^q(\Omega)$ for $q=\frac{np}{n-p}>2n$. Hence, for any ball $B_R(x)\subseteq\Omega$, by the Hölder inequality we have

$$R^{2-n}\int_{B_R(x)}|\psi|^4\leq\left(\int_{B_R(x)}|\psi|^q\right)^{\frac{4}{q}}R^{2-\frac{4n}{q}}\leq\|\nabla\psi\|^4_{L^p(\Omega)}R^{2-\frac{4n}{q}}. \tag{76}$$

Let $\epsilon_0>0$ be the constant given by Theorem 1.4. For a large constant $C(n)>0$ to be chosen later, define

$$\mathcal{S}(\phi)=\bigcap_{R>0}\left\{x\in\Omega: R^{2-n}\int_{B_R(x)}|\nabla\phi|^2>\frac{\epsilon_0^2}{C(n)}\right\}.$$

It is well known (see Evans–Gariepy [10]) that $H^{n-2}(\mathcal{S}(\phi))=0$. For any $x_0\in\Omega\setminus\mathcal{S}(\phi)$, there exists $r_0>0$ such that

$$(2r_0)^{2-n}\int_{B_{2r_0}(x_0)}|\nabla\phi|^2\leq\frac{\epsilon_0^2}{C(n)}.$$

Hence

$$\sup_{x\in B_{r_0}(x_0)}\left\{r_0^{2-n}\int_{B_{r_0}(x)}|\nabla\phi|^2\right\}\leq\frac{2^{n-2}\epsilon_0^2}{C(n)}.$$

Applying the monotonicity inequality (73) implies

$$\sup_{x\in B_{r_0}(x_0),0<r\le r_0}\left\{r^{2-n}\int_{B_r(x)}|\nabla\phi|^2\right\}\le\frac{2^{n-2}\epsilon_0^2}{C(n)}+C_0r_0^{3-\frac{2n}{p}}\le\frac{\epsilon_0^2}{4},\tag{77}$$

provided that we choose $C(n)>2^{n+1}$ and $r_0\le(\frac{\epsilon_0^2}{8C_0})^{\frac{p}{3p-2n}}$. On the other hand, by (76) we have

$$\sup_{x\in B_{r_0}(x_0),0<r\le r_0}\left\{r^{2-n}\int_{B_r(x)}|\psi|^4\right\}\le\|\nabla\psi\|^4_{L^p(\Omega)}r_0^{\frac{6p-4n}{p}}\le\frac{\epsilon_0^2}{4},\tag{78}$$

provided that we choose $r_0<(\frac{\epsilon_0^2}{4\|\nabla\psi\|^4_{L^p}})^{\frac{p}{6p-4n}}$. Combining (77) with (78), we have that there exists $r_0>0$ sufficiently small such that

$$\sup_{x\in B_{r_0}(x_0),0<r\le r_0}\left\{r^{2-n}\int_{B_r(x)}|\nabla\phi|^2+r^{2-n}\int_{B_r(x)}|\psi|^4\right\}\le\frac{\epsilon_0^2}{2}.\tag{79}$$

Thus, Theorem 1.4 implies that $(\phi,\psi)\in C^\infty(B_{\frac{r_0}{2}}(x_0),N)\times C^\infty(B_{\frac{r_0}{2}}(x_0),\mathbb{C}^L\otimes\phi^*TN)$. Note that this also yields $\Omega\setminus\mathcal{S}(\phi)$, which is an open set. The proof of Theorem 1.8 is now complete. ∎

5 Convergence of Approximate Dirac-Harmonic Maps

In dimension two, the weak convergence theorem of approximated harmonic maps or Palais–Smale sequences of Dirichlet energy functional for maps into Riemannian manifolds was first proved by Bethuel [3]. Subsequently, alternative proofs were given by Freire–Müller–Struwe [11], and Wang [25] by employing the moving frame and various techniques including the concentration compactness method. Very recently, Rivière [21] gave another proof using the conservation laws.

In this section, we extend such a convergence theorem to sequences of approximate Dirac-harmonic maps from a spin Riemann surface. The key ingredient is first to use the moving frame to rewrite the equation of approximate harmonic maps into the form similar to (12), and then to use Rivière's Coulomba gauge construction technique to rewrite it into the form (28), in which the concentration compactness method is similar to that of [11], to pass to the limit.

Since $\int|d\phi|^2+|\psi|^4$ is conformally invariant in dimension two, it follows from a scaling argument and a covering argument that Theorem 1.9 follows from Lemma 5.1.

For simplicity, we assume that $(M, g) = (\Omega, g_0)$ for some bounded smooth domain $\Omega \subseteq \mathbb{R}^2$ with the Euclidean metric g_0. Denote by $B_r \subseteq \mathbb{R}^2$ the ball center at 0 with radius r.

Lemma 5.1. There exists $\epsilon_1 > 0$ such that if $(\phi_p, \psi_p) \in H^1(B_1, N) \times \mathcal{S}^{1,\frac{4}{3}}(B_1, \mathbb{C}^2 \otimes \phi_m^* TN)$ is a sequence of approximate Dirac-harmonic maps, i.e.

$$\tau(\phi_p) = \mathcal{R}^N(\phi_p, \psi_p) + u_p; \;\; D\psi_p = v_p \;\text{ on } B_1,$$

$$u_p \to 0 \text{ strongly in } H^{-1}(B_1) \;\text{ and } v_p \rightharpoonup 0 \text{ in } L^{\frac{4}{3}}(B_1),$$

and

$$\int_{B_1} \left(|\nabla \phi_p|^2 + |\psi_p|^4\right) \le \epsilon_1^2. \tag{80}$$

If $(\phi_p, \psi_p) \rightharpoonup (\phi, \psi)$ in $H^1(B_1, N) \times \mathcal{S}^{1,\frac{4}{3}}(B_1, \mathbb{C}^2 \otimes \mathbb{R}^K)$, then $(\phi, \psi) \in H^1(B_1, N) \times \mathcal{S}^{1,\frac{4}{3}}(B_1, \mathbb{C}^2 \otimes \phi^* TN)$ is a weakly Dirac-harmonic map.

Proof. First observe that the argument of Proposition 2.1 can be easily modified to show that if $(\widetilde{N}, \widetilde{h})$ is another Riemannian manifold and $f : (N, h) \to (\widetilde{N}, \widetilde{h})$ is a totally geodesic, isometric embedding, and if we set $\widetilde{\phi} = f(\phi)$ and $\widetilde{\psi} = f_*(\psi)$, then $(\widetilde{\phi}, \widetilde{\psi}) \in H^1(B_1, \widetilde{N}) \times \mathcal{S}^{1,\frac{4}{3}}(B_1, \mathbb{C}^2 \otimes (\widetilde{\phi})^* T\widetilde{N})$ is a sequence of approximate harmonic maps with (u_p, v_p) replaced by $(\widetilde{u}_p, \widetilde{v}_p)$, where $\widetilde{u}_p = f_*(u_p)$ and $\widetilde{v}_p = f_*(v_p)$. Moreover, it is easy to check that

$$\widetilde{u}_p \to 0 \text{ strongly in } H^{-1}(B_1) \;\text{ and } \widetilde{v}_p \rightharpoonup 0 \;\text{ in } L^{\frac{4}{3}}(B_1),$$

$$\widetilde{\phi}_p \rightharpoonup \widetilde{\phi} = f(\phi) \text{ in } H^1(B_1), \; \widetilde{\psi}_p \rightharpoonup \widetilde{\psi} = f_*(\psi) \text{ in } \mathcal{S}^{1,\frac{4}{3}}(B_1),$$

and

$$\int_{B_1} |\nabla \widetilde{\phi}_p|^2 + |\widetilde{\psi}_p|^2 \le \epsilon_0^2.$$

With this reduction, we may assume that there exists a global orthonormal frame $\{\hat{e}_i\}_{i=1}^k$ on (N, h). For any p, let $e_i^p = \hat{e}_i(\phi_p)$, $1 \le i \le k$, be the orthonormal frame along with ϕ_p. Then, similar to Lemma 2.2, Proposition 3.1, and (19), we have

$$d^*\left(\langle d\phi_p, e_i^p\rangle\right) = \sum_{j=1}^k \Theta_{ij}^p \langle d\phi_p, e_j^p\rangle + u_p^i, \tag{81}$$

where

$$\Theta_{ij}^p = \Omega_{ij}^p + \langle de_i^p, e_j^p\rangle, \; 1 \le i, j \le k; \;\; u_p^i = \langle u_p, e_i^p\rangle, \; 1 \le i \le k, \tag{82}$$

and

$$\Omega_{ij}^p = \sum_{\alpha=1}^{n} \left[\sum_{l,m=1}^{k} R^N(\phi_p)\big(e_i^p, e_j^p, e_l^p, e_m^p\big) \left\langle \psi_p^m, \frac{\partial}{\partial x_\alpha} \circ \psi_p^l \right\rangle \right] dx_\alpha, 1 \le i, j \le k. \tag{83}$$

Since $\Theta^p = (\Theta_{ij}^p)$ satisfies

$$\int_{B_1} \left|\Theta^p\right|^2 \le C \int_{B_1} (|\nabla \phi_p|^2 + |\psi_p|^4) \le C\epsilon_1^2 \le \epsilon_0^2,$$

provided $\epsilon_1 \le \frac{\epsilon_0}{\sqrt{C}}$, where ϵ_0 is the same constant as in Lemma 3.3. Hence, Lemma 3.3 implies that there exist $Q^p \in H^1(B_1, \mathrm{SO}(k))$ and $\xi^p \in H^1(B_1, \mathrm{so}(k) \otimes \wedge^2\mathbb{R}^2)$ such that

$$(Q^p)^{-1} dQ^p - (Q^p)^{-1}\Theta^p Q^p = d^*\xi^p \ \text{ in } B_1, \tag{84}$$

$$d\xi^p = 0 \ \text{ in } B_1,\ \xi^p = 0 \ \text{ on } \partial B_1, \tag{85}$$

and

$$\|\nabla Q^p\|_{L^2(B_1)} + \|\nabla \xi^p\|_{L^2(B_1)} \le C\|\Theta^p\|_{L^2(B_1)} \le C\epsilon_0. \tag{86}$$

Multiplying (81) by $(Q^p)^{-1}$ (see also (28)), we obtain

$$d^*\left[(Q^p)^{-1} \begin{pmatrix} \langle d\phi_p, e_1^p\rangle \\ \vdots \\ \langle d\phi_p, e_k^p\rangle \end{pmatrix} \right] = -d^*\xi^p \cdot (Q^p)^{-1} \begin{pmatrix} \langle d\phi_p, e_1^p\rangle \\ \vdots \\ \langle d\phi_p, e_k^p\rangle \end{pmatrix} + (Q^p)^{-1} \begin{pmatrix} u_p^1 \\ \vdots \\ u_p^k \end{pmatrix}. \tag{87}$$

Since $e_i^p = \hat{e}_i(\phi_p)$, it is easy to see that for $1 \le i \le k$, $e_i^p \rightharpoonup e_i = \hat{e}_i(\phi)$ in $H^1(B_1)$ and hence $\{e_i\}$ is an orthonormal frame along the map ϕ.

After passing to possible subsequences, we may now assume that

$$Q^p \to Q \text{ weakly in } H^1(B_1, \mathrm{SO}(k)),\ \text{strongly in } L^2(B_1, \mathrm{SO}(k)),\ \text{and a.e. in } B_1,$$

$$\xi^p \to \xi \text{ weakly in } H^1(B_1, \mathrm{so}(k)),\ \text{strongly in } L^2(B_1, \mathrm{so}(k)),\ \text{and a.e. in } B_1.$$

It is not hard to see that

$$\langle de_i^p, e_j^p\rangle \to \langle de_i, e_j\rangle \text{ weakly in } L^2(B_1).$$

Since (Ω_{ij}^p) is bounded in $L^2(B_1)$ and

$$\Omega_{ij}^p \to \Omega_{ij} \equiv \sum_{\alpha=1}^{n} \left[\sum_{l,m=1}^{k} R^N(\phi)(e_i, e_j, e_l, e_m) \left\langle \psi^m, \frac{\partial}{\partial x_\alpha} \circ \psi^l \right\rangle \right] dx_\alpha, \ \text{ a.e. in } B_1,$$

$\Omega_{ij}^p \to \Omega_{ij}$ weakly in $L^2(B_1)$. Hence, $\Theta_{ij}^p \to \Theta_{ij} \equiv \Omega_{ij} + \langle de_i, e_j\rangle$ weakly in $L^2(B_1)$. Thus, sending $p \to \infty$, (84), and (85) yields that Q, ξ, Θ satisfy

$$Q^{-1}dQ - Q^{-1}\Theta Q = d^*\xi \ \text{ in } B_1, \tag{88}$$

$$d\xi = 0 \ \text{ in } B_1,\ \xi = 0 \ \text{ on } \partial B_1. \tag{89}$$

Since $u_p \to 0$ in $H^{-1}(B_1)$, we have

$$(Q^p)^{-1}\begin{pmatrix} u_p^1 \\ \vdots \\ u_p^k \end{pmatrix} \to 0 \tag{90}$$

in the sense of distribution on B_1. It is also easy to see

$$d^*\left[(Q^p)^{-1}\begin{pmatrix} \langle d\phi_p, e_1^p\rangle \\ \vdots \\ \langle d\phi_p, e_k^p\rangle \end{pmatrix}\right] \to d^*\left[Q^{-1}\begin{pmatrix} \langle d\phi, e_1\rangle \\ \vdots \\ \langle d\phi, e_k\rangle \end{pmatrix}\right] \tag{91}$$

in the sense of distribution on B_1.

Now we want to discuss the convergence of

$$A_p := d^*\xi^p \cdot (Q^p)^{-1}\begin{pmatrix} \langle d\phi_p, e_1^p\rangle \\ \vdots \\ \langle d\phi_p, e_k^p\rangle \end{pmatrix}.$$

Note that the ith component of A_p, A_p^i, is given by

$$A_p^i = d^*\xi_{il}^p \cdot \left(Q_{ml}^p\langle d\phi_p, e_m^p\rangle\right) = \left\langle d^*\xi_{il}^p \cdot d\phi_p, \left(Q_{ml}^p e_m^p\right)\right\rangle.$$

For this, we recall a compensated compactness lemma; see Freire–Müller–Struwe [11] and Lemma 3.4 of Wang [24] for a proof.

Lemma 5.2. For $n = 2$, suppose that $f_p \to f$ weakly in $H^1(B_1)$, $g_p \to g$ weakly in $H^1(B_1, \wedge^2\mathbb{R}^2)$, and $h_p \to h$ weakly in $H^1(B_1)$. Then, after passing to possible subsequences, we have

$$df_p \cdot d^*g_p \cdot h_p \to df \cdot d^*g \cdot h + \nu \tag{92}$$

in the sense of distributions on B_1, where nu is a signed Radon measure given by

$$\nu = \sum_{j\in J} a_j\delta_{x_j},$$

where J is at most countable, $a_j \in \mathbb{R}$, $x_j \in B_1$, and $\sum_{j\in J}|a_j| < +\infty$.

Applying Lemma 5.2, we conclude that for $1 \le i \le k$,

$$A_p^i \to A^i := d^* \xi_{il} \cdot (Q_{ml} \langle d\phi, e_m \rangle) + \nu^i \ \text{ in } B_1 \tag{93}$$

where

$$\nu^i = \sum_{j=1}^{\infty} a_j^i \delta_{x_j^i}, \ \sum_{j=1}^{\infty} |a_j^i| < +\infty.$$

Putting (91), (93), and (90) into (87), we obtain

$$d^* \left[Q^{-1} \begin{pmatrix} \langle d\phi, e_1 \rangle \\ \vdots \\ \langle d\phi, e_k \rangle \end{pmatrix} \right] = -d^* \xi \cdot Q^{-1} \begin{pmatrix} \langle d\phi, e_1 \rangle \\ \vdots \\ \langle d\phi, e_k \rangle \end{pmatrix} + \begin{pmatrix} \nu^1 \\ \vdots \\ \nu^k \end{pmatrix}. \tag{94}$$

Note that (94) implies

$$\begin{pmatrix} \nu^1 \\ \vdots \\ \nu^k \end{pmatrix} \in H^{-1}(B_1) + L^1(B_1),$$

so that $\nu^i = 0$ for all $1 \le i \le k$. Therefore, we have

$$d^* \left[Q^{-1} \begin{pmatrix} \langle d\phi, e_1 \rangle \\ \vdots \\ \langle d\phi, e_k \rangle \end{pmatrix} \right] = -d^* \xi \cdot Q^{-1} \begin{pmatrix} \langle d\phi, e_1 \rangle \\ \vdots \\ \langle d\phi, e_k \rangle \end{pmatrix}. \tag{95}$$

This and (88) imply that

$$d^*(\langle d\phi, e_i \rangle) = \sum_{j=1}^{k} \Theta_{ij} \cdot \langle d\phi, e_j \rangle.$$

Note that this equation is equivalent to the Dirac-harmonic map equation $\tau(\phi) = \mathcal{R}^N(\phi, \psi)$.

Now we want to show $D\psi = 0$. To see this, observe that if we write $\psi_p = \psi_p^i \otimes e_i^p$ and $v_p = v_p^i \otimes e_i^p$, then $D\psi_p = v_p$ becomes

$$\partial \psi_p^i = -\Gamma_{jl}^i(\phi_p) \left\langle \frac{\partial \phi_p}{\partial x_\alpha}, e_j^p \right\rangle \frac{\partial}{\partial x_\alpha} \circ \psi_p^l + v_p^i. \tag{96}$$

It is easy to see that, after taking p to ∞, (96) yields

$$\partial \psi^i = \Gamma_{jl}^i(\phi) \left\langle \frac{\partial \phi}{\partial x_\alpha}, e_j \right\rangle \frac{\partial}{\partial x_\alpha} \circ \psi^l,$$

which is equivalent to $D\psi = 0$. Thus, the proof is complete. □

Proof of Theorem 1.9. Define the possible concentration set

$$C = \bigcap_{R>0} \left\{ x \in M : \liminf_{p\to\infty} \int_{B_R(x)} \left(|\nabla \phi_p|^2 + |\psi_p|^4 \right) > \epsilon_1^2 \right\}.$$

Then, by a simple covering argument, we have that C is at most a finite subset in M. By the definition, we know that for any $x_0 \in M \setminus C$, there exist $r_0 > 0$ and a subsequence of (ϕ_p, ψ_p), denoted by itself, such that

$$\lim_{p\to\infty} \int_{B_{r_0}(x_0)} \left(|\nabla \phi_p|^2 + |\psi_p|^4 \right) \le \epsilon_1^2.$$

Applying Lemma 5.1, we conclude that (ϕ, ψ) is a weakly Dirac-harmonic map on $B_{r_0}(x_0)$. Since $x_0 \in M \setminus C$ is arbitrary, this implies that (ϕ, ψ) is a weakly Dirac-harmonic map on $M \setminus C$. Since C is at most finite, one can easily show that (ϕ, ψ) is also a weakly Dirac-harmonic map on M. Hence, Theorem 1.5 also implies that (ϕ, ψ) is a smooth Dirac-harmonic map on M. The proof is now complete. ■

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