(P1)

Solution to Review problem of Final Exam

1)
$$y' + \frac{4t}{1+t^2} y = \frac{1}{(1+t^2)^3}$$

$$\mu(t) = e^{\int \frac{4t}{1+t^2} dt} = (1+t^2)^2$$

$$\Rightarrow \left((1+t^2)^2 y\right)' = \frac{1}{1+t^2}$$

$$\Rightarrow (1+t^2)^2 y(t) - (1+0^2)^2 y(0) = \int_0^t \frac{1}{1+t^2} dt$$

$$\Rightarrow (1+t^2)^2 y(t) = 1 + \arctan t.$$

$$\Rightarrow \boxed{y(t) = \frac{1}{(1+t^2)^2}\left[1 + \arctan t\right]}$$

2)
$$\int_1^y y^2 dy = \int_0^x \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$$

$$\frac{y^3}{3} - \frac{1}{3} \overset{u = \sin^{-1} x}{\underset{du = \frac{1}{\sqrt{1-x^2}} dx}{=\!=}} \int_0^{\sin^{-1} x} u \, du = \frac{(\sin^{-1} x)^2}{2}.$$

(P2)

3). $M_y = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$

$N_x = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$

$M_y = N_x \Rightarrow$ The equation is exact.

$$F = \int M\,dx + G(y)$$

$$= \int \left(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x\right) dx + G(y)$$

$$= \int \left[\left(e^{xy}\cos 2x\right)_x + 2x\right] dx + G(y)$$

$$= e^{xy}\cos 2x + x^2 + G(y)$$

$$F_y = xe^{xy}\cos 2x + G'(y) = xe^{xy}\cos 2x - 3$$

$$\Rightarrow G'(y) = -3 \Rightarrow G(y) = -3y$$

$$\Rightarrow \boxed{e^{xy}\cos 2x + x^2 - 3y = \text{const}}$$

(P3)

4) $$\frac{y}{K-y} = \frac{y_0}{K-y_0} e^{rt}$$

a). $$\frac{2y_0}{K-2y_0} = \frac{y_0}{K-y_0} e^{r\tau}$$

$$y_0 = K/3$$

$$\Rightarrow \frac{\frac{2K}{3}}{\frac{K}{3}} = \frac{\frac{K}{3}}{\frac{2K}{3}} e^{r\tau}$$

$$\Rightarrow 4 = e^{r\tau} \Rightarrow \tau = \frac{\ln 4}{r}$$

If $r = 0.025 \Rightarrow \boxed{\tau = \frac{\ln 4}{0.025}}$

b) $$\frac{K\beta}{K-\beta K} = \frac{\alpha K}{(1-\alpha)K} e^{rT}$$

$$\Rightarrow \frac{\beta}{1-\beta} \cdot \frac{1-\alpha}{\alpha} = e^{rT} \Rightarrow \boxed{T = \frac{\ln\left(\frac{(1-\alpha)\beta}{(1-\beta)\alpha}\right)}{r}}$$

5). $f_1, \cdots f_n$ L.I $\iff$ $\alpha_1 f_1 + \cdots + \alpha_n f_n \equiv 0$ implies $\alpha_1 = \cdots = \alpha_n = 0$.

P4

$$W[e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}]$$

$$= \begin{vmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & e^{\lambda_3 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \lambda_3 e^{\lambda_3 t} \\ \lambda_1^2 e^{\lambda_1 t} & \lambda_2^2 e^{\lambda_2 t} & \lambda_3^2 e^{\lambda_3 t} \end{vmatrix}$$

$$= e^{(\lambda_1+\lambda_2+\lambda_3)t} \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix}$$

$$= e^{(\lambda_1+\lambda_2+\lambda_3)t} (\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2).$$

6) $\lambda^8 - 1 = 0 \Rightarrow (\lambda^4-1)(\lambda^4+1) = 0$

$\Rightarrow (\lambda-1)(\lambda+1)(\lambda^2+1)(\lambda^4+1) = 0$

$\Rightarrow \lambda_1 = \pm 1$, $\lambda_2 = \pm i$, $\lambda = e^{i\frac{\frac{\pi}{4}+2k\pi}{4}}$ $k = 0, 1, 2, 3$

$\lambda = -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$ $\quad = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

P5

$$\Rightarrow y_c = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$
$$+ e^{\frac{\sqrt{2}}{2}t}\left(c_5 \cos\left(\frac{\sqrt{2}}{2}t\right) + c_6 \sin\left(\frac{\sqrt{2}}{2}t\right)\right)$$
$$+ e^{-\frac{\sqrt{2}}{2}t}\left(c_7 \cos\left(\frac{\sqrt{2}}{2}t\right) + c_8 \sin\left(\frac{\sqrt{2}}{2}t\right)\right).$$

7) $Y = \mathcal{L}\{y\}$

$$s^3 Y - s^2 \cdot 1 - s(-1) + \frac{3}{2} - 3(s^2 Y - s \cdot 1 + \cancel{1})$$
$$+ 2(sY - 1) = \frac{1}{s^2} + \frac{1}{s-1}$$

$$\Rightarrow Y(s) = \frac{1}{s^3 - 3s^2 + 2s}\left(\frac{1}{s^2} + \frac{1}{s-1}\right)$$
$$+ \frac{s^2 - s - \frac{3}{2} - 3s + 3 + 2}{s^3 - 3s^2 + 2s}$$

$$= \frac{1}{s^3(s-1)(s-2)} + \frac{1}{s(s-1)^2(s-2)}$$
$$+ \frac{s^2 - 4s + \frac{7}{2}}{s(s-1)(s-2)} \quad \square$$

(P.6)

$$\frac{1}{s^3(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1} + \frac{E}{s-2}$$

$$A = \frac{7}{8},\quad B = \frac{3}{4},\quad C = \frac{1}{2},\quad D = -1,\quad E = \frac{1}{8}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s-1)(s-2)}\right\} = \frac{7}{8}\cdot 1 + \frac{3}{4}t + \frac{1}{4}t^2 - e^t + \frac{1}{8}e^{2t}.$$

$$\frac{1}{s(s-1)^2(s-2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$A = -1/2$
$B = \frac{1}{2}$,
$C = 0$
$D = -1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)^2(s-2)}\right\} = -\frac{1}{2} + \frac{1}{2}e^{2t} - te^t.$$

$$\frac{s^2 - 4s + 7/2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$A = \frac{7}{4}$, $B = -\frac{1}{2}$,
$C = -\frac{1}{4}$

$$\mathcal{L}^{-1}\left\{\frac{s^2 - 4s + 7/2}{s(s-1)(s-2)}\right\} = \frac{7}{4} - \frac{1}{2}e^t - \frac{1}{4}e^{2t}.$$

$$y(t) = \left(\frac{7}{8} + \frac{3}{4}t + \frac{1}{4}t^2 - e^t + \frac{1}{8}e^{2t}\right) + \left(-\frac{1}{2} + \frac{1}{2}e^{2t} - te^t\right)$$
$$+ \left(\frac{7}{4} - \frac{1}{2}e^t - \frac{1}{4}e^{2t}\right) \quad \square$$

8). $\lambda^4+2\lambda^3+2\lambda^2=0 \Rightarrow \lambda^2(\lambda^2+2\lambda+2)=0$

$\lambda=0\ (m=2)$, $\lambda=-1\pm i$

$y_c = c_1 + c_2 t + e^{-t}(c_3 \cos t + c_4 \sin t)$.

$Y_p = Ae^t + (Bt+C)e^{-t} + te^{-t}(D\cos t + E\sin t)$.

9). $y''' - y'' + y' - y = 0 \Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$

$\Rightarrow (\lambda^2+1)(\lambda-1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = \pm i$

$\Rightarrow y_1 = e^t,\ y_2 = \cos t,\ y_3 = \sin t$

$Y_p = u_1 e^t + u_2 \cos t + u_3 \sin t$

$$W = \begin{vmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{vmatrix} = e^t \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix}$$

$$= e^t \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\sin t & \cos t \\ 2 & 0 & 0 \end{vmatrix} = 2e^t.$$

$$W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1 \qquad W_2 = \begin{vmatrix} e^t & 0 & \sin t \\ e^t & 0 & \cos t \\ e^t & 1 & -\sin t \end{vmatrix}$$

$$W_3 = \begin{vmatrix} e^t & \cos t & 0 \\ e^t & -\sin t & 0 \\ e^t & -\cos t & 1 \end{vmatrix} = -e^t(\cos t + \sin t) \qquad = e^t(\sin t - \cos t)$$

$$\begin{cases} u_1' = \dfrac{W_1}{W} g = \dfrac{1}{2} e^{-t} g(t), \quad u_1 = \dfrac{1}{2}\displaystyle\int e^{-t} g(t)\,dt; \\ u_2' = \dfrac{W_2}{W} g = \dfrac{1}{2}(\sin t - \cos t) g(t), \quad u_2 = \dfrac{1}{2}\displaystyle\int (-\cos t + \sin t) g(t)\,dt \\ u_3' = \dfrac{W_3}{W} g = -\dfrac{1}{2}(\sin t + \cos t) g(t), \quad u_3 = -\dfrac{1}{2}\displaystyle\int (\sin t + \cos t) g(t)\,dt. \end{cases}$$

10). $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 5 a_n x^n = 0$$

$$\boxed{a_0 = 1, \quad a_1 = -1}$$

$$\sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + 5 a_n\right] x^n$$

$$+ (2a_2 + 5a_0) = 0$$

$$\boxed{a_2 = -\frac{5}{2} a_0 = -5/2}$$

$$(n+2)(n+1) a_{n+2} + (n+5) a_n = 0 \qquad n \geq 1$$

$$\boxed{a_{n+2} = -\frac{n+5}{(n+2)(n+1)} a_n, \qquad n \geq 1}$$

11). $y=\sum_{n=0}^{\infty} a_n x^n$, $y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$, $y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$\boxed{y_0 = 1,\ y'(0) = a_1 = 3}$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}(n+1)n\,a_{n+1}x^n$$

$$-\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+1)n\,a_{n+1} - n a_n + 4a_{n-2}\right]x^n$$

$$+\ 2a_2 + 6a_3 x + 2a_2 x - a_1 x = 0$$

$\Rightarrow$ $\underline{a_2 = 0}$ $\quad$ $\underline{a_3 = \frac{1}{6}a_1}$

$$\underline{(n+2)(n+1)a_{n+2} + (n+1)n\,a_{n+1} - n a_n + 4a_{n-2} = 0}$$

P10

12). $\mathcal{L}\{e^{2t}\sinh(3t)\} = \dfrac{3}{(s-2)^2-9}$

$\mathcal{L}\{u_1(t)\, t^n\} = e^{-s}\mathcal{L}\{(t+1)^n\}$

$= e^{-s}\left(\dfrac{n!}{s^{n+1}} + \binom{n}{1}\dfrac{(n-1)!}{s^n} + \cdots + \dfrac{1}{s}\right)$

$\mathcal{L}\{t^2 e^t \cos t\} = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+1}\right)$

$= \dfrac{d}{ds}\left(\dfrac{s^2+1-2s^2}{(s^2+1)^2}\right)$

$= \dfrac{d}{ds}\left(\dfrac{1-s^2}{(s^2+1)^2}\right)$

$= \dfrac{-2s(s^2+1)^2-(1-s^2)4(s^2+1)s}{(s^2+1)^4}$ □

13) . $\dfrac{2s+5}{s^2+2s+17} = \dfrac{2(s+1)+3}{(s+1)^2+4^2}$, $\mathcal{L}^{-1} = 2e^{-t}\cos(4t) + \frac{3}{4}e^{-t}\sin(4$

. $\dfrac{8s^2-4s+12}{(s-1)(s+2)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{B}{s+2} + \dfrac{Cs+D}{s^2+1}$

$A = \frac{8}{3}$, $B = -\frac{52}{15}$, $C = \frac{1}{5}(12-8) = 4/5$

$D = \left(\frac{16}{3} + \frac{52}{15} - 12\right)/2$

$$\rightsquigarrow \mathcal{L}^{-1} = \frac{8}{3}e^{t} - \frac{52}{15}e^{-2t} + \frac{4}{5}\cos t + D\sin t$$

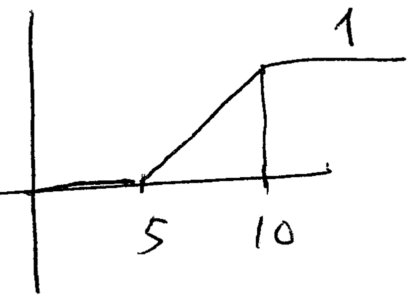
14). $$s^2Y - s + \cancel{1} + sY - \cancel{1} + \frac{5}{4}Y = \frac{1}{s} + \frac{e^{-\pi s}\,s}{s^2+1}$$

$$\Rightarrow Y = \frac{s}{s^2+s+\frac{5}{4}} + \frac{1}{s\left(s^2+s+\frac{5}{4}\right)} + \frac{e^{-\pi s}\,s}{\left(s^2+s+\frac{5}{4}\right)(s^2+1)}$$

$\Rightarrow$ Apply inverse Laplace transformation.

15). $$g(t) = \frac{1}{5}\left[u_5(t)(t-5) - u_{10}(t)(t-10)\right]$$

$$\mathcal{L}\{g\} = \frac{1}{5}\left(\frac{e^{-5s}}{s^2} - \frac{e^{-10s}}{s^2}\right)$$



$$\Rightarrow (s^2+4)Y = \frac{1}{5}\left(\frac{e^{-5s}-e^{-10s}}{s^2}\right)$$

$$\Rightarrow Y = \frac{1}{5}\,\frac{(e^{-5s}-e^{-10s})}{s^2(s^2+4)}$$

$$y(t) = \frac{1}{20}u_5(t)\left[(t-5) - \frac{1}{2}\sin 2(t-5)\right] - \frac{1}{20}u_{10}(t)\left[(t-10) - \frac{1}{2}\sin 2(t-10)\right]$$