

On uniqueness of heat flow of harmonic maps

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Abstract

In this paper, we establish the uniqueness of heat flow of harmonic maps into (N, h) that have sufficiently small renormalized energies, provided that N is either a unit sphere S^{k-1} or a Riemannian homogeneous manifold. For such a class of solutions, we also establish the convexity property of the Dirichlet energy for $t \geq t_0 > 0$ and the unique limit property at time infinity. As a corollary, the uniqueness is shown for heat flow of harmonic maps into any compact Riemannian manifold N whose gradients belong to $L_t^q L_x^l$, for $q > 2$ and $l > n$ satisfying the Serrin's condition.

1 Introduction

It is well-known that for geometric nonlinear evolution equations with critical nonlinearity, the uniqueness and regularity of weak solutions is often a very challenging question. In this paper, we aim to address the issue of uniqueness for heat flow of harmonic maps in dimensions $n \geq 2$.

Let (M, g) be a n -dimensional compact Riemannian manifold possibly with $\partial M \neq \emptyset$ or complete Riemannian manifold with $\partial M = \emptyset$, and $(N, h) \subset \mathbb{R}^k$ be a compact Riemannian manifold without boundary. For $0 < T \leq +\infty$, the heat flow of harmonic maps for $u : M \times [0, T) \rightarrow N$ is:

$$\begin{cases} \partial_t u - \Delta_g u = A(u)(\nabla u, \nabla u) & \text{in } M \times (0, T) \\ u = u_0 & \text{on } \partial_p(M \times [0, T]) \end{cases} \quad (1.1)$$

where Δ_g is the Laplacian operator on (M, g) , $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of $N \subset \mathbb{R}^k$, $\partial_p(M \times [0, T]) = (M \times \{0\}) \cup (\partial M \times (0, T])$ denotes the parabolic boundary of $M \times [0, T]$, and $u_0 : M \rightarrow N$ is a given map.

The heat flow of harmonic maps has been extensively studied in the past several decades. Under certain geometric conditions on (N, h) , the existence of a unique, global smooth solution to (1.1) has been established by Eells-Sampson [13], Hamilton [18], and Hildebrandt-Kaul-Widman [22]. In general, the existence of a unique, global weak solution to (1.1) with finitely many singularities has been obtained by Struwe [35] and Chang [4] for $n = 2$; and the existence of partially regular,

global weak solutions to (1.1) has been established by Chen-Struwe [10] and Chen-Lin [8] for $n \geq 3$. Concerning the uniqueness for weak solutions to (1.1), Freire [17] first proved that in dimension $n = 2$, the uniqueness holds for weak solutions whose Dirichlet energy is monotone decreasing with respect to t (see L.Wang [42] and L. Z. Lin [27] for a new simple proof). For $n \geq 3$, there are non-uniqueness for weak solutions to (1.1), see the examples constructed by Coron [12] and Bethuel-Coron-Ghidaglia-Soyeur [2]. In fact, Coron [12] proved that for suitable initial data, there exist weak solutions to (1.1) that are different from those constructed by Chen-Struwe. Partially motivated by [12], Struwe [38] has raised the following question:

For $M = \mathbb{R}^n$, exhibit a class of functions within which (1.1) posses a unique solution. Certainly the class of functions satisfying the strong monotonicity formula

$$\Phi_{(\bar{x}, \bar{t})}(\rho) \leq \Phi_{(\bar{x}, \bar{t})}(r), \quad \forall \bar{x} \in \mathbb{R}^n, \bar{t} > 0, 0 < \rho \leq r \leq \sqrt{\bar{t}} \quad (1.2)$$

is a likely candidate. Here

$$\Phi_{(\bar{x}, \bar{t})}(\rho) = \rho^2 \int_{\mathbb{R}^n \times \{\bar{t} - \rho^2\}} |\nabla u|^2(x, t) G(x - \bar{x}, t - \bar{t}) dx$$

and

$$G(y, s) = \frac{1}{(4\pi|s|)^{\frac{n}{2}}} \exp\left(-\frac{|y|^2}{4|s|}\right), \quad y \in \mathbb{R}^n, s < 0$$

is the fundamental solution to the backward heat equation on \mathbb{R}^n .

To the best of authors' knowledge, this question is largely open. In this paper, we will obtain some uniqueness results for the heat flow of harmonic maps (1.1), that may shed light on the validity of the Struwe's conjecture as above.

To state the result, we recall some notations. For $1 < p, q < \infty$ and $0 < T \leq +\infty$, define the Sobolev spaces

$$W^{1,p}(M, N) := \left\{ v \in W^{1,p}(M, \mathbb{R}^k) \mid v(x) \in N, \text{ a.e. } x \in M \right\},$$

$$H^1(M \times [0, T], N) := \left\{ v \in W^{1,2}(M \times [0, T], \mathbb{R}^k) \mid v(x, t) \in N, \text{ a.e. } (x, t) \in M \times [0, T] \right\},$$

the $L_t^q L_x^p$ -space

$$L_t^q L_x^p(M \times [0, T], \mathbb{R}^k) := \left\{ f : M \times [0, T] \rightarrow \mathbb{R}^k \mid f \in L^q([0, T], L^p(M)) \right\},$$

and the parabolic Morrey space $\mathcal{M}_R^{l,\lambda}$: For any $1 \leq p < +\infty$, $0 \leq \lambda \leq n+2$, $0 < R < +\infty$, and any open set $U \subset M \times \mathbb{R}$,

$$\mathcal{M}_R^{p,\lambda}(U) = \left\{ f \in L_{\text{loc}}^p(U) : \|f\|_{\mathcal{M}_R^{p,\lambda}(U)} < +\infty \right\},$$

where

$$\|f\|_{\mathcal{M}_R^{p,\lambda}(U)} = \left(\sup_{(x,t) \in U} \sup_{0 < r < \min\{R, d_g(x, \partial U), \sqrt{\bar{t}}\}} r^{\lambda-n-2} \int_{P_r(x,t) \cap U} |f|^p \right)^{\frac{1}{p}}.$$

Here d_g denotes the distance function on M induced by g , and

$$P_r(x, t) = B_r(x) \times (t - r^2, t], \text{ with } B_r(x) = \{y \in M : d_g(y, x) \leq r\}$$

for $(x, t) \in U \times (0, T]$ and $0 < r < d_g(x, \partial U) := \inf_{\{y \in \partial U\}} d_g(x, y)$ ¹.

For $u_0 \in W^{1,2}(M, N)$ and $0 < T \leq +\infty$, $u \in H^1(M \times [0, T], N)$ is a weak solution to (1.1) if u satisfies (1.1)₁ in the sense of distribution and (1.1)₂ in the sense of trace.

Now we state our main theorem on the uniqueness of weak solutions to (1.1).

Theorem 1.1 *For $n \geq 2$ and $1 < p \leq 2$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$ and $R_0 = R_0(M, g, \epsilon_0) > 0$ such that if*

(i) (M, g) is a n -dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;

(ii) $(N, h) \subset \mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and

(iii) $u_1, u_2 \in H^1(M \times [0, T], N)$ are two weak solutions to (1.1), with $u_1 = u_2 = u_0$ on $\partial_p(M \times [0, T])$ for some $u_0 \in W^{1,2}(M, N)$, that satisfy

$$\max_{i=1,2} \left[\|\nabla u_i\|_{\mathcal{M}_{R_0}^{p,p}(M \times (0, T))} + \|\partial_t u_i\|_{\mathcal{M}_{R_0}^{p,2p}(M \times (0, T))} \right] \leq \epsilon_0, \quad (1.3)$$

then $u_1 \equiv u_2$ on $M \times [0, T]$.

Recall that N is a Riemannian homogeneous manifold if there exists a finite dimensional Lie group \mathcal{G} ($\dim \mathcal{G} = s < +\infty$) that acts transitively on N by isometries.

There are two main ideas of proof of Theorem 1.1:

(i) an ϵ_0 -regularity theorem (Theorem 2.1 in §2 below) for the heat flow of harmonic maps that satisfy the smallness condition (1.3), which is new and improves the regularity theorem previously obtained by Chen-Li-Lin [9], Feldman [15], and Chen-Wang [11]. It shall have its own interest. In particular, we have that for $i = 1, 2$, $u_i \in C^\infty(M \times (0, T])$ and satisfies the gradient estimate:

$$\max_{i=1,2} |\nabla u_i|(x, t) \leq C\epsilon_0 \left(\frac{1}{R_0} + \frac{1}{d_g(x, \partial M)} + \frac{1}{\sqrt{t}} \right), \quad \forall (x, t) \in M \times (0, T], \quad (1.4)$$

(ii) applications of (1.4), the Hardy's inequality, and a generalized Gronwall's inequality type argument.

Now a few remarks are in order.

Remark 1.2 i) Due to technical difficulties, it is unknown whether the ϵ -regularity Theorem 2.1 (with $p = 2$) holds for a general Riemannian manifold N . Hence it is an open question that Theorem 1.1, Theorem 1.3, and Corollary 1.4 remain to hold for a general manifold N .

ii) Note that by Hölder inequality, the Morrey norm $\mathcal{E}(p) := (\|\nabla u\|_{\mathcal{M}^{p,p}(\cdot)} + \|\partial_t u\|_{\mathcal{M}^{p,2p}(\cdot)})$ is monotone increasing for $1 < p \leq 2$. The bound of $\mathcal{E}(2)$ for solutions u to (1.1) holds if u satisfies

(a) a local energy inequality (assume $M = \mathbb{R}^n$ for simplicity):

$$\int_{P_r(x,t)} |\partial_t u|^2 \leq \frac{C}{(R-r)^2} \int_{P_R(x,t)} |\nabla u|^2, \quad \forall (x, t) \in \mathbb{R}_+^{n+1}, \quad 0 < r \leq \frac{R}{2}, \quad R \leq \sqrt{t}, \quad (1.5)$$

¹if $\partial U = \emptyset$, then we set $d_g(x, \partial U) = \infty$.

(b) a local energy monotonicity inequality:

$$r^{-n} \int_{P_r(x,t)} |\nabla u|^2 \leq CR^{-n} \int_{P_R(x,t)} |\nabla u|^2, \quad \forall (x,t) \in \mathbb{R}_+^{n+1}, \quad 0 < r \leq \frac{R}{2}, \quad R \leq \sqrt{t}. \quad (1.6)$$

Both properties hold if u is either a smooth solution (see [35] and [10]) or a stationary solution to (1.1) (see [9], [15], and [11]). Therefore, under (1.5) and (1.6), the condition (1.3) is satisfied, provided that there exists $R_0 > 0$ such that there holds

$$\sup \left\{ R_1^{-n} \int_{P_{R_1}(x,t)} |\nabla u|^2 \mid x \in \mathbb{R}^n, R_1 = \min\{R_0, \sqrt{t}\} \right\} \leq \epsilon_0^2. \quad (1.7)$$

Hence Theorem 1.1 implies that the uniqueness does hold for the class of functions that satisfy, in addition to (1.5) and (1.6), the smallness condition (1.7).

iii) For any compact or complete noncompact (M, g) without boundary, there exists $\epsilon_0 > 0$ such that if the initial data $u_0 : M \rightarrow N$ satisfies that for some $R_0 > 0$,

$$\sup \left\{ r^{2-n} \int_{B_r(x)} |\nabla u_0|^2 \mid x \in M, r \leq R_0 \right\} \leq \epsilon_0^2, \quad (1.8)$$

then as a consequence of the local well-posedness theorem by Wang [41], there exists $0 < T_0 (\approx R_0^2)$ and a solution $u \in C^\infty(M \times (0, T_0), N)$ to (1.1) that satisfies the condition (1.3).

Motivated by the proof of Theorem 1.1, we obtain the following convexity property on (1.1).

Theorem 1.3 *For $n \geq 2$, $1 < p \leq 2$, and $1 \leq T \leq \infty$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$, $R_0 = R_0(M, g, \epsilon_0) > 0$, and $0 < T_0 = T_0(\epsilon_0) < T$ such that if*

(i) *(M, g) is a n -dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;*

(ii) *$(N, h) \subset \mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and*

(iii) *$u \in H^1(M \times [0, T], N)$ is a weak solution to (1.1), with $u = u_0$ on $\partial_p(M \times [0, T])$ for some $u_0 \in W^{1,2}(M, N)$, that satisfies*

$$\|\nabla u\|_{\mathcal{M}_{R_0}^{p,p}(M \times (0, T))} + \|\partial_t u\|_{\mathcal{M}_{R_0}^{p,2p}(M \times (0, T))} \leq \epsilon_0, \quad (1.9)$$

then

(i) *the Dirichlet energy $E(u(t)) := \frac{1}{2} \int_M |\nabla u|^2$ is monotone decreasing for $t \geq T_0$; and*

(ii) *for any $t_2 \geq t_1 \geq T_0$,*

$$\int_M |\nabla(u(t_1) - u(t_2))|^2 \leq C \left[\int_M |\nabla u(t_1)|^2 - \int_M |\nabla u(t_2)|^2 \right]. \quad (1.10)$$

We would like to remark that the convexity property has been observed by Schoen [32] for the Dirichlet energy of harmonic maps into manifolds N with nonpositive sectional curvatures. In §5 appendix below, we will show that it also holds for harmonic maps with small renormalized energy, which yields a new proof of the uniqueness theorem by Struwe [37] and Moser [29].

A direct consequence of Theorem 1.3 is the following uniqueness of limit at $t = \infty$ for (1.1).

Corollary 1.4 For $n \geq 2$ and $1 < p \leq 2$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$, and $R_0 = R_0(M, g, \epsilon_0) > 0$ such that if

(i) (M, g) is a n -dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;

(ii) $(N, h) \subset \mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and

(iii) $u \in H^1(M \times [0, \infty), N)$ is a weak solution to (1.1), with $u = u_0$ on $\partial_p(M \times [0, \infty))$ for some $u_0 \in W^{1,2}(M, N)$, that satisfies the condition (1.9).

Then there exists a harmonic map $u_\infty \in C^\infty(M, N) \cap W^{1,2}(M, N)$, with $u_\infty = u_0$ on ∂M , such that

$$\lim_{t \uparrow \infty} \|u(t) - u_\infty\|_{W^{1,2}(M)} = 0, \quad (1.11)$$

and, for any compact subset $K \subset\subset M$ and $m \geq 1$,

$$\lim_{t \uparrow \infty} \|u(t) - u_\infty\|_{C^m(K)} = 0. \quad (1.12)$$

The uniqueness of limit at $t = \infty$ has been proved by Hartman [23] for the smooth solutions to (1.1) when N has nonpositive sectional curvatures. L. Simon in his celebrated work [34] has shown the unique limit at $t = \infty$ for smooth solutions to (1.1) into a target manifold (N, h) that is real analytic. Note that the solution u in Theorem 1.3 is allowed to be singular near the parabolic boundary $\partial_p(M \times (0, \infty))$, as the initial-boundary data u_0 is assumed to be in $W^{1,2}(M, N)$. Also, our proof of Theorem 1.3 depends only on the smallness condition (1.9) and the small energy regularity Theorem 2.1. During the preparation of this work, we have seen two very interesting articles by L. Wang [42] and L.Z. Lin [27], in which it is proved that Theorems 1.1, 1.3, and Corollary 1.4 hold for the Struwe's almost regular solution u to (1.1) in dimension $n = 2$ when the Dirichlet energy of u_0 is sufficiently small. We would like to point that since Struwe's solution u to (1.1) satisfies the energy inequality, the condition in [27] yields the global smallness:

$$\sup_{t \geq 0} E(u(t)) + \int_{M \times [0, t]} |\partial_t u|^2 \leq E(u_0) \leq \epsilon_0^2, \quad \forall t > 0,$$

which is stronger than (1.9) in dimension $n = 2$. There is also an interesting paper by Topping [39] that addressed the rigidity at $t = \infty$ of the heat flow of harmonic maps from S^2 to S^2 .

A natural class of weak solutions that satisfy the smallness condition (1.9) are the so-called Serrin's solutions. Recall that a weak solution $u \in H_{\text{loc}}^1(M \times [0, T], N)$ of (1.1) is called a Serrin's solution if, in addition, $\nabla u \in L_t^q L_x^l(M \times [0, T])$ for some $l \geq n$ and $q \geq 2$ satisfying

$$\frac{n}{l} + \frac{2}{q} = 1. \quad (1.13)$$

In §3 below, we will verify that if u is a Serrin's solution to (1.1) with $l > n$, and $u|_{\partial_p(M \times [0, T])} = u_0$ for a given $u_0 : M \rightarrow N$ with $\nabla u_0 \in L^r(M)$ for some $n < r < \infty$, then u satisfies (1.9) for some $1 < p_0 \leq 2$. We would also like to point out for such an initial and boundary data u_0 , the local existence of Serrin's solutions to (1.1) can be shown by the standard fixed point theory. In fact,

interested readers may verify that the argument by Fabes-Jones-Riviere [16] §4 can be adapted to achieve such an existence. Here we have the following uniqueness result for Serrin's solutions of the heat flow of harmonic maps into a general Riemannian manifold.

Theorem 1.5 *For $n \geq 2$, $0 < T \leq +\infty$, let (M, g) be either a compact or complete Riemannian manifold without boundary or a compact Riemannian manifold with boundary, and N is a compact Riemannian manifold without boundary. Let $u_1, u_2 \in H^1(M \times [0, T], N)$ be two weak solutions of (1.1), with $u_1 = u_2 = u_0$ on $\partial_p(M \times [0, T])$ for some $u_0 \in W^{1,2}(M, N)$, such that $\nabla u_1, \nabla u_2 \in L_t^q L_x^l(M \times [0, T])$ for some (l, q) satisfying (1.13) with $l > n, q > 2$. Then $u_1, u_2 \in C^\infty(M \times (0, T))$, and $u_1 \equiv u_2$ on $M \times [0, T]$.*

Remark 1.6 We conjecture that Theorem 1.5 remains to be true for the end point case $l = n, q = +\infty$. We would like to point out that Lin-Wang [26] have proved the uniqueness holds for two weak solutions u_1, u_2 to (1.1) with the same initial data, provided that $\nabla u_1, \nabla u_2 \in C([0, T], L^n(M))$. Wang [40] has proved that for any $n \geq 4$, any weak solution $u \in H^1(M \times [0, T], N)$ with $\nabla u \in L_t^\infty L_x^n(M \times [0, T])$ belongs to $C^\infty(M \times (0, T])$. However, since $\|\nabla u(t)\|_{L^n(M)}$ may lack continuity at the initial time $t = 0$, the issue of uniqueness for the end point case remains unsolved.

It turns out that we can extend the ideas in this paper to study the uniqueness issue of heat flow of biharmonic maps, which will be discussed in a forthcoming paper [21]. The paper is written as follows. In §2, we will provide an ϵ -regularity theorem on certain weak solutions to (1.1) for N either a unit sphere or a compact Riemannian homogeneous manifold without boundary. In §3, we will outline the proofs of Theorem 1.1, Theorem 1.3, and Corollary 1.4. In §4, we will discuss Serrin's solutions to (1.1) and sketch a proof of Theorem 1.5. In §5, we will provide a simple alternative proof to an improved version of the uniqueness theorem for harmonic maps with small energy, originally due to Struwe [37] ($n = 3$) and Moser [29] ($n \geq 4$).

2 ϵ -regularity Theorem

In this section, we will establish an ϵ -regularity theorem for the heat flow of harmonic maps (1.1), which plays a crucial role in the proof of our main theorems. This regularity theorem seems to be new, whose proof is elementary. It improves the regularity theorem previously obtained by Chen-Li-Lin [9], Feldman [15], Chen-Wang [11] (see also Moser [28] for some related works in dimensions $n \leq 4$). We believe that it shall have its own interests. We would also like to point out the relevant works on the regularity theorem on stationary harmonic maps by Hélein [19], Evans [14], Bethuel [1], Chang-Wang-Yang [7], and Riviere-Struwe [31]. Especially, the proof of the regularity theorem 2.1 below is motivated by that of [7].

From now on, we let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and denote

$$\delta\left((x, t), (y, s)\right) = \max\left\{|x - y|, \sqrt{|t - s|}\right\}, \quad (x, t), (y, s) \in \mathbb{R}^n \times \mathbb{R}$$

as the parabolic distance on $\mathbb{R}^n \times \mathbb{R}$.

Theorem 2.1 *Assume that N is either a unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary. For $1 < p \leq 2$ and $0 < T < +\infty$, there exists $\epsilon_p > 0$ such that if $u \in H^1(\Omega \times [0, T], N)$ is a weak solution to (1.1)₁ and satisfies that, for $z_0 = (x_0, t_0) \in \Omega \times (0, T]$ and $0 < R_0 \leq \frac{1}{2} \min\{d(x_0, \partial\Omega), \sqrt{t_0}\}$,*

$$\|\nabla u\|_{\mathcal{M}_{R_0}^{p,p}(P_{R_0}(z_0))} + \|\partial_t u\|_{\mathcal{M}_{R_0}^{p,2p}(P_{R_0}(z_0))} \leq \epsilon_p. \quad (2.1)$$

Then $u \in C^\infty(P_{\frac{R_0}{4}}(z_0), N)$, and

$$|\nabla u|(x_0, t_0) \leq \frac{C\epsilon_p}{R_0}. \quad (2.2)$$

Remark 2.2 *It remains an open question whether Theorem 2.1 holds for N being any compact Riemannian manifold without boundary, under the condition (2.1) for $p = 2$. The interested readers can refer to Moser [30] for some related works in dimensions $n \leq 4$.*

The proof of Theorem 2.1 is based on the following lemma.

Lemma 2.3 *For any $1 < p \leq 2$, there exists $\epsilon_p > 0$ such that if $N = S^{k-1}$ or a compact Riemannian homogeneous manifold without boundary, and $u \in H^1(P_4, N)$ is a weak solution to (1.1) satisfying*

$$\sup_{(x,t) \in P_2, 0 < r \leq 2} r^{p-(n+2)} \int_{P_r(x,t)} (|\nabla u|^p + r^p |\partial_t u|^p) \leq \epsilon^p. \quad (2.3)$$

Then $u \in C^\infty(P_{\frac{1}{2}}, S^k)$ and satisfies

$$\|\nabla^k u\|_{C^0(P_{\frac{1}{2}})} \leq C(n, p, \epsilon, k), \quad \forall k \geq 1. \quad (2.4)$$

Proof. The crucial step to establish (2.4) is the following decay estimate:

Claim: There exists $\theta \in (0, \frac{1}{2})$ such that for any $z_0 = (x_0, t_0) \in P_1$ and $0 < r \leq 1$, it holds

$$\left(\frac{1}{(\theta r)^{n+2}} \int_{P_{\theta r}(z_0)} |u - u_{z_0, \theta r}|^q \right)^{\frac{1}{q}} \leq \frac{1}{2} \left(\frac{1}{r^{n+2}} \int_{P_r(z_0)} |u - u_{z_0, r}|^q \right)^{\frac{1}{q}}, \quad (2.5)$$

where $f_{z_0, r} = \frac{1}{|P_r(z_0)|} \int_{P_r(z_0)} f$ is the average of f over $P_r(z_0)$, and $q \geq n+2$ is given by $\frac{1}{q} = \frac{n+4}{2(n+2)} - \frac{1}{p}$.

It is standard that (2.5) yields (2.4). In fact, by iterations and the Campanato theory [3], (2.5) implies that there exists $\alpha \in (0, 1)$ such that $u \in C^\alpha(P_{\frac{3}{4}})$ and

$$\left[u \right]_{C^\alpha(P_{\frac{3}{4}})} \leq C(p, \epsilon_p).$$

The higher regularity and (2.4) then follows from the parabolic hole filling type argument and the bootstrap argument (see also [24]).

Now we return to the proof of (2.5). For any $z_0 = (x_0, t_0) \in P_1$ and $0 < r \leq 1$, since $v(y, s) = u(z_0 + (ry, r^2s)) : P_2 \rightarrow N$ satisfies (1.1), and the condition (2.3) yields that v satisfies

$$\sup_{(x,t) \in P_1, 0 < r \leq 1} r^{p-(n+2)} \int_{P_r(x,t)} |\nabla v|^p + r^p |\partial_t v|^p \leq \epsilon^p. \quad (2.6)$$

Thus it suffices to show (2.5) for $z_0 = (0, 0)$ and $r = 1$.

We divide the proof into two cases:

Case 1: $N = S^{k-1}$ is the unit sphere.

Step 1. Rewriting of (1.1). Since $|u| = 1$, we have $u^i u_\alpha^i = 0$. Also, it follows (1.1) that

$$(u^i u_\alpha^j - u^j u_\alpha^i)_\alpha = u^i \Delta u^j - u^j \Delta u^i = u^i u_t^j - u^j u_t^i.$$

Hence we have

$$\begin{aligned} u_t^i - \Delta u^i &= A^i(u)(\nabla u, \nabla u) = |\nabla u|^2 u^i \\ &= u_\alpha^j u_\alpha^j u^i - u_\alpha^j u^j u_\alpha^i = u_\alpha^j (u^i u_\alpha^j - u^j u_\alpha^i) \\ &= \left[(u^j - c^j)(u^i u_\alpha^j - u^j u_\alpha^i) \right]_\alpha - (u^j - c^j)(u^i u_t^j - u^j u_t^i), \end{aligned} \quad (2.7)$$

where $c^j \in \mathbb{R}$ is an arbitrary constant. For convenience, set

$$W^{ij} = u^i u_t^j - u^j u_t^i, \quad V_\alpha^{ij} = u^i u_\alpha^j - u^j u_\alpha^i, \quad 1 \leq i, j, \alpha \leq n.$$

Step 2. Construction of auxiliary functions. Let $\eta \in C_0^\infty(P_2)$ such that

$$0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } P_1, \quad \text{and } |\nabla \eta| \leq C.$$

Define $v, w : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{k+1}$ by

$$v_t^i - \Delta v^i = \left[\eta^2 (u^j - c^j) V_\alpha^{ij} \right]_\alpha, \quad v^i \Big|_{t=0} = 0 \quad (2.8)$$

and

$$w_t^i - \Delta w^i = -\eta^2 (u^j - c^j) W^{ij}, \quad w^i \Big|_{t=0} = 0. \quad (2.9)$$

Set $h = u - (v + w) : P_1 \rightarrow \mathbb{R}^{k+1}$. Then

$$h_t - \Delta h = 0 \text{ in } P_1. \quad (2.10)$$

Step 3. Estimation of v, w , and u . By the Duhamel's formula, we have that

$$\begin{aligned} v^i(x, t) &= \int_0^t \int_{\mathbb{R}^n} H(x - y, t - s) \left[\eta^2 (u^j - c^j) V_\alpha^{ij} \right]_\alpha (y, s) \\ &= \int_0^t \int_{\mathbb{R}^n} \nabla_x H(x - y, t - s) \eta^2 (u^j - c^j) V_\alpha^{ij} (y, s), \end{aligned}$$

where H denotes the heat kernel on \mathbb{R}^n . Then, as in [24], we have

$$|\nabla_x H|(x - y, t - s) \lesssim \delta((x, t), (y, s))^{-(n+1)}, \quad (x, t), (y, s) \in \mathbb{R}^{n+1},$$

where $\delta((x, t), (y, s))$ denotes the parabolic distance on \mathbb{R}^{n+1} . Hence

$$|v^i|(x, t) \lesssim I_1(\eta^2 |u^j - c^j| |V_\alpha^{ij}|)(x, t),$$

where

$$I_1(f)(x, t) := \int_{\mathbb{R}^{n+1}} \frac{f(y, s)}{\delta((x, t), (y, s))^{n+1}}, \quad \forall f \in L^1_{\text{loc}}(\mathbb{R}^{n+1}),$$

is the parabolic Riesz potential of order 1. By the Riesz potential estimate (see [24]), we have

$$\|v^i\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|\eta^2|u^j - c^j| \|V_\alpha^{ij}\|_{L^r(\mathbb{R}^{n+1})} \lesssim \|V_\alpha^{ij}\|_{L^p(P_2)} \|u^j - c^j\|_{L^{n+2}(P_2)}, \quad (2.11)$$

where $\frac{1}{p} = \frac{1}{r} - \frac{1}{n+2}$.

To estimate w , we multiply (2.9) by w^i and integrate over \mathbb{R}^n so that

$$\frac{d}{dt} \int_{\mathbb{R}^n} |w^i|^2 + 2 \int_{\mathbb{R}^n} |\nabla w^i|^2 \lesssim \int_{\mathbb{R}^n} \eta^2 |u^j - c^j| |W^{ij}| |w^i|.$$

Integrating this inequality with respect to $-2 \leq t \leq 0$, we arrive

$$\begin{aligned} \sup_{-2 \leq t \leq 0} \int_{\mathbb{R}^n} |w^i|^2 + 2 \int_{\mathbb{R}^n \times [-2, 0]} |\nabla w^i|^2 &\lesssim \int_{\mathbb{R}^n \times [-2, 0]} \eta^2 |u^j - c^j| |W^{ij}| |w^i| \\ &\lesssim \|w^i\|_{L^{\frac{2(n+2)}{n}}(P_2)} \|W^{ij}\|_{L^p(P_2)} \|u^j - c^j\|_{L^q(P_2)}, \end{aligned}$$

where $q \geq n + 2$ satisfies

$$\frac{n}{2(n+2)} + \frac{1}{p} + \frac{1}{q} = 1. \quad (2.12)$$

Recall the interpolation inequality:

$$\|w^i\|_{L^{\frac{2(n+2)}{n}}(P_2)}^2 \lesssim \left[\sup_{-2 \leq t \leq 0} \int_{B_2} |w^i|^2 + \int_{P_2} |\nabla w^i|^2 \right].$$

Hence we have

$$\|w^i\|_{L^p(P_2)} \lesssim \|w^i\|_{L^{\frac{2(n+2)}{n}}(P_2)} \lesssim \|W^{ij}\|_{L^p(P_2)} \|u^j - c^j\|_{L^q(P_2)}. \quad (2.13)$$

For h , by the standard theory on the heat equation we have that for any $0 < \theta < 1$,

$$\left(\frac{1}{\theta^{n+2}} \int_{P_\theta} |h - h_\theta|^p \right)^{\frac{1}{p}} \lesssim \theta \left(\int_{P_1} |h - h_1|^p \right)^{\frac{1}{p}}, \quad (2.14)$$

where h_r is the average of h over P_r .

Now we let $c^j = u_2^j$, the average of u^j over P_2 . Combining the estimates (2.11), (2.13), and (2.14) together yields

$$\begin{aligned} &\left(\frac{1}{\theta^{n+2}} \int_{P_\theta} |u - u_\theta|^p \right)^{\frac{1}{p}} \\ &\leq C \left[\theta + \theta^{-\frac{n}{2}} (\|V_\alpha^{ij}\|_{L^p(P_2)} + \|W^{ij}\|_{L^p(P_2)}) \right] \left(\frac{1}{2^{n+2}} \int_{P_2} |u - u_2|^q \right)^{\frac{1}{q}} \\ &\leq C [\theta + \theta^{-\frac{n}{2}} \epsilon_p] \left(\frac{1}{2^{n+2}} \int_{P_2} |u - u_2|^q \right)^{\frac{1}{q}}, \end{aligned} \quad (2.15)$$

where q is given by (2.12), and we have used in the last step the condition (2.3) that implies

$$\|V_\alpha^{ij}\|_{L^p(P_2)} + \|W^{ij}\|_{L^p(P_2)} \leq C \epsilon_p.$$

Recall also that the condition (2.3) and the Poincaré inequality imply that $u \in \text{BMO}(P_2)$, and

$$\left[u \right]_{\text{BMO}(P_2)} := \sup_{P_r(x,t) \subset P_2} \inf_{c \in \mathbb{R}^k} \left\{ \frac{1}{r^{n+2}} \int_{P_r(x,t)} |u - c| \right\} \leq C\epsilon. \quad (2.16)$$

Applying the celebrated John-Nirenberg's inequality [25], (2.16) implies

$$\left(\frac{1}{\theta^{n+2}} \int_{P_\theta} |u - u_\theta|^q \right)^{\frac{1}{q}} \leq C(\epsilon, n) \left(\frac{1}{\theta^{n+2}} \int_{P_\theta} |u - u_\theta|^p \right)^{\frac{1}{p}}. \quad (2.17)$$

Combining (2.15) and (2.17) yields

$$\begin{aligned} \left(\frac{1}{\theta^{n+2}} \int_{P_\theta} |u - u_\theta|^q \right)^{\frac{1}{q}} &\leq C[\theta + \theta^{-\frac{n}{2}}\epsilon] \left(\frac{1}{2^{n+2}} \int_{P_2} |u - u_2|^q \right)^{\frac{1}{q}} \\ &\leq \frac{1}{2} \left(\frac{1}{2^{n+2}} \int_{P_2} |u - u_2|^q \right)^{\frac{1}{q}}, \end{aligned} \quad (2.18)$$

provided that θ and ϵ_p are chosen to be sufficiently small. This implies (2.5).

Case 2: N is a compact Riemannian homogeneous manifold without boundary. We will indicate that (1.1) can be written into the same form as (2.7). In fact, according to Hélein [20], there exist s smooth tangential vector fields Y_1, \dots, Y_s and s smooth tangential killing vector fields X_1, \dots, X_s on N such that for any $y \in N$ and $V \in T_y N$, it holds

$$V = \sum_{i=1}^s \langle V, X_i(y) \rangle Y_i(y).$$

Thus, as in [11] Lemma 4.2, (1.1) is equivalent to

$$\begin{aligned} \partial_t u - \Delta u &= - \sum_{i=1}^s \langle \nabla u, X_i(u) \rangle \nabla(Y_i(u)) \\ &= - \sum_{i=1}^s \text{div}(\langle \nabla u, X_i(u) \rangle (Y_i(u) - c^i)) - \sum_{i=1}^s \langle \partial_t u, X_i(u) \rangle (Y_i(u) - c^i), \end{aligned} \quad (2.19)$$

where $c^i \in \mathbb{R}^k$ is an arbitrary constant. Here we have used the killing property of X_i that yields $\langle \nabla u, \nabla(X_i(u)) \rangle = 0$ in the derivation of (2.19). It is clear that the rest of proof follows exactly as Case 1. This completes the proof. \square

Proof of Theorem 2.1: It is easy to see that (2.1) implies

$$r^{p-(n+2)} \int_{P_r(z)} (|\nabla u|^p + r^p |\partial_t u|^p) \leq \epsilon_p^p, \quad \forall z = (x, t) \in P_{\frac{R_0}{2}}(z_0) \text{ and } 0 < r \leq \frac{R_0}{2}. \quad (2.20)$$

Hence Lemma 2.3 implies that $u \in C^\infty(P_{\frac{R_0}{4}}(z_0))$, and (2.4) holds. \square

3 Proof of Theorems 1.1, 1.3, and Corollary 1.4

In this section, we will provide proofs for our main theorems. The idea is based on Theorem 2.1, and application of both the Hardy inequality and a generalized Gronwall's inequality.

Proof of Theorem 1.1. For simplicity, we will focus on the case that (M, g) is a compact Riemannian manifold with boundary and remark on the other two cases at the end of the proof.

Assume $(M, g) = (\Omega, g_0)$, with g_0 the standard metric. By Theorem 2.1, we have that $u_i \in C^\infty(\Omega \times (0, T])$ for $i = 1, 2$, and

$$\max \left\{ |\nabla u_1|(x, t), |\nabla u_2|(x, t) \right\} \leq C\epsilon_0 \left(\frac{1}{R_0} + \frac{1}{d(x, \partial\Omega)} + \frac{1}{\sqrt{t}} \right), \quad \forall (x, t) \in \Omega \times (0, T]. \quad (3.1)$$

Set $w = u - v$. Then w satisfies

$$\begin{cases} w_t - \Delta w = A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v) & \text{in } \Omega \times (0, T] \\ w = 0 & \text{on } \partial_p(\Omega \times [0, T]). \end{cases} \quad (3.2)$$

Multiplying (3.2) by w and integrating over Ω yields

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |w|^2 + 2 \int_{\Omega} |\nabla w|^2 &\leq C \int_{\Omega} (|\nabla u_1|^2 + |\nabla u_2|^2) |w|^2 + \int_{\Omega} (|\nabla u_1| + |\nabla u_2|) |\nabla w| |w| \\ &\leq \int_{\Omega} |\nabla w|^2 + C \int_{\Omega} (|\nabla u_1|^2 + |\nabla u_2|^2) |w|^2. \end{aligned}$$

By (3.1), the Poincaré inequality, and the Hardy inequality

$$\int_{\Omega} \frac{|f(x)|^2}{d^2(x, \partial\Omega)} \lesssim \int_{\Omega} |\nabla f|^2, \quad \forall f \in H_0^1(\Omega),$$

we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |w|^2 + \int_{\Omega} |\nabla w|^2 &\leq \frac{C\epsilon_0^2}{R_0^2} \int_{\Omega} |w|^2 + C\epsilon_0^2 \int_{\Omega} \frac{|w(x)|^2}{d^2(x, \partial\Omega)} + \frac{C\epsilon_0^2}{t} \int_{\Omega} |w|^2 \\ &\leq C \left(\frac{\epsilon_0^2}{R_0^2} + \epsilon_0^2 \right) \int_{\Omega} |\nabla w|^2 + \frac{C\epsilon_0^2}{t} \int_{\Omega} |w|^2. \end{aligned}$$

If we choose $\epsilon_0 \leq (2C)^{-\frac{1}{2}}$ and $R_0 \geq \sqrt{2C}\epsilon_0$, then we have $C \left(\frac{\epsilon_0^2}{R_0^2} + \epsilon_0^2 \right) \leq 1$ so that

$$\frac{d}{dt} \int_{\Omega} |w|^2 \leq \frac{C\epsilon_0^2}{t} \int_{\Omega} |w|^2. \quad (3.3)$$

This yields

$$\begin{aligned} \frac{d}{dt} \left(t^{-\frac{1}{2}} \int_{\Omega} |w|^2 \right) &= t^{-\frac{1}{2}} \frac{d}{dt} \int_{\Omega} |w|^2 - \frac{1}{2} t^{-\frac{3}{2}} \int_{\Omega} |w|^2 \\ &\leq \left(C\epsilon_0^2 - \frac{1}{2} \right) t^{-\frac{3}{2}} \int_{\Omega} |w|^2 \leq 0. \end{aligned} \quad (3.4)$$

Thus we obtain that for any $0 < t \leq T$,

$$t^{-\frac{1}{2}} \int_{\Omega} |w(x, t)|^2 \leq \lim_{s \downarrow 0^+} s^{-\frac{1}{2}} \int_{\Omega} |w(x, s)|^2. \quad (3.5)$$

Since $w(\cdot, 0) = 0$, we have

$$w(x, s) = \int_0^s w_{\tau}(x, \tau) d\tau, \text{ a.e. } x \in \Omega$$

so that by the Hölder inequality,

$$s^{-\frac{1}{2}} \int_{\Omega} |w(x, s)|^2 \leq s^{\frac{1}{2}} \int_0^s \int_{\Omega} |w_{\tau}|^2(x, \tau) dx d\tau \leq C s^{\frac{1}{2}} \rightarrow 0, \text{ as } s \downarrow 0^+.$$

Thus we conclude that $w \equiv 0$ in $\Omega \times [0, T]$.

When (M, g) is either compact or complete non-compact with $\partial M = \emptyset$, observe that we can substitute $d(x, \partial M) = \infty$ into the above proof and obtain the same result without applying the Hardy's inequality. This completes the proof. \square

Proof of Theorem 1.3. For simplicity, we only consider the difficult case (M, g) is compact with boundary. First by Theorem 2.1, we have that $u \in C^{\infty}(M \times (0, T))$ and

$$|\nabla u|(x, t) \leq C\epsilon_0 \left(\frac{1}{R_0} + \frac{1}{d(x, \partial M)} + \frac{1}{\sqrt{t}} \right), \quad \forall (x, t) \in M \times (0, T). \quad (3.6)$$

First we need two claims.

Claim 1. There exists $T_0 > 0$ such that $\int_M |u_t(x, t)|^2$ is monotone nonincreasing for $t \geq T_0$:

$$\int_M |\partial_t u|^2(t_2) + C \int_{M \times [t_1, t_2]} |\nabla \partial_t u|^2 \leq \int_M |\partial_t u|^2(t_1), \quad T_0 \leq t_1 \leq t_2 < T. \quad (3.7)$$

To show it, we introduce the finite quotient for u in the t -variable. For sufficiently small $h > 0$, set

$$u^h(x, t) = \frac{u(x, t+h) - u(x, t)}{h}, \quad (x, t) \in M \times (0, T-h).$$

Since $u^h = 0$ on ∂M , we see that $u^h \in L^2([0, T-h], H_0^1(M))$, $\partial_t u^h \in L^2([0, T-h], L^2(M))$, and

$$\lim_{h \downarrow 0^+} \left\| u^h - u_t \right\|_{L^2(M \times [0, T-h])} = 0.$$

Since u satisfies (1.1), we have

$$u_t^h - \Delta u^h = \frac{1}{h} [A(u(t+h))(\nabla u(t+h), \nabla u(t+h)) - A(u(t))(\nabla u(t), \nabla u(t))]. \quad (3.8)$$

Multiplying (3.8) by u^h , integrating over M , and applying Hölder's inequality and (3.6), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_M |u^h|^2 + \int_M |\nabla u^h|^2 \\
& \leq C \int_M |u^h|^2 (|\nabla u(t+h)|^2 + |\nabla u(t)|^2) + |u^h| (|\nabla u(t+h)| + |\nabla u(t)|) |\nabla u^h| \\
& \leq \frac{1}{2} \int_M |\nabla u^h|^2 + C \int_M |u^h|^2 (|\nabla u(t+h)|^2 + |\nabla u(t)|^2) \\
& \leq \frac{1}{2} \int_M |\nabla u^h|^2 + C\epsilon_0^2 \int_M \left(\frac{|u^h|^2}{R_0^2} + \frac{|u^h|^2}{d^2(x, \partial M)} + \frac{|u^h|^2}{t} \right) \\
& \leq \frac{1}{2} \int_M |\nabla u^h|^2 + C\epsilon_0^2 \int_M \left(\frac{|u^h|^2}{R_0^2} + \frac{|u^h|^2}{d^2(x, \partial M)} + \frac{|u^h|^2}{T_0} \right) \\
& \leq \frac{1}{2} \int_M |\nabla u^h|^2 + C\epsilon_0 \int_M |\nabla u^h|^2 \leq \frac{3}{4} \int_M |\nabla u^h|^2,
\end{aligned}$$

where we have used both Poincaré inequality and Hardy inequality in the last step, and we have chosen $R_0 \geq \sqrt{\epsilon_0}$, $T_0 \geq \epsilon_0$, and $C\epsilon_0 \leq \frac{1}{4}$. Integrating this inequality from $T_0 \leq t_1 \leq t_2$ yields

$$\int_M |u^h|^2(t_2) + C \int_{M \times [t_1, t_2]} |\nabla u^h|^2 \leq \int_M |u^h|^2(t_1). \quad (3.9)$$

Sending h to zero in (3.9) yields (3.7).

Next we have

Claim 2. *There exists $T_0 > 0$ such that $E(u(t))$ is monotone decreasing for $t \geq T_0$:*

$$\int_{M \times [t_1, t_2]} |\partial_t u|^2 + E(u(t_2)) \leq E(u(t_1)), \quad T_0 \leq t_1 \leq t_2 < T. \quad (3.10)$$

For $\delta > 0$, let $\phi_\delta \in C_0^\infty(M)$ be a test function such that

$$0 \leq \phi_\delta \leq 1, \quad \phi_\delta(x) = 1 \text{ for } d(x, \partial M) \geq \delta, \quad |\nabla \phi_\delta| \leq C\delta^{-1}.$$

Since $u \in C^\infty(M \times (0, T))$, multiplying (1.1) by $\partial_t u \phi_\delta^2$ and integrating over $M \times [t_1, t_2]$, we obtain the following local energy inequality:

$$\int_{M \times [t_1, t_2]} |\partial_t u|^2 \phi_\delta^2 + \frac{1}{2} \int_M |\nabla u(t_2)|^2 \phi_\delta^2 \leq \frac{1}{2} \int_M |\nabla u(t_1)|^2 \phi_\delta^2 + 2 \int_{M \times [t_1, t_2]} \nabla u \cdot \partial_t u \phi_\delta \nabla \phi_\delta. \quad (3.11)$$

It is clear that (3.10) follows from (3.11), if we can show

$$\lim_{\delta \rightarrow 0} \int_{M \times [t_1, t_2]} \nabla u \cdot \partial_t u \phi_\delta \nabla \phi_\delta = 0. \quad (3.12)$$

To see (3.12), observe that (3.7) implies $\partial_t u(t) \in H_0^1(M)$ for $t \in [t_1, t_2]$ so that

$$\begin{aligned}
\int_{M \times [t_1, t_2]} |\partial_t u|^2 |\nabla \phi_\delta|^2 & \lesssim \delta^{-2} \int_{t_1}^{t_2} \int_{\{x \in M: d(x, \partial M) \leq \delta\}} |\partial_t u|^2 \\
& \lesssim \int_{t_1}^{t_2} \int_{\{x \in M: d(x, \partial M) \leq \delta\}} |\nabla \partial_t u|^2 \rightarrow 0, \text{ as } \delta \rightarrow 0.
\end{aligned}$$

It is clear that by Hölder inequality, (3.12) follows from this. Thus (3.10) holds.

Choose $T_0 > 0$ such that both claims hold. Then by (1.1) we can estimate

$$\begin{aligned}
& \int_M |\nabla u(t_1)|^2 - \int_M |\nabla u(t_2)|^2 - \int_M |\nabla(u(t_1) - u(t_2))|^2 \\
&= 2 \int_M \nabla u(t_2) \cdot \nabla(u(t_1) - u(t_2)) \\
&= 2 \int_M A(u(t_2))(\nabla u(t_2), u(t_2)) \cdot (u(t_1) - u(t_2)) + 2 \int_M u_t(t_2) \cdot (u(t_2) - u(t_1)) \\
&= I + II.
\end{aligned} \tag{3.13}$$

We first estimate I . Recall that for $y \in N$, let $P^\perp(y) : \mathbb{R}^k \rightarrow (T_y N)^\perp$ denote the orthogonal projection map from \mathbb{R}^k to the normal space of N at y . Since N is compact, a simple geometric argument implies that there exists $C > 0$ depending on N such that

$$|P^\perp(y)(z - y)| \leq C|z - y|^2, \quad \forall z \in N. \tag{3.14}$$

Thus

$$\begin{aligned}
|I| &\lesssim \int_M |\nabla u(t_2)|^2 |u(t_1) - u(t_2)|^2 \\
&\leq C\epsilon_0^2 \int_M \left(\frac{1}{R_0^2} + \frac{1}{T_0} + \frac{1}{d^2(x, \partial M)} \right) |u(t_1) - u(t_2)|^2 \\
&\lesssim C\epsilon_0 \int_M |\nabla(u(t_1) - u(t_2))|^2,
\end{aligned}$$

where we have used both Poincaré inequality and Hardy inequality in the last step.

By (3.7), we have

$$\int_M |\partial_t u(t_2)|^2 \leq \frac{1}{t_2 - t_1} \int_{M \times [t_1, t_2]} |\partial_t u|^2.$$

This, combined with Hölder inequality and (3.10), implies

$$\begin{aligned}
|II| &\leq \|\partial_t u(t_2)\|_{L^2(M)} \|u(t_1) - u(t_2)\|_{L^2(M)} \\
&\leq \sqrt{t_2 - t_1} \|\partial_t u(t_2)\|_{L^2(M)} \|\partial_t u\|_{L^2(M \times [t_1, t_2])} \\
&\leq \int_{M \times [t_1, t_2]} |\partial_t u|^2 \\
&\leq \frac{1}{2} \left[\int_M |\nabla u(t_1)|^2 - \int_M |\nabla u(t_2)|^2 \right].
\end{aligned}$$

Putting the estimates of I, II into (3.13) yields (1.10) so that the conclusions of Theorem 1.3 hold.

The proof is now complete. \square

Proof of Corollary 1.4. It follows from Theorem 1.3 that $E(u(t))$ is monotone decreasing for $T_0 \leq t < +\infty$. Hence

$$\lim_{t \rightarrow \infty} E(u(t)) = c < +\infty.$$

Let $\{t_i\}$ be any monotone increasing sequence such that $\lim_{i \rightarrow \infty} t_i = +\infty$. Then (1.10) implies that for any $j \geq 1$,

$$\int_M |\nabla(u(t_{i+j}) - u(t_i))|^2 \leq C \left[\int_M |\nabla u(t_i)|^2 - \int_M |\nabla u(t_{i+j})|^2 \right] \rightarrow 0$$

as $i \rightarrow \infty$. This implies that there exists a map $u_\infty \in H^1(M, N)$, with $u_\infty = u_0$ on ∂M , such that

$$\lim_{t \rightarrow \infty} \int_M |\nabla(u(t) - u_\infty)|^2 = 0.$$

Since (3.10) implies there exists $t_i \uparrow \infty$ such that

$$\lim_{i \rightarrow \infty} \|\partial_t u(t_i)\|_{L^2(M)} = 0,$$

we see that u_∞ is a weak harmonic map. Moreover, by the gradient estimate (3.6), we have that for any compact set $K \subset\subset M$ and $m \geq 1$, one has that for t sufficiently large,

$$\|\nabla^m u(t)\|_{C^0(K)} \leq C(\epsilon_0, m, K),$$

which clearly implies that $u(t) \rightarrow u_\infty$ in $C^m(K)$, as $t \rightarrow \infty$. This completes the proof. \square

4 Serrin's solutions and proof of Theorem 1.5

In this section, we will indicate that any Serrin's solution to (1.1), under a suitable initial-boundary data u_0 , satisfies the condition (2.1) in Theorem 2.1. Hence the ϵ -regularity and the uniqueness for Serrin's solutions follows from Theorem 2.1 and Theorem 1.1, when N is either a unit sphere or a compact Riemannian homogeneous manifold without boundary. However, it turns out that with a different argument on the ϵ -regularity, the uniqueness holds for Serrin's solution to (1.1) into an arbitrary Riemannian manifold N . We will also sketch it in this section.

We start with the following proposition.

Proposition 4.1 *For $n \geq 2$, $0 < T < +\infty$, and an arbitrary compact Riemannian manifold $N \subset \mathbb{R}^k$ without boundary, suppose $u \in H^1(M \times [0, T], N)$ is a weak solution to (1.1), with the initial and boundary value $u_0 : M \rightarrow N$ satisfying $\nabla u_0 \in L^r(M)$ for some $n < r < +\infty$, such that $\nabla u \in L_t^q L_x^l(M \times [0, T])$ for some (l, q) satisfying (1.13) with $l > n, q > 2$. Then*

(i) $\partial_t u \in L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(M \times [0, T])$; and

(ii) for any $\epsilon > 0$, there exists $R = R(u, \epsilon) > 0$ such that for any $1 < s < \min\{\frac{l}{2}, \frac{q}{2}\}$,

$$\sup \left\{ r^{s-(n+2)} \int_{P_r(x,t) \cap (M \times [0, T])} (|\nabla u|^s + r^s |\partial_t u|^s) \mid (x, t) \in M \times [0, T], 0 < r \leq R \right\} \leq \epsilon^s. \quad (4.1)$$

Proof. We consider the case that (M, g) is complete and noncompact, and leave the discussion of the other cases to interested readers. For simplicity, assume $(M, g) = (\mathbb{R}^n, g_0)$.

Let H be the heat kernel in \mathbb{R}^n . Then by the Duhamel's formula, we have

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}^n} H(x - y, t) u_0(y) + \int_0^t \int_{\mathbb{R}^n} H(x - y, t - s) A(u)(\nabla u, \nabla u)(y, s) \\ &= u_1(x, t) + u_2(x, t). \end{aligned} \quad (4.2)$$

It is easy to see that

$$\nabla^2 u_1(x, t) = \int_{\mathbb{R}^n} \nabla_x H(x - y, t) \nabla_y u_0(y).$$

Hence by the standard integral estimates (see [16] page 234), we have

$$\left\| \nabla^2 u_1 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n \times [0, T])} \leq CT^{\frac{1}{2} - \frac{n}{2r}} \left\| \nabla u_0 \right\|_{L^r(\mathbb{R}^n)}. \quad (4.3)$$

For u_2 , since

$$\nabla^2 u_2(x, t) = \int_0^t \int_{\mathbb{R}^n} \nabla_x^2 H(x - y, t - s) A(u)(\nabla u, \nabla u)(y, s),$$

we can apply the Calderon-Zygmund's $L_t^s L_x^{s'}$ -theory to obtain

$$\left\| \nabla^2 u_2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n \times [0, T])} \leq C \left\| |\nabla u|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n \times [0, T])} \leq C \left\| \nabla u \right\|_{L_t^q L_x^l(\mathbb{R}^n \times [0, T])}^2. \quad (4.4)$$

Substituting (4.3) and (4.4) into (4.2) yields $\nabla^2 u \in L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n \times [0, T])$. This, combined with the equation (1.1), then implies (i).

To see (ii), observe that by Hölder inequality, we have that for any $1 < s < \min\{\frac{l}{2}, \frac{q}{2}\}$,

$$\left(r^{s-(n+2)} \int_{P_r(x,t) \cap (M \times [0, T])} |\nabla u|^s \right)^{\frac{1}{s}} \leq \left\| \nabla u \right\|_{L_t^q L_x^l(P_r(x,t) \cap (M \times [0, T]))},$$

and

$$\left(r^{2s-(n+2)} \int_{P_r(x,t) \cap (M \times [0, T])} |\partial_t u|^s \right)^{\frac{1}{s}} \leq \left\| \partial_t u \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(P_r(x,t) \cap (M \times [0, T]))}.$$

These two inequalities clearly imply (4.1). \square

Now we give a proof of ϵ -regularity of Serrin's solutions to (1.1) for any Riemannian manifold N . For $x \in \mathbb{R}^n$, $t > 0$, and $r > 0$, let $B_r(x) \subset \mathbb{R}^n$ be the ball with center x and radius r , and $P_r(x, t) = B_r(x) \times [t - r^2, t]$. When $(x, t) = (0, 0)$, denote $P_r = P_r(0, 0)$.

Lemma 4.2 *There is an $\epsilon_0 > 0$ such that if $u \in H^1(P_1, N)$, with $\nabla u \in L_t^q L_x^l(P_1)$ for some $l \geq n$ and $q \geq 2$ satisfying (1.13), is a weak solution to (1.1) and*

$$\left\| \nabla u \right\|_{L_t^q L_x^l(P_1)} \leq \epsilon_0, \quad (4.5)$$

then $u \in C^\infty(P_{\frac{1}{2}}, N)$ and

$$\left\| \nabla u \right\|_{C^m(P_{\frac{1}{2}})} \leq C(m, p, q) \left\| \nabla u \right\|_{L^2(P_1)}, \quad (4.6)$$

for any positive integer m .

We need the following inequality, due to Serrin ([33] Lemma 1).

Lemma 4.3 *For any open set $U \subset \mathbb{R}^n$ and any open interval $I \subset \mathbb{R}$, let $f, g, h \in L_t^2 H_x^1(U \times I)$ and $f \in L_t^q L_x^l(U \times I)$ with $l \geq n$ and $q \geq 2$ satisfying (1.13). Then we have*

$$\int_{U \times I} |f| |g| |\nabla h| \leq C \|\nabla h\|_{L^2(U \times I)} \|g\|_{L_t^2 H_x^1(U \times I)}^{\frac{n}{l}} \left\{ \int_I \|f\|_{L^l(U)}^q \|g\|_{L^2(U)}^2 dt \right\}^{\frac{1}{q}}, \quad (4.7)$$

where $C > 0$ depends only on n .

Proof of Lemma 4.2. For any $(x, t) \in P_{\frac{1}{2}}$ and $0 < r \leq \frac{1}{2}$, by (4.5) we have

$$\|\nabla u\|_{L_t^q L_x^l(P_r(x, t))} \leq \epsilon_0. \quad (4.8)$$

Let $v : P_r(x, t) \rightarrow \mathbb{R}^k$ solve

$$\begin{cases} v_t - \Delta v = 0, & \text{in } P_r(x, t) \\ v = u, & \text{on } \partial_p P_r(x, t). \end{cases} \quad (4.9)$$

Denote $w = u - v$. Multiplying (1.1) and (4.9) by w , subtracting the resulting equations and integrating over $P_r(x, t)$, we obtain

$$\begin{aligned} & \sup_{t-r^2 \leq s \leq t} \int_{B_r(x)} |w|^2(\cdot, s) + 2 \int_{P_r(x, t)} |\nabla w|^2 \leq C \int_{P_r(x, t)} |\nabla u|^2 |w| \\ & \leq C \|\nabla u\|_{L^2(P_r(x, t))} \|\nabla w\|_{L^2(P_r(x, t))}^{\frac{n}{l}} \left\{ \int_{t-r^2}^t \|\nabla u\|_{L^l(B_r(x))}^q \|w\|_{L^2(B_r(x))}^2 \right\}^{\frac{1}{q}}, \end{aligned} \quad (4.10)$$

where we have used (4.7) and Poincaré inequality in last step. Since

$$\frac{1}{2} + \frac{1}{q} + \frac{n}{2l} = 1,$$

by Young inequality we obtain

$$\begin{aligned} & \|\nabla u\|_{L^2(P_r(x, t))} \|\nabla w\|_{L^2(P_r(x, t))}^{\frac{n}{l}} \left\{ \int_{t-r^2}^t \|\nabla u\|_{L^l(B_r(x))}^q \|w\|_{L^2(B_r(x))}^2 \right\}^{\frac{1}{q}} \\ & \leq \epsilon_0 \|\nabla w\|_{L^2(P_r(x, t))}^2 + \epsilon_0 \|\nabla u\|_{L^2(P_r(x, t))}^2 + \frac{C}{\epsilon_0^{\frac{q}{2}}} \int_{t-r^2}^t \|\nabla u\|_{L^l(B_r(x))}^q \|w\|_{L^2(B_r(x))}^2. \end{aligned} \quad (4.11)$$

Combining (4.10) with (4.11) and applying (4.8), we have

$$\begin{aligned} & \sup_{t-r^2 \leq s \leq t} \int_{B_r(x)} |w|^2 + \int_{P_r(x, t)} |\nabla w|^2 \\ & \leq \epsilon_0 \|\nabla w\|_{L^2(P_r(x, t))}^2 + \left(\frac{C}{\epsilon_0^{\frac{q}{2}}} \int_{t-r^2}^t \|\nabla u\|_{L^l(B_r(x))}^q \right) \left(\sup_{t-r^2 \leq s \leq t} \int_{B_r(x)} |w|^2 \right) \\ & \leq \epsilon_0 \|\nabla w\|_{L^2(P_r(x, t))}^2 + C \epsilon_0^{\frac{q}{2}} \left(\sup_{t-r^2 \leq s \leq t} \int_{B_r(x)} |w|^2 \right). \end{aligned} \quad (4.12)$$

Choose $\epsilon_0 > 0$ so that $C\epsilon_0^{\frac{q}{2}} \leq \frac{1}{2}$, we obtain

$$\int_{P_r(x,t)} |\nabla w|^2 \leq 2\epsilon_0 \|\nabla u\|_{L^2(P_r(x,t))}^2. \quad (4.13)$$

On the other hand, by the standard estimate on the heat equation, we obtain that for any $0 < \theta < 1$,

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla v|^2 \leq C\theta^2 r^{-n} \int_{P_r(x,t)} |\nabla u|^2. \quad (4.14)$$

(4.13) and (4.14) imply that

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla u|^2 \leq C(\theta^2 + \theta^{-n}\epsilon_0) r^{-n} \int_{P_r(x,t)} |\nabla u|^2. \quad (4.15)$$

For any $0 < \alpha < 1$, first choose $\theta_0 > 0$ such that $C\theta_0^2 \leq \frac{1}{2}\theta_0^{2\alpha}$ and then choose

$$\epsilon_0 \leq \min \left\{ \frac{\theta_0^{2\alpha+n}}{2C}, \left(\frac{1}{2C} \right)^{\frac{2}{q}} \right\},$$

we obtain that for any $(x, t) \in P_{\frac{1}{2}}$ and $0 < r \leq \frac{1}{2}$, it holds

$$(\theta_0 r)^{-n} \int_{P_{\theta_0 r}(x,t)} |\nabla u|^2 \leq \theta_0^{2\alpha} r^{-n} \int_{P_{r_0}(x,t)} |\nabla u|^2. \quad (4.16)$$

Iterating (4.16), we obtain for any positive integer l ,

$$(\theta_0^l r)^{-n} \int_{P_{\theta_0^l r}(x,t)} |\nabla u|^2 \leq \theta_0^{2l\alpha} r^{-n} \int_{P_r(x,t)} |\nabla u|^2. \quad (4.17)$$

It is standard that (4.17) implies

$$r^{-n} \int_{P_r(x,t)} |\nabla u|^2 \leq Cr^{2\alpha} \int_{P_1} |\nabla u|^2, \quad \forall (x, t) \in P_{\frac{1}{2}}, \quad 0 < r \leq \frac{1}{2}. \quad (4.18)$$

By (4.18), we have that $\nabla u \in \mathcal{M}^{2,2-2\alpha}(P_1)$ for any $0 < \alpha < 1$. Now we can apply the regularity theorem by Huang-Wang (see [24] Theorem 1.5) to conclude that $u \in C^\infty(P_{\frac{1}{2}})$ and (4.6) holds.

This completes the proof. \square

By suitable scaling, we have the following estimate on the possible blow-up rate of $\|\nabla u(t)\|_{L^\infty}$ as t tends to zero.

Lemma 4.4 *For $T > 0$ and a compact or complete manifold (M, g) without boundary, suppose that $u \in H^1(M \times [0, T], N)$ is a weak solution to (1.1), with $\nabla u \in L_t^q L_x^l(M \times [0, T])$ for some $l > n$ and $q > 2$ satisfying (1.13), then $u \in C^\infty(M \times (0, T], N)$ and there exists $t_0 > 0$ such that*

$$\sup_{0 < t \leq t_0} \sqrt{t} \left\| \nabla u(t) \right\|_{L^\infty(M)} \leq C \left\| \nabla u \right\|_{L_t^q L_x^l(M \times [0, t_0])}. \quad (4.19)$$

In particular,

$$\lim_{t \downarrow 0^+} \sqrt{t} \left\| \nabla u(t) \right\|_{L^\infty(M)} = 0. \quad (4.20)$$

Proof. For simplicity, we assume that $(M, g) = (\mathbb{R}^n, g_0)$. Since $\nabla u \in L_t^q L_x^l(\mathbb{R}^n \times [0, T])$ for some $l > n$ and $q > 2$ satisfying (1.13), we have that for $\epsilon_0 > 0$ given by Lemma 4.2, there exists $\delta_0 > 0$ such that

$$\sup_{(x_0, t_0) \in \mathbb{R}^n \times [0, T]} \left\| \nabla u \right\|_{L_t^q L_x^l(P_{\delta_0}(x_0, t_0) \cap \mathbb{R}_+^{n+1})} \leq \epsilon_0,$$

In particular, for any $0 < \tau \leq \delta_0$ and any $x_0 \in \mathbb{R}^n$, we have

$$\left\| \nabla u \right\|_{L_t^q L_x^l(B_\tau(x_0) \times [0, \tau^2])} \leq \epsilon_0. \quad (4.21)$$

Define $v(y, s) = u(x_0 + \tau y, \tau^2 + \tau^2 s)$ for $(y, s) \in P_1(0, 0)$. Then v solves (1.1) on $P_1(0, 0)$, and satisfies

$$\left\| \nabla v \right\|_{L_t^q L_x^l(P_1(0, 0))} \leq \epsilon_0.$$

Hence Lemma 4.2 implies

$$\|\nabla v\|_{L^\infty(P_{\frac{1}{2}}(0, 0))} \leq C \|\nabla v\|_{L^2(P_1(0, 0))}. \quad (4.22)$$

After scaling, (4.22) implies that $u \in C^\infty(P_{\frac{\tau}{2}}(x_0, \tau^2))$ and

$$\tau \left\| \nabla u \right\|_{L^\infty(P_{\frac{\tau}{2}}(x, \tau^2))} \leq C \tau^{-\frac{n}{2}} \left\| \nabla u \right\|_{L^2(P_\tau(x, \tau^2))}. \quad (4.23)$$

By Hölder's inequality and (1.13), we have

$$\tau^{-\frac{n}{2}} \left\| \nabla u \right\|_{L^2(P_\tau(x_0, \tau^2))} \leq \left\| \nabla u \right\|_{L_t^q L_x^l(P_\tau(x_0, \tau^2))}, \quad (4.24)$$

Putting (4.24) together with (4.23), we obtain

$$\tau \left\| \nabla u(\tau^2) \right\|_{L^\infty(\mathbb{R}^n)} \leq C \left\| \nabla u \right\|_{L_t^q L_x^l(\mathbb{R}^n \times [0, \tau^2])}. \quad (4.25)$$

After sending $\tau \rightarrow 0$, (4.25) clearly implies (4.20). It is not hard to see that (4.19) also follows. This completes the proof. \square

The next lemma handles the case that (M, g) is a compact Riemannian manifold with boundary.

Lemma 4.5 *For $T > 0$ and a compact manifold (M, g) with boundary, suppose that $u \in H^1(M \times [0, T], N)$ is a weak solution to (1.1), with $\nabla u \in L_t^q L_x^l(M \times [0, T])$ for some $l > n$ and $q > 2$ satisfying (1.13), then $u \in C^\infty(M \times (0, T], N)$. Moreover, for any sufficiently small $\epsilon_0 > 0$ there exists $T_0 > 0$ depending only on ϵ_0 and u such that*

$$|\nabla u(x, t)| \leq C \epsilon_0 \left(\frac{1}{d(x, \partial M)} + \frac{1}{\sqrt{t}} \right), \forall (x, t) \in M \times (0, T_0]. \quad (4.26)$$

Proof. Let $\epsilon_0 > 0$ be given by Lemma 4.2. Since $\nabla u \in L_t^q L_x^l(M \times [0, T])$ with $l > n, q > 2$, there exists $T_0 > 0$ such that

$$\left\| \nabla u \right\|_{L_t^q L_x^l(M \times [0, T_0])} \leq \epsilon_0.$$

For any $x_0 \in M$ and $0 < t_0 \leq T_0$, we divide the proof into two cases:

(i) $d(x_0, \partial M) > \sqrt{t_0}$; and

(ii) $d(x_0, \partial M) \leq \sqrt{t_0}$.

For (i), since $P_{\sqrt{t_0}}(x_0, t_0) \subset M \times (0, T_0]$, we have $\|\nabla u\|_{L_t^q L_x^l(P_{\sqrt{t_0}}(x_0, t_0))} \leq \epsilon_0$. As in Lemma 4.4, we conclude that $u \in C^\infty(P_{\frac{\sqrt{t_0}}{2}}(x_0, t_0))$ and

$$|\nabla u|(x_0, t_0) \leq \frac{C\epsilon_0}{\sqrt{t_0}}.$$

For (ii), set $r_0 = \min\{d(x_0, \partial M), \sqrt{t_0}\}$. Then $P_{r_0}(x_0, t_0) \subset M \times (0, T_0]$ and $\|\nabla u\|_{L_t^q L_x^l(P_{r_0}(x_0, t_0))} \leq \epsilon_0$. Hence we can conclude that $u \in C^\infty(P_{\frac{r_0}{2}}(x_0, t_0))$ and

$$|\nabla u|(x_0, t_0) \leq \frac{C\epsilon_0}{r_0} \leq C\epsilon_0 \left(\frac{1}{d(x_0, \partial M)} + \frac{1}{\sqrt{t_0}} \right).$$

Thus (4.26) holds. This completes the proof. \square

Proof of Theorem 1.5. It follows from Lemma 4.4 and Lemma 4.5 that there exists $T_0 > 0$ such that both the condition (1.3) of Theorem 1.1 and the gradient estimate (2.2) of Theorem 2.1 hold on $M \times [0, T_0]$. Thus we can apply the same proof of Theorem 1.1 to obtain that $u = v$ on $M \times [0, T_0]$. One can repeat the same argument to show that $u = v$ on $M \times [T_0, T]$. \square

5 Appendix

As a byproduct of the proof of Theorem 1.1, we will prove a convexity property on certain weak harmonic maps that yields an alternative, simple proof of the uniqueness theorem on the Dirichlet problem of weak harmonic maps, due to Struwe [37] for $n = 3$ and Moser [29] for $n \geq 4$. Furthermore, the statement of the uniqueness theorem for N either a unit sphere or a compact Riemannian homogeneous manifold without boundary is an improvement of that by [37] and [29].

To do it, we introduce the Morrey spaces in \mathbb{R}^n : For $1 \leq l < +\infty$, $0 < \lambda \leq n$, $0 < R < +\infty$, and $U \subset \mathbb{R}^n$, $f \in \mathcal{M}_R^{l,\lambda}(U)$ iff $f \in L_{\text{loc}}^l(U)$ satisfies

$$\|f\|_{\mathcal{M}_R^{l,\lambda}(U)}^l := \sup_{x \in U} \sup_{0 < r \leq \min\{R, d(x, \partial U)\}} \left\{ r^{\lambda-n} \int_{B_r(x)} |f|^l \right\} < +\infty.$$

For any bounded smooth domain $\Omega \subset \mathbb{R}^n$, we have

Theorem 5.1 *For $n \geq 2$, $\delta \in (0, 1)$, and $1 < p \leq 2$, there exist $\epsilon_p = \epsilon(p, \delta) > 0$ and $R_p = R(p, \delta) > 0$ such that if $u \in H^1(\Omega, N)$ is a weak harmonic map satisfying either*

- (i) $\|\nabla u\|_{\mathcal{M}_{R_p}^{2,2}(\Omega)} \leq \epsilon_2$, when N is an arbitrary compact Riemannian manifold without boundary, or
- (ii) $\|\nabla u\|_{\mathcal{M}_{R_p}^{p,p}(\Omega)} \leq \epsilon_p$ for some $1 < p \leq 2$, when $N = S^{k-1}$ or a compact Riemannian homogeneous manifold without boundary. Then

$$\int_{\Omega} |\nabla v|^2 \geq \int_{\Omega} |\nabla u|^2 + (1 - \delta) \int_{\Omega} |\nabla(v - u)|^2 \tag{5.1}$$

holds for any $v \in H^1(\Omega, N)$ with $v = u$ on $\partial\Omega$.

Proof. First, as observed by [37] and [29], for an arbitrary manifold N under the condition (i), the small energy regularity theorem on stationary harmonic maps by Bethuel [1] holds. While, for $N = S^{k-1}$ under the condition (ii), the small energy regularity theorem on weak harmonic maps by Moser [30] is applicable. Thus we have $u \in C^\infty(\Omega, N)$ and, for any $x \in \Omega$, it holds

$$|\nabla u|(x) \leq C\epsilon_p \left(\frac{1}{d(x, \partial\Omega)} + \frac{1}{R_p} \right). \quad (5.2)$$

Here $p = 2$ for an arbitrary N .

Now multiplying the equation of u by $(u - v)$ and integrating over Ω , we obtain

$$\int_{\Omega} \nabla u \cdot \nabla(u - v) = \int_{\Omega} \langle A(u)(\nabla u, \nabla u), u - v \rangle. \quad (5.3)$$

This, combined with (5.2), Poincaré inequality, and the Hardy's inequality, implies

$$\begin{aligned} \left| \int_{\Omega} \langle A(u)(\nabla u, \nabla u), u - v \rangle \right| &\leq C \int_{\Omega} |\nabla u|^2 |u - v|^2 \\ &\leq C\epsilon_p^2 \int_{\Omega} \frac{|u - v|^2}{R_p^2} + \frac{|u - v|^2}{d(x, \partial\Omega)^2} \\ &\leq C\epsilon_p^2 \left(1 + \frac{1}{R_p^2}\right) \int_{\Omega} |\nabla(u - v)|^2 \\ &\leq \frac{\delta}{2} \int_{\Omega} |\nabla(u - v)|^2 \end{aligned} \quad (5.4)$$

provided that we have chosen $\epsilon_p \leq \left(\frac{\delta}{4C}\right)^{\frac{1}{2}}$ and R_p such $\frac{C\epsilon_p^2}{R_p^2} \leq \frac{\delta}{4}$. Thus, by (5.3) and (5.4) we obtain

$$\begin{aligned} &\int_{\Omega} |\nabla v|^2 - \int_{\Omega} |\nabla u|^2 - \int_{\Omega} |\nabla(v - u)|^2 \\ &= 2 \int_{\Omega} \nabla u \cdot \nabla(v - u) = -2 \int_{\Omega} \langle A(u)(\nabla u, \nabla u), u - v \rangle \\ &\geq -\delta \int_{\Omega} |\nabla(v - u)|^2. \end{aligned}$$

This clearly implies (5.1), provided that $\epsilon > 0$ is sufficiently small. This proof is complete. \square

Corollary 5.2 *For $n \geq 2$ and $1 < p \leq 2$, there exist $\epsilon_p > 0$ and $R_p > 0$ such that if $u_1, u_2 \in H^1(\Omega, N)$ are two weak harmonic maps satisfying either*

(i) $\max_{i=1}^2 \|\nabla u_i\|_{\mathcal{M}_{R_2}^{2,2}(\Omega)} \leq \epsilon_2$, when N is an arbitrary compact Riemannian manifold without boundary, or

(ii) $\max_{i=1}^2 \|\nabla u_i\|_{\mathcal{M}_{R_p}^{p,p}(\Omega)} \leq \epsilon_p$ for some $1 < p \leq 2$, when $N = S^{k-1}$ or a compact Riemannian homogeneous manifold without boundary.

Then $u_1 \equiv u_2$ in Ω , provided that $u_1 - u_2 \in W_0^{1,2}(\Omega, \mathbb{R}^k)$.

Proof. Choosing $\delta = \frac{1}{2}$, we can apply Theorem 5.1 to u_1 and u_2 by choosing sufficiently small $\epsilon_p > 0$ and $R_p > 0$. Thus Theorem 5.1 implies

$$\int_{\Omega} |\nabla u_2|^2 \geq \int_{\Omega} |\nabla u_1|^2 + \frac{1}{2} \int_{\Omega} |\nabla(u_2 - u_1)|^2,$$

and

$$\int_{\Omega} |\nabla u_1|^2 \geq \int_{\Omega} |\nabla u_2|^2 + \frac{1}{2} \int_{\Omega} |\nabla(u_1 - u_2)|^2.$$

Adding these two inequalities together yields

$$\int_{\Omega} |\nabla(u_1 - u_2)|^2 = 0.$$

Therefore, $u_1 = u_2$ in Ω . □

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