Weak solution to compressible hydrodynamic flow of liquid crystals in 1-D

Shijin Ding* Changyou Wang† Huanyao Wen*

Abstract

We consider the equation modeling the compressible hydrodynamic flow of liquid crystals in one dimension. As mentioned in [12], the weak solution was obtained with the initial density having a positive lower bound and H^1-integrable. In this paper, we get a weak solution with initial density nonnegative and L^γ-integrable.

Key Words: Liquid crystal, compressible hydrodynamic flow, global weak solution.

1 Introduction

In this paper, we consider the one dimensional initial-boundary value problem:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + P_x = \mu u_{xx} - \lambda(|n_x|^2)_x, \\ n_t + u n_x = \theta(n_{xx} + |n_x|^2 n), \end{cases} \tag{1.1}$$

for $(x,t) \in (0,1) \times (0,+\infty)$, with the initial condition:

$$(\rho,\ \rho u,\ n)\big|_{t=0} = (\rho_0,\ m_0,\ n_0) \text{ in } [0,1], \tag{1.2}$$

*School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China
†Corresponding author. Email: cywang@ms.uky.edu. Department of Mathematics, University of Kentucky, Lexington, KY 40506

where $n_0 : [0,1] \to S^2$ and the boundary condition:

$$(u,\ n_x)\big|_{\partial I} = (0,\ 0),\ t > 0, \tag{1.3}$$

where $\rho \geq 0$ denotes the density function, u denotes the velocity field, n denotes the optical axis vector of the liquid crystal that is a unit vector (i.e., $|n| = 1$), $\mu > 0, \lambda > 0, \theta > 0$ are viscosity of the fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. $P = a\rho^\gamma$, for some constants $\gamma > 1$ and $a > 0$, is the pressure function.

The hydrodynamic flow of compressible (or incompressible) liquid crystals was first derived by Ericksen [1] and Leslie [2] in 1960's. However, its rigorous mathematical analysis was not taken place until 1990's, when Lin [3] and Lin-Liu [4, 5, 6] made some very important progress towards the existence of global weak solutions and partial regularity of the incompressible hydrodynamic flow equation of liquid crystals.

When the Ossen-Frank energy configuration functional reduces to the Dirichlet energy functional, the hydrodynamic flow equation of liquid crystals in $\Omega \subset \mathbf{R}^d$ can be written as follows (see Lin [3]):

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mu\Delta u - \lambda \mathrm{div}(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} I_d), \\ n_t + u \cdot \nabla n = \theta(\Delta n + |\nabla n|^2 n), \end{cases} \tag{$\star$}$$

where $u \otimes u = (u^i u^j)_{1 \leq i,j \leq d}$, and $\nabla n \odot \nabla n = (n_{x_i} \cdot n_{x_j})_{1 \leq i,j \leq d}$.

Observe that for $d = 1$, the system ($\star$) reduces to (1.1). When the density function ρ is a positive constant, then ($\star$) becomes the hydrodynamic flow equation of incompressible liquid crystals (i.e., div $u = 0$). In a series of papers, Lin [3] and Lin-Liu [4, 5, 6] addressed the existence and partial regularity theory of suitable weak solution to the incompressible hydrodynamic flow of liquid crystals of variable length. More precisely, they considered the approximate equation of incompressible hydrodynamic flow of liquid crystals: (i.e., $\rho = 1$, and $|\nabla n|^2$ in $(\star)_3$ is replaced by $\dfrac{(1 - |n|^2)n}{\epsilon^2}$), and proved [4], among many other results, the local existence of classical solutions and the global existence of weak solutions in dimension two and three. For any fixed $\epsilon > 0$, they also showed the existence and uniqueness of global classical solution either in dimension two or dimension three when the fluid viscosity μ is sufficiently large; in [6], Lin and Liu extended the classical theorem by Caffarelli-Kohn-Nirenberg [7] on the Navier-Stokes equation that asserts the one dimensional parabolic Hausdorff measure of the singular set of any *suitable* weak solution is zero. See also [8, 9] for relevant results. For the incompressible case $\rho = 1$ and div $u = 0$, it remains to be an open problem that for $\epsilon \downarrow 0$ whether a sequence of solutions (u_ϵ, n_ϵ) to the approximate equation converges to a solution of the original equation $(\star)$. It is also a very interesting question to ask whether there exists a global weak solution to the incompressible hydrodynamic flow equation $(\star)$ similar to the Leray-Hopf type solutions in the context of Naiver-Stokes equation. We answer this question in [10] for $d = 2$. Moreover, for $\rho \geq 0$, div u=0, and $d = 2$ or 3, we also get an unique local strong solution to $(\star)$ in [11]. Particularly, if initial density has a positive lower bound and d=2, we get an unique global strong solution with small initial data.

When dealing with the compressible hydrodynamic flow equation $(\star)$, in [12], we get the existence and uniqueness of global classical, strong solution and the existence of weak solution for $0 < c_0^{-1} \leq \rho_0 \leq c_0$ and the existence of strong solution for $\rho_0 \geq 0$ when the dimension $d = 1$. As mentioned in the Remark 1.1 of [12], the global weak solution may be obtained with improved initial conditions. We answer this question in the paper.

We remark that when the optical axis n is a constant unit vector, (1.1) reduces to the Navier-Stokes equation for compressible isentropic flow. Let's review some previous results about the weak solution to the compressible isentropic Navier-Stokes equation. In 1998, P.L. Lions in [13] got a weak solution with $\gamma \geq \frac{9}{5}$ for dimension $d = 3$. In 2001, S. Jiang and P. Zhang in [14] obtained a weak solution to the Cauchy problem with spherically symmetric initial data for any $\gamma > 1$, and d=2 or 3. For general initial data, and d=3, E. Feireisl et al in [15] improved the condition of γ given by P.L. Lions in [13], i.e. $\gamma > \frac{3}{2}$.

Our ideas mainly come from [15]. While the proof in this paper is simpler, since we exploit the one-dimensional feature, and use integrals instead of commutators. Moreover, we only need $\gamma > 1$.

Since the constant a and μ, λ, θ in (1.1) don't play any role in the analysis, we assume henceforth that

$$\mu = \lambda = \theta = a = 1.$$

Notations:

(1) $I = (0,1)$, $\partial I = \{0,1\}$, $Q_T = I \times (0,T)$ for $T > 0$.

(2) $\widehat{f}$: $\widehat{f}(x) = f(x)$ for $x \in I$, and $\widehat{f}(x) = 0$ for $x \in \mathbb{R} \setminus I$.

(3) $\eta_\sigma(\cdot) = \frac{1}{\sigma^d}\eta(\frac{\cdot}{\sigma})$, where η is a standard mollifier.

(4) $C([0,T]; X - \omega)$: $f \in C([0,T]; X - \omega) \Leftrightarrow \forall g \in X'$, $\langle f(t), g\rangle_{X\times X'} \in C([0,T])$.

Definition 1.1 *For any $T > 0$, we call (ρ, u, n) a global weak solution of (1.1)-(1.3), if*

$$(1) \qquad \rho \in L^\infty(0,T; L^\gamma(I)),\ \rho u^2 \in L^\infty(0,T; L^1(I)),\ \rho \ge 0 \ a.e.\ in\ Q_T,$$

$$u \in L^2(0,T; H_0^1(I)),\ n \in L^\infty(0,T; H^1(I)) \cap L^2(0,T; H^2(I)),$$

$$n_t \in L^2(0,T; L^2(I)),\ |n| = 1 \ in\ Q_T,$$

$$(\rho, \rho u)(x,0) = (\rho_0(x), m_0(x)),\ weakly\ in\ L^\gamma(I) \times L^{\frac{2\gamma}{\gamma+1}}(I),$$

$$n(x,0) = n_0(x)\ in\ \overline{I},\ (n_x(0,t), n_x(1,t)) = 0\ a.e.\ in\ (0,T).$$

(2) $(1.1)_1$, $(1.1)_2$ *are satisfied in* $\mathscr{D}'(Q_T)$, *and* $(1.1)_3$ *holds a.e. in* Q_T.

$$(3) \qquad \int_I \Big(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + |n_x|^2\Big)(t) + \int_{Q_T} \big(u_x^2 + 2\left|n_{xx} + |n_x|^2 n\right|^2\big)$$

$$\le \int_I \Big(\frac{m_0^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} + |(n_0)_x|^2\Big),\ for\ a.e.\ t \in (0,T). \qquad (1.4)$$

Our main result is as follows

Theorem 1.1 *If $\rho_0 \ge 0$, $\rho_0 \in L^\gamma(I)$, $\frac{m_0}{\sqrt{\rho_0}} \in L^2(I)$, and $n_0 \in H^1(I, S^2)$, then there exists a global weak solution $(\rho, u, n) : [0,1] \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R} \times S^2$ to (1.1)-(1.3) such that for any $T > 0$,*

$$\int_{Q_T} \rho^{2\gamma} \le c(E_0, T),$$

where

$$E_0 := \int_I \Big(\frac{m_0^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} + |(n_0)_x|^2\Big)$$

denotes the total energy of the initial data.

The rest of the paper is organized as follows. In section 2, we present some useful Lemmas which will be needed. In section 3, we derive some a priori estimates for the approximate solutions of (1.1)-(1.3), and prove the existence of weak solution.

2 Preliminaries

Lemma 2.1 *([16]). Assume $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow E$. Then the following embedding are compact:*

$$(i) \quad \left\{\varphi : \varphi \in L^q(0,T;X), \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y)\right\} \hookrightarrow\hookrightarrow L^q(0,T;E), \quad \text{if} \quad 1 \le q \le \infty;$$

$$(ii) \quad \left\{\varphi : \varphi \in L^\infty(0,T;X), \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y)\right\} \hookrightarrow\hookrightarrow C([0,T];E), \quad \text{if} \quad 1 < r \le \infty.$$

Lemma 2.2 *([15]). Let $\rho \in L^2(\Omega \times (0,T))$, $u \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^d))$ solve*

$$\rho_t + div(\rho u) = 0, \ in \ \mathscr{D}'(\Omega \times (0,T)). \tag{2.1}$$

Then

$$\partial_t b(\rho) + div[b(\rho)u] + [b'(\rho)\rho - b(\rho)]divu = 0, \ in \ \mathscr{D}'(\Omega \times (0,T)), \tag{2.2}$$

for any $b \in C^1(\mathbb{R})$ such that $b'(z) \equiv 0$ for all $z \in \mathbb{R}$ large enough.

Lemma 2.3 *([20]). Let $\rho \in L^2(\mathbb{R}^d)$, $u \in H^1(\mathbb{R}^d)$ Then*

$$\|\eta_\sigma * div(\rho u) - div(u(\rho * \eta_\sigma))\|_{L^1(\mathbb{R}^d)} \le C\|u\|_{H^1(R^d)}\|\rho\|_{L^2(\mathbb{R}^d)},$$

for some $C > 0$ independent of σ, ρ, u. In addition,

$$\eta_\sigma * div(\rho u) - div(u(\rho * \eta_\sigma)) \to 0, \ in \ L^1(\mathbb{R}^d) \ as \ \sigma \to 0.$$

Lemma 2.4 *([19]). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\rho \in L^2(\Omega \times (0,T))$, $u \in L^2(0,T;H_0^1(\Omega;\mathbb{R}^d))$ solve (2.1). Then ρ, u solve (2.1) in $\mathscr{D}'(\mathbb{R}^d \times (0,T))$ provided they were extended to be zero outside Ω.*

Lemma 2.5 *([19]). Let $\overline{O} \subset \mathbb{R}^M$ be compact and let X be a separable Banach space. Assume that $v_m : \overline{O} \to X^*$, $m \in \mathbb{Z}_+$, is a sequence of measurable functions such that*

$$\operatorname*{esssup}_{t \in \overline{O}} \|v_m(t)\|_{X^*} \le N, \ uniformly \ in \ m.$$

Moreover, let the family of functions

$$\langle v_m, \Phi\rangle : t \to \langle v_m(t), \Phi\rangle,\ t \in \overline{O}$$

be equi-continuous for any Φ *belonging to a dense subset in* X*. Then* $v_m \in C(\overline{O}; X-\omega)$ *for* $m \in \mathbb{Z}_+$*, and there exists* $v \in C(\overline{O}; X-\omega)$ *such that after taking possible subsequences,*

$$v_m \to v \text{ in } C(\overline{O}; X-\omega),\ \text{as } m \to \infty.$$

Lemma 2.6 *([19]). Let* $O \subset \mathbb{R}^N$ *be a measurable set and* $v_m \in L^1(O;\mathbb{R}^M)$ *for* $m \in \mathbb{Z}_+$ *such that*

$$v_m \to v \text{ weakly in } L^1(O;\mathbb{R}^M).$$

Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ *be a lower semi-continuous convex function such that* $\Phi(v_m) \in L^1(O)$ *for any* m*, and*

$$\Phi(v_m) \to \overline{\Phi(v)},\ \text{weakly in } L^1(O).$$

Then

$$\Phi(v) \le \overline{\Phi(v)},\ \text{a.e. in } O.$$

3 Existence of weak solution

In this section, we mollify the initial data, give the initial density a positive lower bound, and get a sequence of classical approximate solutions. Then we derive some a priori estimates of the approximate solutions, and take limits. The main difficulties come from the convergence of the pressure. We overcome these difficulties by Lemmas 3.2-3.4.

By Sobolev's extension theorem in [18], there exists a $\widetilde{n}_0 \in H^1_c(\mathbb{R})$ such that $\widetilde{n}_0 = n_0$ in I. We mollify the initial data as follows.

$$\begin{cases} \rho_{0\delta} = \eta_\delta \star \widehat{\rho}_0 + \delta, \\ u_{0\delta} = \dfrac{1}{\sqrt{\rho_{0\delta}}}\eta_\delta \star (\widehat{\dfrac{m_0}{\sqrt{\rho_0}}}), \\ n_{0\delta} = \dfrac{\eta_\delta \star \widetilde{n}_0}{|\eta_\delta \star \widetilde{n}_0|}. \end{cases}$$

Then $\rho_{0\delta} \geq \delta > 0$, $\rho_{0\delta}, u_{0\delta}, n_{0\delta} \in C^{2+\alpha}(\bar{I})$ for $0 < \alpha < 1$, satisfy as $\delta \to 0$

$$\begin{cases} \rho_{0\delta} \to \rho_0, \ in\ L^\gamma(I), \\ \sqrt{\rho_{0\delta}} u_{0\delta} \to \frac{m_0}{\sqrt{\rho_0}}, \ in\ L^2(I), \\ \rho_{0\delta} u_{0\delta} \to m_0, \ in\ L^{\frac{2\gamma}{\gamma+1}}(I), \\ n_{0\delta} \to n_0 \ in\ H^1(I). \end{cases} \tag{3.1$'$}$$

From [12], we get a sequence of global classical solutions $(\rho_\delta, u_\delta, n_\delta)$ such that

$$\begin{cases} (\rho_\delta)_t + (\rho_\delta u_\delta)_x = 0, \ \rho_\delta > 0, \\ (\rho_\delta u_\delta)_t + (\rho_\delta u_\delta^2)_x + (\rho_\delta^\gamma)_x = (u_\delta)_{xx} - (|(n_\delta)_x|^2)_x, \\ (n_\delta)_t + u_\delta (n_\delta)_x = (n_\delta)_{xx} + |(n_\delta)_x|^2 n_\delta, \ |n_\delta| = 1, \end{cases} \tag{3.1}$$

for $(x, t) \in [0, 1] \times (0, +\infty)$, with the initial and boundary conditions:

$$(\rho_\delta, \ u_\delta, \ n_\delta)\big|_{t=0} = (\rho_{0\delta}, \ u_{0\delta}, \ n_{0\delta}) \text{ in } [0, 1],$$

$$(u_\delta, \ \partial_x n_\delta)\big|_{\partial I} = (0, \ 0).$$

Lemma 3.1 *([12]) For any $T > 0$ and $0 \leq t \leq T$, it holds*

$$\begin{aligned} &\int_I \left(\frac{\rho_\delta u_\delta^2}{2} + \frac{\rho_\delta^\gamma}{\gamma - 1} + |(n_\delta)_x|^2\right)(t) + \int_0^t \int_I \left(|(u_\delta)_x|^2 + 2\left|(n_\delta)_{xx} + |(n_\delta)_x|^2 n_\delta\right|^2\right) \\ = &\int_I \left(\frac{\rho_{0\delta} u_{0\delta}^2}{2} + \frac{\rho_{0\delta}^\gamma}{\gamma - 1} + |(n_{0\delta})_x|^2\right), \end{aligned} \tag{3.2}$$

and

$$\int_{Q_T} |(n_\delta)_t|^2 + |(n_\delta)_{xx}|^2 \leq c(E_0, T). \tag{3.3}$$

From (3.2), we have $\rho_\delta^\gamma \in L^\infty(0, T; L^1(I))$. To take limits of ρ_δ^γ as $\delta \to 0$, we need more regularity of ρ_δ with respect to the space variable. Namely,

Lemma 3.2

$$\int_{Q_T} \rho_\delta^{2\gamma} \leq c(E_0, T).$$

Proof. Multiplying $(3.1)_2$ by $\int_0^x \rho_\delta^\gamma - x\int_I \rho_\delta^\gamma$, integrating the resulting equation over Q_T, and using integration by parts, we get

$$\begin{aligned}
\int_{Q_T} \rho_\delta^{2\gamma} &= \int_I \rho_\delta u_\delta (\int_0^x \rho_\delta^\gamma - x\int_I \rho_\delta^\gamma)|_0^T - \int_0^T \int_I \rho_\delta u_\delta [\int_0^x (\rho_\delta^\gamma)_t - x\int_I (\rho_\delta^\gamma)_t] \\
&\quad - \int_0^T \int_I (\rho_\delta u_\delta^2)(\rho_\delta^\gamma - \int_I \rho_\delta^\gamma) + \int_0^T (\int_I \rho_\delta^\gamma)^2 + \int_0^T \int_I (u_\delta)_x (\rho_\delta^\gamma - \int_I \rho_\delta^\gamma) \\
&\quad - \int_0^T \int_I |(n_\delta)_x|^2 (\rho_\delta^\gamma - \int_I \rho_\delta^\gamma) \\
&= I + II + III + IV + V + VI.
\end{aligned}$$

$$\begin{aligned}
I &= \int_I \rho_\delta u_\delta (\int_0^x \rho_\delta^\gamma - x\int_I \rho_\delta^\gamma)|_0^T \\
&\le c \sup_{0\le t\le T} (\int_I \rho_\delta |u_\delta| \int_I \rho_\delta^\gamma) \\
&\le c \sup_{0\le t\le T} (\int_I \rho_\delta u_\delta^2 \int_I \rho_\delta^\gamma) + c \sup_{0\le t\le T} (\int_I \rho_\delta \int_I \rho_\delta^\gamma) \\
&\le c(E_0),
\end{aligned}$$

where we have used (3.2). To estimate II, we multiply $(3.1)_1$ by $\gamma\rho_\delta^{\gamma-1}$ and get

$$(\rho_\delta^\gamma)_t + (\rho_\delta^\gamma u_\delta)_x + (\gamma-1)\rho_\delta^\gamma (u_\delta)_x = 0. \tag{3.4}$$

Therefore, we have from (3.4) that

$$\begin{aligned}
II &= \int_0^T \int_I \rho_\delta u_\delta \int_0^x [(\rho_\delta^\gamma u_\delta)_x + (\gamma-1)\rho_\delta^\gamma (u_\delta)_x] \\
&\quad - \int_0^T \int_I x\rho_\delta u_\delta \int_I [(\rho_\delta^\gamma u_\delta)_x + (\gamma-1)\rho_\delta^\gamma (u_\delta)_x] \\
&= \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + (\gamma-1)\int_0^T \int_I \rho_\delta u_\delta \int_0^x \rho_\delta^\gamma (u_\delta)_x \\
&\quad -(\gamma-1)\int_0^T \int_I x\rho_\delta u_\delta \int_I \rho_\delta^\gamma (u_\delta)_x \\
&\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c\int_0^T \int_I \rho_\delta |u_\delta| \int_I \rho_\delta^\gamma |(u_\delta)_x| \\
&\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c\int_0^T \int_I \rho_\delta^\gamma |(u_\delta)_x| (\int_I \rho_\delta + \rho_\delta u_\delta^2) \\
&\le \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c(E_0)\int_0^T \int_I \rho_\delta^\gamma |(u_\delta)_x|.
\end{aligned}$$

By Cauchy's inequality, Hölder's inequality, and (3.2), we have

$$
\begin{aligned}
II &\leq \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c(E_0) \int_0^T \|\rho_\delta\|_{L^{2\gamma}(I)}^\gamma \|(u_\delta)_x\|_{L^2(I)} \\
&\leq \int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + \frac{1}{4} \int_{Q_T} \rho_\delta^{2\gamma} + c(E_0).
\end{aligned}
$$

$$
\begin{aligned}
III + IV &= -\int_0^T \int_I (\rho_\delta u_\delta^2)(\rho_\delta^\gamma - \int_I \rho_\delta^\gamma) + \int_0^T (\int_I \rho_\delta^\gamma)^2 \\
&= -\int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + \int_0^T \int_I \rho_\delta u_\delta^2 \int_I \rho_\delta^\gamma + \int_0^T (\int_I \rho_\delta^\gamma)^2 \\
&\leq -\int_{Q_T} \rho_\delta^{\gamma+1} u_\delta^2 + c(E_0, T).
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_0^T \int_I (u_\delta)_x \rho_\delta^\gamma - \int_0^T \int_I (u_\delta)_x \int_I \rho_\delta^\gamma \\
&\leq \int_0^T \|(u_\delta)_x\|_{L^2(I)} \|\rho_\delta\|_{L^{2\gamma}(I)}^\gamma + \frac{1}{2} \sup_{0 \leq t \leq T} \int_I \rho_\delta^\gamma \int_{Q_T} (|(u_\delta)_x|^2 + 1) \\
&\leq \frac{1}{4} \int_{Q_T} \rho_\delta^{2\gamma} + c(E_0, T).
\end{aligned}
$$

$$
\begin{aligned}
VI &= -\int_0^T \int_I |(n_\delta)_x|^2 (\rho_\delta^\gamma - \int_I \rho_\delta^\gamma) \\
&\leq \sup_{0 \leq t \leq T} \int_I \rho_\delta^\gamma \int_0^T \int_I |(n_\delta)_x|^2 \\
&\leq c(E_0, T).
\end{aligned}
$$

Putting these inequalities together, we have

$$
\begin{aligned}
\int_{Q_T} \rho_\delta^{2\gamma} &= I + II + III + IV + V + VI \\
&\leq \frac{1}{2} \int_{Q_T} \rho_\delta^{2\gamma} + c(E_0, T).
\end{aligned}
$$

This completes the proof of the lemma. □

It follows from Lemma 3.1-3.2 that there exists a subsequence of $(\rho_\delta, u_\delta, n_\delta)$, still denoted by $(\rho_\delta, u_\delta, n_\delta)$, such that as $\delta \to 0$, for any $T > 0$,

$$
\rho_\delta \to \rho \;\; weak^\star \; in \; L^\infty(0,T; L^\gamma(I)), \; and \; weakly \; in \; L^{2\gamma}(Q_T), \tag{3.5}
$$

$$
\rho_\delta^\gamma \to \overline{\rho^\gamma}, \; weakly \; in \; L^2(Q_T), \tag{3.6}
$$

$$u_\delta \to u,\ weakly\ in\ L^2(0,T;H_0^1(I)), \tag{3.7}$$

$$n_\delta \to n,\ weak^\star\ in\ L^\infty(Q_T), \tag{3.8}$$

$$(n_\delta)_x \to n_x,\ weak^\star\ in\ L^\infty(0,T;L^2), \tag{3.9}$$

$$((n_\delta)_t,\ (n_\delta)_{xx}) \to (n_t,\ n_{xx}),\ weakly\ in\ L^2(Q_T). \tag{3.10}$$

Since $\rho_\delta \in L^{2\gamma}(Q_T), u_\delta \in L^2(0,T;H_0^1(I)) \subset L^2(0,T;L^\infty(I))$, we have

$$\rho_\delta u_\delta \in L^{\frac{2\gamma}{\gamma+1}}(0,T;L^{2\gamma}(I)).$$

Therefore, $\partial_t \rho_\delta = -(\rho_\delta u_\delta)_x \in L^{\frac{2\gamma}{\gamma+1}}(0,T;H^{-1}(I))$. Since $\frac{2\gamma}{\gamma+1} > 1$, $\rho_\delta \in L^\infty(0,T;L^\gamma(I))$, and $L^\gamma \hookrightarrow\hookrightarrow H^{-1}(I)$, we have from Lemma 2.1 and Lemma 2.5

$$\rho_\delta \to \rho,\ in\ C([0,T];L^\gamma - \omega), \tag{3.11}$$

$$\rho_\delta \to \rho\ in\ C([0,T];H^{-1}). \tag{3.12}$$

(3.7) and (3.12) imply

$$\rho_\delta u_\delta \to \rho u,\ in\ \mathscr{D}'(Q_T). \tag{3.13}$$

Hence,

$$\rho_t + (\rho u)_x = 0,\ in\ \mathscr{D}'(Q_T). \tag{3.14}$$

Moreover, $\sqrt{\rho_\delta}u_\delta \in L^\infty(0,T;L^2)$ and $\sqrt{\rho_\delta} \in L^\infty(0,T;L^{2\gamma})$ give

$$\rho_\delta u_\delta \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}).$$

From $(3.1)_2$, we get $(\rho_\delta u_\delta)_t = -(\rho_\delta u_\delta^2)_x - (\rho_\delta^\gamma)_x + (u_\delta)_{xx} - (|(n_\delta)_x|^2)_x \in L^2(0,T;W^{-1,\frac{2\gamma}{\gamma+1}})$. By (3.13), Lemma 2.1 and Lemma 2.5, we obtain

$$\rho_\delta u_\delta \to \rho u,\ in\ C([0,T];L^{\frac{2\gamma}{\gamma+1}} - \omega), \tag{3.15}$$

$$\rho_\delta u_\delta \to \rho u,\ in\ C([0,T];H^{-1}),\ (also\ weak^\star\ in\ L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}})). \tag{3.16}$$

We have from (3.7) and (3.16)

$$\rho_\delta u_\delta^2 \to \rho u^2,\ in\ \mathscr{D}'(Q_T),\ (also\ weakly\ in\ L^2(0,T;L^{\frac{2\gamma}{\gamma+1}})). \tag{3.17}$$

Similar to the above argument, we have from (3.8)-(3.10) and Lemma 2.1

$$n_\delta \to n,\ in\ C(\overline{Q}_T), \tag{3.18}$$

$$n_\delta \to n, \ in \ L^2(0,T;C^1([0,1])). \tag{3.19}$$

This, together with (3.6), (3.7), (3.9), (3.10), (3.13), and (3.17), implies

$$(\rho u)_t + (\rho u^2)_x + (\overline{\rho^\gamma})_x = u_{xx} - (|n_x|^2)_x, \ in \ \mathscr{D}'(Q_T), \tag{3.20}$$

$$n_t + u n_x = n_{xx} + |n_x|^2 n, \ in \ L^2(0,T;L^2). \tag{3.21}$$

It follows from $(3.1)'_1$, $(3.1)'_3$, (3.11), and (3.15) that

$$(\rho, \rho u)(x,0) = (\rho_0(x), m_0(x)), \ weakly \ in \ L^\gamma(I) \times L^{\frac{2\gamma}{\gamma+1}}(I).$$

By $(3.1)'_4$ and (3.18), and the fact that $|n_\delta| = 1$, we have

$$n(x,0) = n_0(x) \ in \ [0,1], \ and \ |n| = 1 \ in \ \overline{Q}_T.$$

(1.3) follows from (3.7) and (3.19). Since $\rho_\delta > 0$ in Q_T, we have from (3.5)

$$\int_{Q_T} \rho f = \lim_{\delta \to 0} \int_{Q_T} \rho_\delta f \geq 0,$$

for any nonnegative $f \in C_0^\infty(Q_T)$. For f is arbitrary, we get

$$\rho \geq 0 \ a.e. \ in \ Q_T.$$

From (3.17), we have

$$\begin{aligned} \frac{1}{\epsilon} \int_t^{t+\epsilon} \int_I \rho u^2 &= \frac{1}{\epsilon} \lim_{\delta \to 0} \int_t^{t+\epsilon} \int_I \rho_\delta u_\delta^2 \\ &\leq \frac{1}{\epsilon} \int_t^{t+\epsilon} \overline{\lim_{\delta \to 0}} \int_I \rho_\delta u_\delta^2(s), \end{aligned}$$

for $t \in (0,T)$ and $\epsilon > 0$. Let $\epsilon \to 0^+$, and use Lebesgue theorem, we get

$$\int_I \rho u^2(t) \leq \overline{\lim_{\delta \to 0}} \int_I \rho_\delta u_\delta^2(t), \ for \ a.e. \ t \in \ Q_T.$$

This, together with $(3.1)'$, (3.2), and the lower semi-continuity, implies the energy inequality (1.4). □

We still need to prove $\overline{\rho^\gamma} = \rho^\gamma$. This requires the following lemmas.

Lemma 3.3 . *It holds as* $\delta \to 0$

$$[(u_\delta)_x - \rho_\delta^\gamma]\rho_\delta \to (u_x - \overline{\rho^\gamma})\rho, \ in \ \mathscr{D}'(Q_T).$$

Proof. For any $\varphi \in C_0^\infty((0,T))$, $\phi \in C_0^\infty((0,1))$, multiplying $(3.1)_2$ by $\varphi\phi\int_0^x \rho_\delta$, integrating the resulting equation over Q_T, and using integration by parts, we have

$$\begin{aligned}
& \int_{Q_T} \varphi(t)\phi(x)[(u_\delta)_x - \rho_\delta^\gamma]\rho_\delta \\
= & \int_{Q_T} \varphi'(t)\phi(x)\rho_\delta u_\delta \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi(x)\rho_\delta u_\delta (\int_0^x \rho_\delta)_t + \int_{Q_T} \varphi(t)\phi(x)\rho_\delta^2 u_\delta^2 \\
& + \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta u_\delta^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta^\gamma \int_0^x \rho_\delta - \int_{Q_T} \varphi(t)\phi'(x)(u_\delta)_x \int_0^x \rho_\delta \\
& + \int_{Q_T} \varphi(t)\phi'(x)|(n_\delta)_x|^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi(x)|(n_\delta)_x|^2 \rho_\delta \\
= & \int_{Q_T} \varphi'(t)\phi(x)\rho_\delta u_\delta \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta u_\delta^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)\rho_\delta^\gamma \int_0^x \rho_\delta \\
& - \int_{Q_T} \varphi(t)\phi'(x)(u_\delta)_x \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi'(x)|(n_\delta)_x|^2 \int_0^x \rho_\delta + \int_{Q_T} \varphi(t)\phi(x)|(n_\delta)_x|^2 \rho_\delta,
\end{aligned}$$

where we have used $(3.1)_1$.

Since $\int_0^x \rho_\delta \in L^\infty(0,T;W^{1,\gamma})$, $\partial_t(\int_0^x \rho_\delta) = -\rho_\delta u_\delta \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}})$, we have from Lemma 2.1 and (3.5)

$$\int_0^x \rho_\delta \to \int_0^x \rho, \ in\ C(\overline{Q}_T), \ as\ \delta \to 0. \tag{3.22}$$

This, combined with (3.5)-(3.7), (3.16), (3.17), and (3.19), gives

$$\begin{aligned}
& \lim_{\delta\to 0} \int_{Q_T} \varphi(t)\phi(x)[(u_\delta)_x - \rho_\delta^\gamma]\rho_\delta \\
& = \int_{Q_T} \varphi'(t)\phi(x)\rho u \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)\rho u^2 \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)\overline{\rho^\gamma} \int_0^x \rho - \\
& \int_{Q_T} \varphi(t)\phi'(x)u_x \int_0^x \rho + \int_{Q_T} \varphi(t)\phi'(x)|n_x|^2 \int_0^x \rho + \int_{Q_T} \varphi(t)\phi(x)|n_x|^2 \rho.
\end{aligned} \tag{3.23}$$

To complete the proof, it suffices to show that the right side of (3.23) is equal to $\int_{Q_T} \varphi(t)\phi(x)(u_x - \overline{\rho^\gamma})\rho$. The main difficulty is $\rho u \notin L^2(Q_T)$. To overcome it, take

$\varphi\phi\int_0^x\langle\widehat{\rho}\rangle_\sigma$ as a test function of (3.20), where $\langle\widehat{\rho}\rangle_\sigma = \eta_\sigma * \widehat{\rho}$. Then

$$
\begin{aligned}
&\int_{Q_T}\varphi(t)\phi(x)(u_x-\overline{\rho^\gamma})\langle\widehat{\rho}\rangle_\sigma \\
&= \int_{Q_T}\varphi'(t)\phi(x)\rho u\int_0^x\langle\widehat{\rho}\rangle_\sigma + \int_{Q_T}\varphi(t)\phi(x)\rho u(\int_0^x\langle\widehat{\rho}\rangle_\sigma)_t + \\
&\int_{Q_T}\varphi(t)\phi(x)\rho u^2\langle\widehat{\rho}\rangle_\sigma + \int_{Q_T}\varphi(t)\phi'(x)\rho u^2\int_0^x\langle\widehat{\rho}\rangle_\sigma + \\
&\int_{Q_T}\varphi(t)\phi'(x)\overline{\rho^\gamma}\int_0^x\langle\widehat{\rho}\rangle_\sigma - \int_{Q_T}\varphi(t)\phi'(x)u_x\int_0^x\langle\widehat{\rho}\rangle_\sigma + \\
&\int_{Q_T}\varphi(t)\phi'(x)|n_x|^2\int_0^x\langle\widehat{\rho}\rangle_\sigma + \int_{Q_T}\varphi(t)\phi(x)|n_x|^2\langle\widehat{\rho}\rangle_\sigma. \qquad (3.24)
\end{aligned}
$$

Since $\rho\in L^2(Q_T), u\in L^2(0,T;H_0^1(I))$, we have from Lemma 2.4

$$
(\widehat{\rho})_t + (\widehat{\rho}\widehat{u})_x = 0,\ \ in\ \mathscr{D}'(\mathbb{R}\times(0,T)). \qquad (3.25)
$$

Denote $r_\sigma = (\langle\widehat{\rho}\rangle_\sigma\widehat{u})_x - \langle(\widehat{\rho}\widehat{u})_x\rangle_\sigma$. It follows from Lemma 2.3 that $r_\sigma\in L^1(\mathbb{R}\times(0,T))$, and

$$
r_\sigma\to 0,\ \ in\ L^1(\mathbb{R}\times(0,T)),\ \ as\ \sigma\to 0. \qquad (3.26)
$$

Take $\eta_\sigma(x-\cdot)$ as a test function of (3.25), then

$$
(\langle\widehat{\rho}\rangle_\sigma)_t + (\langle\widehat{\rho}\rangle_\sigma\widehat{u})_x = r_\sigma,\ \ a.e.\ in\ \mathbb{R}\times(0,T). \qquad (3.27)
$$

Integrating (3.27) over $(0,x)$, for $0<x\le 1$, we have

$$
(\int_0^x\langle\widehat{\rho}\rangle_\sigma)_t = -\langle\widehat{\rho}\rangle_\sigma\widehat{u} + \int_0^x r_\sigma.
$$

Therefore,

$$
\begin{aligned}
&\int_{Q_T}\varphi(t)\phi(x)\rho u(\int_0^x\langle\widehat{\rho}\rangle_\sigma)_t + \int_{Q_T}\varphi(t)\phi(x)\rho u^2\langle\widehat{\rho}\rangle_\sigma \\
= &-\int_{Q_T}\varphi(t)\phi(x)\rho u\langle\widehat{\rho}\rangle_\sigma\widehat{u} + \int_{Q_T}\varphi(t)\phi(x)\rho u\int_0^x r_\sigma + \int_{Q_T}\varphi(t)\phi(x)\rho u^2\langle\widehat{\rho}\rangle_\sigma \\
= &\int_{Q_T}\varphi(t)\phi(x)\rho u\int_0^x r_\sigma,
\end{aligned}
$$

where we have used $\widehat{u}=u$ in Q_T. This, together with (3.24), implies

$$\begin{aligned}
&\int_{Q_T}\varphi(t)\phi(x)(u_x-\overline{\rho^\gamma})\langle\widehat{\rho}\rangle_\sigma\\
&=\int_{Q_T}\varphi'(t)\phi(x)\rho u\int_0^x\langle\widehat{\rho}\rangle_\sigma+\int_{Q_T}\varphi(t)\phi(x)\rho u\int_0^x r_\sigma+\\
&\int_{Q_T}\varphi(t)\phi'(x)\rho u^2\int_0^x\langle\widehat{\rho}\rangle_\sigma+\int_{Q_T}\varphi(t)\phi'(x)\overline{\rho^\gamma}\int_0^x\langle\widehat{\rho}\rangle_\sigma\\
&-\int_{Q_T}\varphi(t)\phi'(x)u_x\int_0^x\langle\widehat{\rho}\rangle_\sigma+\int_{Q_T}\varphi(t)\phi'(x)|n_x|^2\int_0^x\langle\widehat{\rho}\rangle_\sigma\\
&+\int_{Q_T}\varphi(t)\phi(x)|n_x|^2\langle\widehat{\rho}\rangle_\sigma. \qquad (3.28)
\end{aligned}$$

By the regularities of (ρ,u,n), (3.26), Lebesgue's Dominated convergence theorem, we get, after taking $\sigma\to 0$ in (3.28),

$$\begin{aligned}
&\int_{Q_T}\varphi(t)\phi(x)(u_x-\overline{\rho^\gamma})\rho\\
&=\int_{Q_T}\varphi'(t)\phi(x)\rho u\int_0^x\rho+\int_{Q_T}\varphi(t)\phi'(x)\rho u^2\int_0^x\rho+\int_{Q_T}\varphi(t)\phi'(x)\overline{\rho^\gamma}\int_0^x\rho-\\
&\int_{Q_T}\varphi(t)\phi'(x)u_x\int_0^x\rho+\int_{Q_T}\varphi(t)\phi'(x)|n_x|^2\int_0^x\rho+\int_{Q_T}\varphi(t)\phi(x)|n_x|^2\rho. \qquad (3.29)
\end{aligned}$$

The conclusion follows from (3.23) and (3.29). This completes the proof of the Lemma. □

Lemma 3.4 . *It holds*

$$\overline{\lim_{\delta\to 0}}\int_{Q_T}\rho_\delta(u_\delta)_x\le\int_{Q_T}\rho u_x.$$

Proof. Since $\rho\in L^{2\gamma}(Q_T), u\in L^2(0,T;H_0^1)$, we replace b in (2.2) by b_j^l, where $j,l\in Z_+$, $b_j^l\in C^1(R)$ and $b_j^l(z)=(z+\frac{1}{l})\log(z+\frac{1}{l})$ for $0\le z\le j$, and $b_j^l(z)=(j+1+\frac{1}{l})\log(j+1+\frac{1}{l})$ for $z\ge j+1$. Since $\rho\in L^\infty(0,T;L^\gamma)$, we have $\rho<+\infty$ a.e. in Q_T. This implies $b_j^l(\rho)\to(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})$ a.e. in Q_T, as $j\to\infty$. Let $j\to\infty$ in (2.2), the Lebesgue's Dominated convergence theorem implies

$$\partial_t[(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})]+[(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})u]_x+\rho u_x-\frac{1}{l}u_x\log(\rho+\frac{1}{l})=0,\ in\ \mathscr{D}'(Q_T). \qquad (3.30)$$

We obtain from $\rho\in L^{2\gamma}(Q_T)$ that $(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})\in L^2(Q_T)$. Similar to (3.25)-(3.27), we extend ρ,u in (3.30) to be zero outside I, mollify (3.30), integrate the resulting equation over Q_T, and take limits, we obtain

$$\begin{aligned}\int_{Q_T}\rho u_x &= \int_I(\rho_0+\frac{1}{l})\log(\rho_0+\frac{1}{l})-\int_I(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})(T)+\\ &\quad \frac{1}{l}\int_{Q_T}u_x\log(\rho+\frac{1}{l}) \qquad (3.31)\end{aligned}$$

Since $(3.1)_1$ is valid in classical sense, a direct calculation gives

$$\partial_t[(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})]+[(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})u_\delta]_x+\rho_\delta(u_\delta)_x-\frac{1}{l}(u_\delta)_x\log(\rho_\delta+\frac{1}{l})=0. \qquad (3.32)$$

Integrating (3.32) over Q_T, we have

$$\begin{aligned}\int_{Q_T}\rho_\delta(u_\delta)_x &= \int_I(\rho_{0\delta}+\frac{1}{l})\log(\rho_{0\delta}+\frac{1}{l})-\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T)+\\ &\quad \frac{1}{l}\int_{Q_T}(u_\delta)_x\log(\rho_\delta+\frac{1}{l})\\ &\le \int_I(\rho_{0\delta}+\frac{1}{l})\log(\rho_{0\delta}+\frac{1}{l})-\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T)+\\ &\quad \frac{1}{l}\|(u_\delta)_x\|_{L^2(Q_T)}\|\rho_\delta+1\|_{L^2(Q_T)}\\ &\le \int_I(\rho_{0\delta}+\frac{1}{l})\log(\rho_{0\delta}+\frac{1}{l})-\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T)\\ &\quad +\frac{1}{l}c(E_0,T), \qquad (3.33)\end{aligned}$$

where we have used Hölder inequality, Lemma 3.1, and Lemma 3.2.

Since $\rho_\delta\in L^\infty(0,T;L^\gamma)$, we have

$$(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})\in L^\infty(0,T;L^\tau), \qquad (3.34)$$

for some $\tau>1$. From (3.32), we get

$$\partial_t[(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})]\in L^{\frac{2\gamma}{\gamma+1}}(0,T;W^{-1,\frac{2\gamma}{\gamma+1}}). \qquad (3.35)$$

(3.34), (3.35), and Lemma 2.5 give

$$(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})\to\overline{(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})},\ in\ C([0,T];L^\tau-\omega),$$

as $\delta\to0$. This implies

$$\lim_{\delta\to0}\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T)=\int_I\overline{(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})}(T).$$

Since the function $(z+\frac{1}{l})\log(z+\frac{1}{l})$ is convex for $z\geq 0$, Lemma 2.6 implies

$$(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})\leq \overline{(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})},\ a.e.\ in\ Q_T.$$

Therefore,

$$\lim_{\delta\to 0}\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T)\geq\int_I(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})(T). \tag{3.36}$$

Take $\overline{\lim_{\delta\to 0}}$ in (3.33), and use (3.36), we get

$$\begin{aligned}\overline{\lim_{\delta\to 0}}\int_{Q_T}\rho_\delta(u_\delta)_x &\leq \int_I(\rho_0+\frac{1}{l})\log(\rho_0+\frac{1}{l})-\underline{\lim_{\delta\to 0}}\int_I(\rho_\delta+\frac{1}{l})\log(\rho_\delta+\frac{1}{l})(T)+\frac{1}{l}c(E_0,T)\\ &\leq \int_I(\rho_0+\frac{1}{l})\log(\rho_0+\frac{1}{l})-\int_I(\rho+\frac{1}{l})\log(\rho+\frac{1}{l})(T)+\frac{1}{l}c(E_0,T)\\ &= \int_{Q_T}\rho u_x-\frac{1}{l}\int_{Q_T}u_x\log(\rho+\frac{1}{l})+\frac{1}{l}c(E_0,T)\\ &\leq \int_{Q_T}\rho u_x+\frac{1}{l}\|u_x\|_{L^2(Q_T)}\|\rho+1\|_{L^2(Q_T)}+\frac{1}{l}c(E_0,T).\end{aligned}$$

Since $u_x\in L^2(Q_T)$, and $\rho\in L^2(Q_T)$, sending $l\to\infty$ yields

$$\overline{\lim_{\delta\to 0}}\int_{Q_T}\rho_\delta(u_\delta)_x\leq\int_{Q_T}\rho u_x.$$

The proof of the Lemma is complete. □

Now we return to the proof of $\rho^\gamma=\overline{\rho^\gamma}$. Assume $\varphi_m\in C_0^\infty(0,T)$, $\phi_m\in C_0^\infty(0,1)$, $0\leq\varphi_m,\phi_m\leq 1$, and $\varphi_m,\phi_m\to 1$, as $m\to\infty$. For any $\psi\in C_0^\infty(Q_T)$, denote $v=\rho-\epsilon\psi$ for $\epsilon>0$, then

$$\begin{aligned}&\int_{Q_T}(\overline{\rho^\gamma}-v^\gamma)(\rho-v)\\ =&\int_{Q_T}\varphi_m\phi_m(\overline{\rho^\gamma}-v^\gamma)(\rho-v)+\int_{Q_T}(1-\varphi_m\phi_m)(\overline{\rho^\gamma}-v^\gamma)(\rho-v)\\ =&\int_{Q_T}\varphi_m\phi_m(\overline{\rho^\gamma}\rho-\overline{\rho^\gamma}v-v^\gamma\rho+v^{\gamma+1})+\int_{Q_T}(1-\varphi_m\phi_m)(\overline{\rho^\gamma}-v^\gamma)(\rho-v)\\ =&\int_{Q_T}\varphi_m\phi_m(\overline{\rho^\gamma}-u_x)\rho+\int_{Q_T}(\varphi_m\phi_m-1)\rho u_x+\int_{Q_T}\rho u_x\\ +&\int_{Q_T}\varphi_m\phi_m(-\overline{\rho^\gamma}v-v^\gamma\rho+v^{\gamma+1})+\int_{Q_T}(1-\varphi_m\phi_m)(\overline{\rho^\gamma}-v^\gamma)(\rho-v).\end{aligned}$$

Denote $A_m=\int_{Q_T}(\varphi_m\phi_m-1)\rho u_x+\int_{Q_T}(1-\varphi_m\phi_m)(\overline{\rho^\gamma}-v^\gamma)(\rho-v)$. Together with

Lemma 3.3-3.4, (3.5), and (3.6), we have

$$\begin{aligned}
&\int_{Q_T}(\overline{\rho^\gamma}-v^\gamma)(\rho-v)\\
\geq\ & \overline{\lim_{\delta\to0}}\int_{Q_T}\varphi_m\phi_m[\rho_\delta^\gamma-(u_\delta)_x]\rho_\delta+\overline{\lim_{\delta\to0}}\int_{Q_T}\rho_\delta(u_\delta)_x+\\
&\overline{\lim_{\delta\to0}}\int_{Q_T}\varphi_m\phi_m(-\rho_\delta^\gamma v-v^\gamma\rho_\delta+v^{\gamma+1})+A_m\\
\geq\ & \overline{\lim_{\delta\to0}}[\int_{Q_T}\varphi_m\phi_m[\rho_\delta^\gamma-(u_\delta)_x]\rho_\delta+\int_{Q_T}\varphi_m\phi_m\rho_\delta(u_\delta)_x\\
&+\int_{Q_T}\varphi_m\phi_m(-\rho_\delta^\gamma v-v^\gamma\rho_\delta+v^{\gamma+1})]-\overline{\lim_{\delta\to0}}\int_{Q_T}\rho_\delta|1-\varphi_m\phi_m||(u_\delta)_x|+A_m\\
=\ & \overline{\lim_{\delta\to0}}\int_{Q_T}\varphi_m\phi_m(\rho_\delta^\gamma-v^\gamma)(\rho_\delta-v)-\overline{\lim_{\delta\to0}}\int_{Q_T}\rho_\delta|1-\varphi_m\phi_m||(u_\delta)_x|+A_m.
\end{aligned}$$

By the monotonicity of z^γ, we have

$$\int_{Q_T}\varphi_m\phi_m(\rho_\delta^\gamma-v^\gamma)(\rho_\delta-v)\geq0.$$

Therefore,

$$\begin{aligned}
\int_{Q_T}(\overline{\rho^\gamma}-v^\gamma)(\rho-v)\ &\geq\ -\overline{\lim_{\delta\to0}}\int_{Q_T}\rho_\delta|1-\varphi_m\phi_m||u_{\delta x}|+A_m\\
&\geq\ -\overline{\lim_{\delta\to0}}\|1-\varphi_m\phi_m\|_{L^{\frac{2\gamma}{\gamma-1}}(Q_T)}\|\rho_\delta\|_{L^{2\gamma}(Q_T)}\|u_{\delta x}\|_{L^2(Q_T)}+A_m\\
&\geq\ -c(E_0,T)\|1-\varphi_m\phi_m\|_{L^{\frac{2\gamma}{\gamma-1}}(Q_T)}+A_m, \qquad (3.37)
\end{aligned}$$

where we have used Hölder inequality, Lemma 3.1, and Lemma 3.2. By the Lebesgue's Dominated Convergence Theorem, we have

$$\|1-\varphi_m\phi_m\|_{L^{\frac{2\gamma}{\gamma-1}}(Q_T)}\to0,A_m\to0\text{ as }m\to\infty.$$

Let $m\to\infty$ in (3.37), we get

$$\int_{Q_T}(\overline{\rho^\gamma}-v^\gamma)(\rho-v)\geq0.$$

Since $v=\rho-\epsilon\psi$, and $\epsilon>0$, we have

$$\int_{Q_T}[\overline{\rho^\gamma}-(\rho-\epsilon\psi)^\gamma]\psi\geq0. \qquad (3.38)$$

Sending $\epsilon\downarrow0$ yields

$$\int_{Q_T}(\overline{\rho^\gamma}-\rho^\gamma)\psi\geq0.$$

This clearly implies

$$\overline{\rho^\gamma} = \rho^\gamma, \ a.e.\ in\ Q_T.$$

The proof of Theorem 1.1 is complete. □

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