Phase Transition for Potentials of High-Dimensional Wells

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Abstract

For a potential function $F : \mathbb{R}^k \to \mathbb{R}_+$ that attains its global minimum value at two disjoint compact connected submanifolds $N^{\pm}$ in $\mathbb{R}^k$, we discuss the asymptotics, as $\epsilon \to 0$, of minimizers u_ϵ of the singular perturbed functional $\mathbf{E}_\epsilon(u) = \int_\Omega (|\nabla u|^2 + \frac{1}{\epsilon^2} F(u))dx$ under suitable Dirichlet boundary data $g_\epsilon : \partial\Omega \to \mathbb{R}^k$. In the expansion of $\mathbf{E}_\epsilon(u_\epsilon)$ with respect to $\frac{1}{\epsilon}$, we identify the first-order term by the area of the sharp interface between the two phases, an area-minimizing hypersurface Γ, and the energy c_0^F of minimal connecting orbits between N^+ and N^-, and the zeroth-order term by the energy of minimizing harmonic maps into $N^{\pm}$ both under the Dirichlet boundary condition on $\partial\Omega$ and a very interesting partially constrained boundary condition on the sharp interface Γ. © 2011 Wiley Periodicals, Inc.

1 Introduction

Because of both its important applications to many subjects in sciences and its intrinsic mathematical interest and challenges, the theory of singular perturbation for scalar-valued phase transition problems has drawn great interest in analysis and computations. A typical energy that models the phase separation phenomena of a two-phase fluid is given by the Cahn-Hilliard energy functional

$$\int_\Omega \left(\epsilon|\nabla v|^2 + \frac{1}{\epsilon} W(v)\right) dx,$$

Communications on Pure and Applied Mathematics, 0001–0056 (PREPRINT)

where $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded, smooth domain in $\mathbb{R}^n$ throughout this paper, $v : \Omega \to \mathbb{R}$ is the density of the fluid, and $W : \mathbb{R} \to \mathbb{R}_+$ is a double-well potential function that has two minima (zeros) at ± 1. The term $\epsilon|\nabla v|^2$ is the interfacial energy contribution that penalizes the formation of interfaces (see Gurtin [10]). The asymptotic behavior of minimizers v_ϵ of the above Cahn-Hilliard energy functional under the constraint $\int_\Omega v_\epsilon = c$, as $\epsilon \to 0$, was first studied by Modica and Mortola [19], Modica [18], and Luckhaus and Modica [17]; they show that the separation region between the two stable phases has $O(\epsilon)$ thickness and the phase transition converges to a minimal hypersurface. Sternberg [26, 27], Kohn and Sternberg [14], Fonseca and Tartar [9], and Modica and Mortola [19] have studied it within the framework of De Giorgi's Γ-convergence.

To model certain chemical reaction processes, Rubinstein, Sternberg, and Keller [21, 22] introduced the vector-valued system of fast reaction and slow diffusion (under the Neumann boundary condition):

$$\partial_t v_\epsilon = \epsilon \Delta v_\epsilon - \epsilon^{-1} W_v(v_\epsilon) \ \text{ in } \Omega, \quad \frac{\partial v_\epsilon}{\partial \nu} = 0 \ \text{ on } \partial\Omega,$$

where the order parameter $v_\epsilon : \Omega \to \mathbb{R}^k$ represents the concentration vector of reactants. By the law of mass action during the chemical reaction process, the potential function $W : \mathbb{R}^k \to \mathbb{R}_+$ can vanish on two disjoint submanifolds in $\mathbb{R}^k$. In this case, a front develops in Ω. By the formal WKB analysis on the asymptotic expansion for potential functions vanishing on two submanifolds, it was found in [21, 22] that with respect to the slow time variables, the front moves by its mean curvature, and v_ϵ approximates the heat flow of harmonic maps away from the front.

Although there have been many studies for the rigorous analysis of such an asymptotics for the scalar case $k = 1$ (see, for example, [7, 13]), the corresponding analysis has remained an open problem for $k \geq 2$ (see Bronsard and Stoth [5] for some preliminary results). In this paper, we intend to analyze the corresponding time-independent case of the so-called Keller-Rubinstein-Sternberg problem, associated with suitable Dirichlet boundary conditions. The aim is twofold: (1) the time-independent case is very delicate and it also arises in various applications; (2) a complete understanding of the time-independent case sheds light on the dynamics of the general cases that we plan to investigate in the near future.

First let's describe the problem. For $k \geq 1$, let

$$N = N^+ \sqcup N^- \subseteq \mathbb{R}^k$$

be the union of two disjoint, compact, connected, smooth Riemannian submanifolds $N^\pm \subseteq \mathbb{R}^k$ without boundaries. See Figure 1.1.

For $\delta > 0$, let

$$N_\delta \equiv \left\{ p \in \mathbb{R}^k : \mathrm{d}(p, N) \equiv \inf_{y \in N} |p - y| \leq \delta \right\}$$

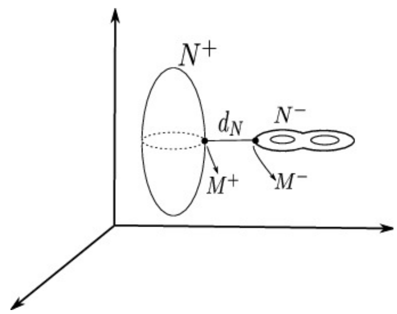

FIGURE 1.1. Target manifold.

denote the δ-neighborhood of N. It is well-known that there exists $\delta_N > 0$ depending only on N such that

(1) the nearest-point projection map $\Pi : N_{\delta_N} \to N$ is smooth, and
(2) $\mathrm{d}^2(p, N) = |p - \Pi(p)|^2 \in C^\infty(N_{\delta_N})$.

Throughout this paper, we will consider the class of double-well potential functions depending only on the distance function from the target manifolds.[1] Namely, we assume

$$F(p) = f(\mathrm{d}^2(p, N)),$$

where $f \in C^\infty(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the property that there exist $c_1, c_2, c_3 > 0$ such that

$$\begin{cases} c_1 t \le f(t) \le c_2 t & \text{if } 0 \le t \le \delta_N^2, \\ f(t) \ge c_3 & \text{if } t \ge \delta_N^2. \end{cases} \tag{1.1}$$

We will consider the family of energy functionals

$$E_\epsilon(u) = \int_\Omega (\epsilon^2 |\nabla u|^2 + F(u)) dx, \quad u \in H^1(\Omega, \mathbb{R}^k), \ \ \epsilon > 0,$$

that are singular perturbations of the functional of phase transitions of high-dimensional wells:

$$E_0(u) = \int_\Omega F(u) dx, \quad u \in L^1(\Omega, \mathbb{R}^k).$$

We will discuss the gradient theory of phase transitions, namely, study the behavior of minimizers u_ϵ of $E_\epsilon(\cdot)$ under well-prepared Dirichlet boundary data as $\epsilon \to 0$.

Now we prescribe the boundary data. Let $\Sigma^\pm \subseteq \partial\Omega$ be two disjoint, connected, open subsets of $\partial\Omega$ such that

[1] However, Theorem 1.1 actually holds for general potential functions of high-dimensional double wells satisfying (1.8), which are not necessarily functions of $d(\cdot, N)$; see Remark 1.2(3) and the discussions in Section 3 and Section 7.

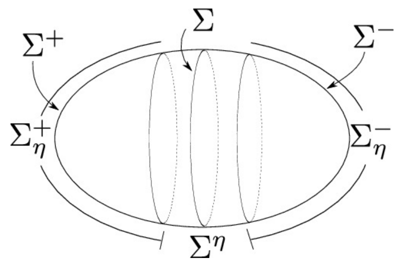

FIGURE 1.2. Boundary of the domain Ω.

(1) $\partial\Sigma^+ = \partial\Sigma^- = \Sigma$ is a connected $(n-2)$-dimensional smooth manifold and
(2) $\partial\Omega = \Sigma^+ \sqcup \Sigma^- \sqcup \Sigma$.

For $R > 0$, let $B^k_R = \{y \in \mathbb{R}^k : |y| \le R\}$. For any small $\eta > 0$, let $\Sigma^\eta = \{x \in \mathbb{R}^n : d(x, \Sigma) < \eta\}$ be the η-neighborhood of Σ, and denote

$$\Sigma^\pm_\eta = \Sigma^\pm \setminus \Sigma^\eta (\equiv \{x \in \Sigma^\pm : \mathrm{d}(x, \Sigma) \ge \eta\}).$$

See Figure 1.2. We assume that there exist $\beta > 0$, $R > 0$, $L > 0$, and $C > 0$ such that $g_\epsilon : \partial\Omega \to \mathbb{R}^k$ satisfy the following two conditions for all $0 < \epsilon < 1$:

(1)

$$\begin{aligned} g_\epsilon(\Sigma^\pm_{\epsilon^\beta}) \subset N^\pm, \quad g_\epsilon(\partial\Omega) \subset B^k_R, \\ \int_{\partial\Omega} \left(\epsilon|\nabla_\tau g_\epsilon|^2 + \frac{1}{\epsilon} F(g_\epsilon)\right) dH^{n-1} \le L; \end{aligned} \tag{1.2}$$

(2) the extension property: For any $p^\pm \in N^\pm$, there exists $\beta \in (\frac{1}{2}, 1]$ and

$$G^\pm_\epsilon : \Sigma^\pm_{\epsilon^\beta} \times [0, \epsilon^\beta] \to N^\pm$$

such that

$$\begin{aligned} G^\pm_\epsilon|_{\Sigma^\pm_{\epsilon^\beta}\times\{0\}} = g_\epsilon, \quad G^\pm_\epsilon|_{\Sigma^\pm_{\epsilon^\beta}\times\{\epsilon^\beta\}} = p^\pm, \\ \int_{\Sigma^\pm_{\epsilon^\beta}\times[0,\epsilon^\beta]} |\nabla G^\pm_\epsilon|^2 \, dx \le \\ C\left\{\epsilon^\beta \int_{\Sigma^\pm_{\epsilon^\beta}} |\nabla_\tau g_\epsilon|^2 \, dH^{n-1} + \frac{1}{\epsilon^\beta} \int_{\Sigma^\pm_{\epsilon^\beta}} |g_\epsilon - p^\pm|^2 \, dH^{n-1}\right\}, \end{aligned} \tag{1.3}$$

where ∇_τ is the tangential derivative on hypersurfaces in $\mathbb{R}^n$.

For such a boundary data g_ϵ, set

$$\mathbf{E}(\epsilon) := \min\left\{\int_\Omega \left(|\nabla u|^2 + \frac{1}{\epsilon^2} F(u)\right) dx : u\big|_{\partial\Omega} = g_\epsilon\right\}. \tag{1.4}$$

We are mainly interested in the asymptotic behavior of $\mathbf{E}(\epsilon)$ as $\epsilon \to 0$.

Now we state our first main theorem.

THEOREM 1.1. *Assume that the potential function $F \in C^\infty(\mathbb{R}^k)$ satisfies* (1.1), *$\Gamma \subset \Omega$ is an area-minimizing hypersurface with $\partial\Gamma = \Sigma$, and $g_\epsilon : \partial\Omega \to \mathbb{R}^k$ satisfies conditions* (1.2) *and* (1.3). *Then*

$$\lim_{\epsilon\to 0} \epsilon\mathbf{E}(\epsilon) = c_0^F H^{n-1}(\Gamma), \tag{1.5}$$

where c_0^F is the energy of the minimal connecting orbits between N^+ and N^- defined by

$$c_0^F = \inf\{c^F(p^+, p^-) : p^\pm \in N^\pm\} \tag{1.6}$$

and

$$c^F(p^+, p^-) = \inf\left\{\int_{\mathbb{R}} (|\xi'(t)|^2 + F(\xi))dt : \xi \in H^1(\mathbb{R}, \mathbb{R}^k),\ \xi(\pm\infty) = p^\pm\right\}. \tag{1.7}$$

A few remarks are in order.

Remark 1.2.

(1) Based on the assumption that $F = 0$ on $N^\pm$, it is not hard to see that $c^F(p^+, p^-) : N^+ \times N^- \to \mathbb{R}_+$ is Lipschitz-continuous; hence the infimum c_0^F is attained.

(2) For general target manifolds $N^\pm$, (1.2) is a minor condition, while the extension property (1.3) does pose some restrictions on g_ϵ. However, (1.3) does hold for an arbitrary map $g_\epsilon \in H^1(\partial\Omega, \mathbb{R}^k)$, provided either

(a) N^+ and N^- are simply connected, i.e.,

$$\Pi_1(N^+) = \Pi_1(N^-) = \{0\},$$

or

(b) there exists a small $\eta_0 > 0$ depending only on N such that for some $p^\pm \in N^\pm$,

$$\left(\int_{\Sigma^\pm_{\epsilon\alpha}} |\nabla_\tau g_\epsilon|^2\, dH^{n-1}\right)\left(\int_{\Sigma^\pm_{\epsilon\alpha}} |g_\epsilon - p^\pm|^2\, dH^{n-1}\right) \le \eta_0.$$

(a) follows from Hardt and Lin's extension lemma [11, p. 580, lemma 6.1], and (b) follows from the extension lemmas due to Schoen and Uhlenbeck [23, p. 24, lemma 4.4] and Luckhaus [16, p. 354, lemma 1]. Both are well-known facts for harmonic maps.

(3) The conclusion of Theorem 1.1 remains true if we only assume that the potential functions F satisfy the following condition: there are constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned} c_1 d^2(p,N) \le F(p) \le c_2 d^2(p,N) \qquad & \text{for } p \in N_{\delta_N}, \\ F(p) \ge c_3 \qquad & \text{for } d(p,N) \ge \delta_N. \end{aligned} \tag{1.8}$$

Notice that potential functions satisfying (1.8) are not necessarily functions of distance to the target manifold, and the corresponding minimal connecting orbits may not be straight lines. In fact, the proof of the lower bound estimate can be done exactly in the same way as Proposition 3.1, and the construction for the upper bound estimate can be done by slight modifications of that of Proposition 3.2, namely, replace $\widetilde{\alpha}_\epsilon$ by a truncated version of a minimizer β for c_0^F. We leave the details to the interested reader.

(4) Theorem 1.1 also remains true if we consider the problem on a manifold M without boundary (or with boundary ∂M) by imposing a condition of volume constraint type. In fact, in this case it is readily seen that the limiting problem reduces to the one-dimensional problem: the two phases are simply two points $p^\pm \in N^\pm$ that attain c_0^F, and they are separated by the sharp interface $\Gamma \subset M$ (with $\partial\Gamma \subset \partial M$ when $\partial M \neq \varnothing$) that is a hypersurface of constant mean curvature due to the volume constraint.

(5) Under the Neumann boundary condition, or equivalently under the condition of volume constraints,

$$\int_\Omega u\,dx = c;$$

the conclusion of Theorem 1.1 was previously proven by [9, 14, 27] when $N = \{p, q\} \subset \mathbb{R}^k$ is a set of two points, by [26] when $N \subset \mathbb{R}^2$ is the union of two concentric, simple closed curves, and by André and Shafrir [3] for N, the disjoint union of two simple closed curves in $\mathbb{R}^2$.

(6) There are very few works available in the literature that consider the above asymptotics problem associated with Dirichlet boundary conditions even in the scalar case. To the best of the authors' knowledge, an earlier work by Sternberg, Rubinstein, and Owen [20] that discussed the Gamma-convergence of the scalar-valued problem subject to a Dirichlet boundary condition seems to be the rare one.

(7) We would like to point out that the analysis of such a problem under the Dirichlet boundary condition for high-dimensional well potentials is much more delicate than the Neumann boundary condition, mentioned in (5). It is partially

because in many cases, even in higher dimensions with higher-dimensional potential wells, the Neumann boundary condition case often reduces to effectively one-dimensional problems. We should also point out that when we need to establish an optimal upper bound estimate through the construction of suitable comparison maps, we have to employ schemes to glue an almost optimal map profile from the fast transition region near the sharp interface Γ with the Dirichlet boundary data on $\partial\Omega$ in the complement of the fast transition region that contributes a much smaller scale of energy (see Proposition 3.2 and Lemma 5.1 below for more details).

(8) We would also like to briefly discuss an earlier work by Ambrosio [1, theorem 4.2, p. 468] in which he considered the Gamma-convergence and convergence of minimizers of a vector-valued singular perturbation problem for a class of potential functions g. However, the cases he considered are rather different from what we consider here. In particular, our Theorem 1.1 is independent of his theorem 4.2. For example, the boundary condition considered in his theorem 4.2 is the Neumann boundary condition. And thus in many cases, the sharp interfaces in that problem reduce again to a one-dimensional problem. It is unknown what additional properties the set of jump discontinuities for the limit map enjoy in his theorem 4.2. One would expect that the limiting problems are isoperimetric problems or partition problems with volume-type constraints, and the maps describing sharp transitions between various regions behave essentially like in a one-dimensional problem. The Dirichlet boundary conditions often capture sharp transitions between higher-dimensional wells, and thus it is more interesting and technically more involved.

Theorem 1.1 implies that for small ϵ, we can expand $\mathbf{E}(\epsilon)$ as

$$\mathbf{E}(\epsilon) = \frac{c_0^F}{\epsilon} H^{n-1}(\Gamma) + \mathbf{D}(\epsilon), \quad \mathbf{D}(\epsilon) = \frac{o(1)}{\epsilon}. \tag{1.9}$$

In particular, the leading order of $\mathbf{E}(\epsilon)$ is

$$\frac{c_0^F}{\epsilon} H^{n-1}(\Gamma).$$

The next important question is whether $\mathbf{D}(\epsilon)$ is of zero order $O(1)$ as $\epsilon \to 0$ and how to characterize $\mathbf{D}(\epsilon)$.

There are some major new difficulties to attack the asymptotics problem for the phase transition of high-dimensional wells. For example, let $\{u_\epsilon\}$ be a sequence of minimizers of $\mathbf{E}(\epsilon)$, and $u : \Omega \to \mathbb{R}^k$ be a limit of u_ϵ in suitable spaces. Then we have the following:

(1) For $k = 1$ (the case of double-point wells), it is rather easy to see that the limiting map u is completely determined by the condition $F(u) = 0$ (in fact, $u = \pm 1$).

(2) For $k \geq 2$ and double wells that are higher dimensional, since the phase fields are the manifolds $N^{\pm}$, the limiting map u cannot be determined by $F(u) = 0$ alone. In particular, the analysis of the limiting map u over $\Omega^{\pm}$,

its one-sided traces on Γ, and its role in the expansion of $\mathbf{E}(\epsilon)$ are very delicate and constitute the second main result in this paper.

It turns out that for general potential functions F vanishing on $N^{\pm}$, the shape of minimal connecting orbits ξ between N^{+} and N^{-} that attain the minimal connecting energy c_0^F given by (1.6) can be very complicated (see, for example, [26]). To characterize $\mathbf{D}(\epsilon)$, we will assume that F satisfies (1.1). For this class of potential functions, we will show in Theorem 2.1, that the minimal connecting orbit ξ attaining c_0^F is a straight-line segment. This property plays very important roles in the proof of our main theorem, Theorem 1.3: First, it asserts that the value of c_0^F equals that of the corresponding scalar double-well potential function, which is needed to show the optimal lower bound estimate in Section 4. Second, the optimal upper bound estimate in Section 5 relies heavily on the straight-line profile of the minimal connecting orbit.

Now let's briefly describe some basic properties of minimal orbits. Let

$$d_N = \inf\{|p^+ - p^-| : \ p^{\pm} \in N^{\pm}\} \quad (> 2\delta_N)$$

be the euclidean distance between N^+ and N^-, and

$$\begin{aligned} M^+ &= \{p^+ \in N^+ : \exists p^- \in N^- \text{ s.t. } |p^+ - p^-| = d_N\}, \\ M^- &= \{q^- \in N^- : \exists q^+ \in N^+ \text{ s.t. } |q^+ - q^-| = d_N\} \end{aligned} \tag{1.10}$$

be the pair of sets of points in $N^{\pm}$ achieving d_N. Theorem 2.1 shows that

$$c_0^F = 4\int_0^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda, \tag{1.11}$$

$$c_0^F = c^F(p^+, p^-) \tag{1.12}$$
$$\text{for } p^{\pm} \in N^{\pm} \iff p^{\pm} \in M^{\pm} \text{ and } |p^+ - p^-| = d_N,$$

and the corresponding minimal connecting orbit $\gamma \in H^1(\mathbb{R}, \mathbb{R}^k)$ attaining c_0^F is

$$\gamma(t) = \frac{p^+ + p^-}{2} + \alpha(t)\frac{p^+ - p^-}{|p^+ - p^-|}, \quad t \in \mathbb{R}, \tag{1.13}$$

where $\alpha \in H^1(\mathbb{R})$, with $\alpha(\pm\infty) = \pm\frac{d_N}{2}$, is a solution to the associated scalar-valued minimal connection problem (see (2.1)–(2.6)):

$$\min\left\{\int_{\mathbb{R}} \left(|\zeta'(t)|^2 + \widetilde{F}(\zeta(t))\right)dt : \zeta \in H^1(\mathbb{R}),\ \zeta(\pm\infty) = \pm\frac{d_N}{2}\right\}, \tag{1.14}$$

$$\widetilde{F}(\lambda) = \begin{cases} f\left((\frac{d_N}{2} + \lambda)^2\right) & \text{if } \lambda \le 0, \\ f\left((\frac{d_N}{2} - \lambda)^2\right) & \text{if } \lambda \ge 0. \end{cases} \tag{1.15}$$

To characterize $\mathbf{D}(\epsilon)$, we also require additional properties on the sharp interface Γ and better-prepared boundary data g_ϵ. For Γ, we assume:

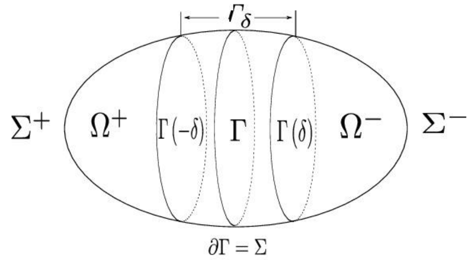

FIGURE 1.3. Domain Ω.

(A1) $\Gamma \subset \Omega$ is the unique area-minimizing hypersurface with $\partial\Gamma = \Sigma$ and is smooth.

(A2) Γ is strictly stable, i.e.,

$$\frac{d^2}{dt^2}\bigg|_{t=0} H^{n-1}(\{x + t\phi(x)\nu_\Gamma(x) : x \in \Gamma\}) > 0 \quad \forall 0 \not\equiv \phi \in C_0^\infty(\Gamma), \tag{1.16}$$

where ν_Γ is the unit normal vector field of Γ.

For g_ϵ, we assume that the behavior of g_ϵ near Σ is almost optimal in the sense that it gives the minimal connecting orbits between N^+ and N^- as $\epsilon \to 0$. In particular, the limiting map

$$g = \lim_{\epsilon \to 0} g_\epsilon : \partial\Omega \to \mathbb{R}^k$$

satisfies

$$\begin{gathered} g \in H^1(\Sigma^\pm, N^\pm), \quad g(x^\pm) \in M^\pm, \\ \text{with } |g(x^+) - g(x^-)| = d_N \text{ for } H^{n-2} \text{ a.e. } x \in \Sigma, \end{gathered} \tag{1.17}$$

where $g(x^\pm)$, for $x \in \Sigma$, is the one-sided trace value of $g \in H^1(\Sigma^\pm)$ on Σ ($= \partial\Sigma^\pm$). We note that conditions such as (1.15) above for g_ϵ also seem to be necessary in order to establish that $D(\epsilon)$ is uniformly bounded. One can easily construct a large class of such boundary data g_ϵ. In fact, let $\Omega^\pm$ be the connected components of $\Omega \setminus \Gamma$ such that

$$\partial\Omega^\pm = \Sigma^\pm \cup \Gamma.$$

Define the signed distance function to Γ on Ω by

$$\mathrm{d}_\Gamma(x) = \begin{cases} \mathrm{d}(x, \Gamma) & \text{if } x \in \Omega^+, \\ -\mathrm{d}(x, \Gamma) & \text{if } x \in \Omega^-. \end{cases}$$

For any $\delta > 0$, let

$$\Gamma_\delta = \bigcup_{-\delta \le \lambda \le \delta} \Gamma(\lambda), \quad \Gamma(\lambda) = \{x \in \Omega : \mathrm{d}_\Gamma(x) = \lambda\} \text{ for } -\delta \le \lambda \le \delta.$$

See Figure 1.3.

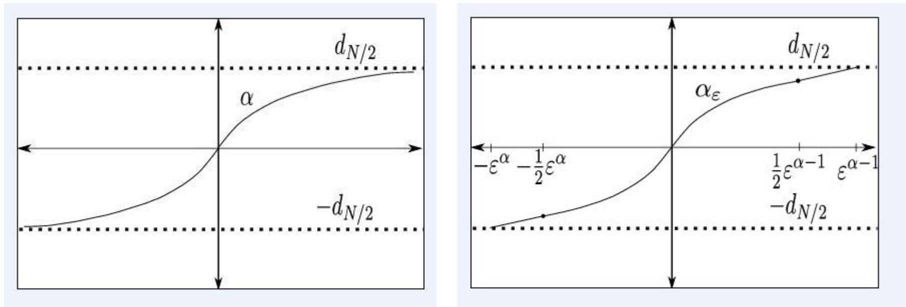

FIGURE 1.4. Graphs of both α and α_ϵ

For small $\delta_0 > 0$, let

$$\mathbf{\Phi} : \Gamma \times [-\delta_0, \delta_0] \to \Gamma_{\delta_0}$$

be a C^2-diffeomorphism such that $\mathbf{\Phi}(\cdot, t)(\Gamma) = \Gamma(t)$ for any $t \in [-\delta_0, \delta_0]$. For $\beta \in (\frac{1}{2}, 1)$ and $0 < \epsilon^\beta \le \delta_0$, let

$$\mathbf{\Psi}_\epsilon : \Omega^\pm \setminus \Gamma_{\epsilon^\beta} \to \Omega^\pm$$

be a C^2-diffeomorphism such that

$$\mathbf{\Psi}_\epsilon(\partial\Omega^\pm \setminus \Gamma_{\epsilon^\beta}) = \Sigma^\pm, \quad \mathbf{\Psi}_\epsilon\big|_{\Gamma(\pm\epsilon^\beta)} = \mathbf{\Phi}^{-1}(\cdot, \pm\epsilon^\beta),$$
$$\text{and} \quad |\nabla \mathbf{\Psi}_\epsilon - \mathrm{Id}| \le C\epsilon^\beta.$$

For a minimizer α of (1.14), let

$$\alpha_\epsilon : [-\epsilon^{\beta-1}, \epsilon^{\beta-1}] \to \left[-\frac{d_N}{2}, \frac{d_N}{2}\right]$$

be a truncation of α defined by (see also (A.7)):

$$\alpha_\epsilon(t) = \begin{cases} \big(\frac{2t+2\epsilon^{\beta-1}}{\epsilon^{\beta-1}}\big)\alpha(-\frac{\epsilon^{\beta-1}}{2}) + \big(\frac{\epsilon^{\beta-1}+2t}{\epsilon^{\beta-1}}\big)\frac{d_N}{2} & \text{if } -\epsilon^{\beta-1} \le t \le -\frac{\epsilon^{\beta-1}}{2}, \\ \alpha(t) & \text{if } -\frac{\epsilon^{\beta-1}}{2} \le t \le \frac{\epsilon^{\beta-1}}{2}, \\ \frac{2t}{\epsilon^{\beta-1}}\alpha(\frac{\epsilon^{\beta-1}}{2}) + \big(\frac{2t-\epsilon^{\beta-1}}{\epsilon^{\beta-1}}\big)\frac{d_N}{2} & \text{if } \frac{\epsilon^{\beta-1}}{2} \le t \le \epsilon^{\beta-1}. \end{cases} \tag{1.18}$$

See Figure 1.4.

For g given by (1.17), define $g_\epsilon : \partial\Omega \to \mathbb{R}^k$ by

$$g_\epsilon(x) = \begin{cases} g(\mathbf{\Psi}_\epsilon(x)) & \text{if } x \in \partial\Omega^\pm \setminus \Gamma_{\epsilon^\beta}, \\ \frac{g(x_*^+)+g(x_*^-)}{2} + \alpha_\epsilon\big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\big)\frac{g(x_*^+)-g(x_*^-)}{|g(x_*^+)-g(x_*^-)|} & \text{if } x \in \partial\Omega \cap \Gamma_{\epsilon^\beta}, \end{cases} \tag{1.19}$$

where $x_* \in \Sigma\ (= \partial\Gamma)$ is uniquely determined by

$$\mathbf{\Phi}(x_*, \mathrm{d}_\Gamma(x)) = x, \quad x \in \partial\Omega \cap \Gamma_{\epsilon^\beta}.$$

For these boundary values g_ϵ, we will show the uniform boundedness of $D(\epsilon)$.

Throughout the paper, we will let $v(x^{\pm})$, for $x \in \Gamma$, denote the one-sided trace value of v on $\Gamma = \partial\Omega^{\pm} \cap \Omega$ for any map $v \in H^1(\Omega^{\pm})$. For g given by (1.17), define the space $\mathbf{A}$ by

$$\mathbf{A} \equiv \left\{ v \in H^1(\Omega^{\pm}, N^{\pm}) : v\big|_{\partial\Omega} = g,\ |v(x^+) - v(x^-)| = d_N \text{ a.e. } x \in \Gamma \right\}. \tag{1.20}$$

Now we state our second main theorem.

THEOREM 1.3. *Assume* $F(p) = F(d^2(p, N))$ *satisfies* (1.1), Γ *satisfies* (A1) *and* (A2)*, and* $g_\epsilon : \partial\Omega \to \mathbb{R}^k$ *is given by* (1.19). *If* $\mathbf{A}$ *is nonempty and there exists at least one minimizer* $v \in \mathbf{A}$ *attaining*

$$\mathbf{D} = \inf\left\{ \int_{\Omega^+} |\nabla v|^2 \, dx + \int_{\Omega^-} |\nabla v|^2 \, dx : v \in \mathbf{A} \right\} \tag{1.21}$$

that satisfies $v(x^{\pm}) \in H^1(\Gamma, N^{\pm})$*, then*

$$\mathbf{E}(\epsilon) = \frac{c_0^F}{\epsilon} H^{n-1}(\Gamma) + \mathbf{D} + o(1). \tag{1.22}$$

Furthermore, if $\{u_\epsilon\}$ *is a sequence of minimizers of* $\mathbf{E}(\epsilon)$*, then there exists* $u \in \mathbf{A}$ *attaining the minimum of* $\mathbf{D}$ *such that after taking possible subsequences,* u_ϵ *converges to* u *in* $L^1(\Omega, \mathbb{R}^k)$.

We would like to make a few remarks on the problem of minimizing harmonic maps under the Dirichlet and partially constrained boundary conditions as in (1.20).

Remark 1.4.

(1) By the direct method in the calculus of variations, we know that $\mathbf{D}$ is finite and can be attained whenever the space $\mathbf{A}$ is nonempty. Furthermore, we can show by slightly modifying the argument of [12] that $\mathbf{A} \neq \varnothing$ provided that

$$\Pi_1(N^+) = \Pi_1(N^-) = \{0\}. \tag{1.23}$$

(2) In a forthcoming article [15], we will study the boundary regularity of minimizing harmonic maps $u \in \mathbf{A}$ that attain the minimum of $\mathbf{D}$. We will prove that for $n \geq 3$, if

$$g \in C^\infty(\Sigma^+ \cup \Sigma, N^+) \cap C^\infty(\Sigma^- \cup \Sigma, N^-),$$

then there exist closed subsets $\mathcal{S}_1$ and $\mathcal{S}_2^{\pm}$, where $\mathcal{S}_1 \Subset \Gamma$ is discrete for $n = 3$ and $\dim_{\mathbb{H}}(\mathcal{S}_1) \leq n - 3$ for $n > 3$, and $\mathcal{S}_2^{\pm} \Subset \Omega^{\pm}$ with $\dim_{\mathbb{H}}(\mathcal{S}_2^{\pm}) \leq n - 3$ such that

$$u \in C^\infty\big(\overline{\Omega^+} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2^+), N^+\big) \cap C^\infty\big(\overline{\Omega^-} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2^-), N^-\big).$$

(3) In [15], we will also show that if $M^{\pm} \subset N^{\pm}$ are Lipschitz submanifolds and

$$\Pi_1(M^+) = \Pi_1(M^-) = \{0\}, \tag{1.24}$$

then any minimizing harmonic map $u \in \mathbf{A}$ attaining the mimimum of $\mathbf{D}$ satisfies the property that its one-sided traces $u(x^{\pm})$ on Γ satisfy $u(x^{\pm}) \in H^1(\Gamma, M^{\pm})$.

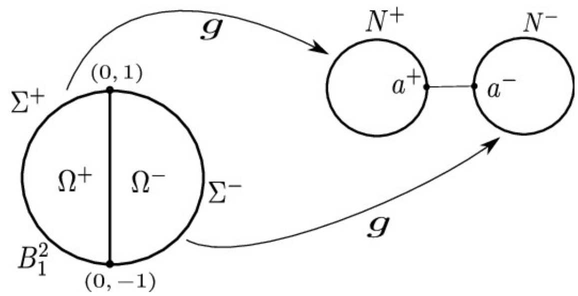

FIGURE 1.5. Two-dimensional example.

(4) For $n = 2$, $\mathbf{A}$ can be empty and $\mathbf{D}(\epsilon)$ may be unbounded. For example, let $N^{\pm} \approx \mathbb{S}^1$ be two disjoint unit circles in $\mathbb{R}^2$, $\Omega = B_1$ be the unit ball of $\mathbb{R}^2$, $\Sigma = \{(0, \pm 1)\}$, $\Sigma^+ = \partial B_1 \cap \{(x_1, x_2) : x_1 < 0\}$, and $\Sigma^- = \partial B_1 \cap \{(x_1, x_2) : x_1 > 0\}$. Then $\Gamma = \{(0, x_2) : |x_2| < 1\}$, $\Omega^+ = B_1 \cap \{(x_1, x_2) : x_1 < 0\}$, $\Omega^- = B_1 \cap \{(x_1, x_2) : x_1 > 0\}$, and the pair of minimal sets $M^{\pm} = \{a^{\pm}\} \subset N^{\pm}$. If we choose a boundary data $g \in H^1(\Sigma^{\pm}, N^{\pm})$ such that

$$g((0, \pm 1)^+) = a^+, \quad g((0, \pm 1)^-) = a^-, \quad \deg(g, \partial\Omega^{\pm}) = d \neq 0,$$

then $\mathbf{A} = \varnothing$, and by the work of Brezis, Bethuel, and Hélein [4] we have

$$\mathbf{D}(\epsilon) \approx 2\pi |d| \log\left(\frac{1}{\epsilon}\right).$$

See Figure 1.5.

The rest of the paper is organized as follows. In Section 2, we will classify the minimal connecting orbits. In Section 3, we will prove Theorem 1.1. In Section 4, we will establish the lower bound estimate of $\mathbf{E}(\epsilon)$. In Section 5, we will show the upper bound estimate of $\mathbf{E}(\epsilon)$ by explicit constructions. In Section 6, we will prove Theorem 1.3. In Appendix A, we will provide some needed properties for the semidistance function $d^F(\cdot, \cdot)$ that seem to be standard to experts. For the convenience of readers, we will provide a list of notations in Appendix B.

2 Minimal Connecting Orbits

In this section, we will assume the potential function $F : \mathbb{R}^k \to \mathbb{R}_+$ satisfies (1.1), and will prove an explicit formula for c_0^F and characterize the minimal connecting orbits, which play crucial roles in the optimal estimate of upper bounds of $\mathbf{E}(\epsilon)$.

First we review some properties of the associated, scalar-valued minimal connection problem:

$$c_0^{\widetilde{F}} := \min\left\{\int_{\mathbb{R}} \left(|\zeta'(t)|^2 + \widetilde{F}(\zeta(t))\right) dt : \zeta \in H^1(\mathbb{R}),\ \zeta(\pm\infty) = \pm\frac{d_N}{2}\right\}, \tag{2.1}$$

where $\widetilde{F} : \mathbb{R} \to \mathbb{R}_+$ is the even function induced from F and defined by (1.15) in Section 1. Then it is well-known [9, 26] that the following statements are true:

(1) It holds that

$$c_0^{\widetilde{F}} = 4\int_0^{\frac{d_N}{2}} \sqrt{\widetilde{F}(\lambda)}\, d\lambda \quad \left(= 4\int_0^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda\right). \tag{2.2}$$

(2) There exists a minimizer

$$\alpha \in C^\infty\left(\mathbb{R}, \left(-\frac{d_N}{2}, \frac{d_N}{2}\right)\right)$$

of (2.1) that is odd and strictly monotone increasing, and satisfies

$$-\alpha''(t) + \frac{1}{2}\widetilde{F}'(\alpha(t)) = 0,\ \ t \in \mathbb{R}, \quad \alpha(\pm\infty) = \pm\frac{d_N}{2}, \tag{2.3}$$

$$c_0^{\widetilde{F}} = \int_{\mathbb{R}} \left(|\alpha'(t)|^2 + \widetilde{F}(\alpha(t))\right) dt, \tag{2.4}$$

$$\alpha'(t) = \sqrt{\widetilde{F}(\alpha(t))} \quad \forall t \in \mathbb{R}, \tag{2.5}$$

$$\begin{aligned} |\alpha'(t)| + \left|\alpha(t) + \frac{d_N}{2}\right| &\le C_1 e^{C_2 t} \qquad \text{as } t \to -\infty, \\ |\alpha'(t)| + \left|\alpha(t) - \frac{d_N}{2}\right| &\le C_3 e^{-C_4 t} \quad \text{as } t \to +\infty, \end{aligned} \tag{2.6}$$

for some $C_i > 0$ $(1 \le i \le 4)$.

Now we have the following:

THEOREM 2.1. *Consider the following two statements*:

(i) $c_0^F = 4\int_0^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda$.
(ii) $(p^+, p^-) \in N^+ \times N^-$ *and* $\xi \in H^1(\mathbb{R}, \mathbb{R}^k)$, *with* $\xi(\pm\infty) = p^\pm$, *satisfy*

$$c_0^F = c^F(p^+, p^-) = \int_{\mathbb{R}} \left(|\xi'(t)|^2 + F(\xi(t))\right) dt. \tag{2.7}$$

Statements (i) *and* (ii) *hold if and only if*

$$(p^+, p^-) \in M^+ \times M^- \quad \textit{and} \quad |p^+ - p^-| = d_N, \tag{2.8}$$

$$\xi(t) = \frac{p^+ + p^-}{2} + \alpha(t)\frac{p^+ - p^-}{|p^+ - p^-|} \quad \forall t \in \mathbb{R}, \tag{2.9}$$

where $\alpha \in H^1(\mathbb{R}, \mathbb{R})$, *with* $\alpha(\pm\infty) = \pm\frac{d_N}{2}$, *is the minimizer for* $c_0^{\widetilde{F}}$ *given by* (2.1).

PROOF.

(i) For $(p^+, p^-) \in N^+ \times N^-$ and $\xi \in H^1(\mathbb{R}, \mathbb{R}^k)$ with $\xi(\pm\infty) = p^\pm$, let

$$-\infty < t_1 \le t_2 < t_3 \le t_4 < t_5 \le t_6 < +\infty$$

be such that

$$\begin{cases} \mathrm{d}(\xi(t), N^-) \le \delta_N & \text{for } -\infty \le t \le t_1,\ \mathrm{d}(\xi(t_1), N^-) = \delta_N, \\ \delta_N \le \mathrm{d}(\xi(t), N^-) \le \frac{d_N}{2} & \text{for } t_2 \le t \le t_3, \\ & \quad \mathrm{d}(\xi(t_2), N^-) = \delta_N,\ \mathrm{d}(\xi(t_3), N^-) = \frac{d_N}{2}, \\ \delta_N \le \mathrm{d}(\xi(t), N^+) \le \frac{d_N}{2} & \text{for } t_4 \le t \le t_5, \\ & \quad \mathrm{d}(\xi(t_4), N^+) = \frac{d_N}{2},\ \mathrm{d}(\xi(t_5), N^+) = \delta_N, \\ \mathrm{d}(\xi(t), N^+) \le \delta_N & \text{for } t_6 \le t \le +\infty;\ \mathrm{d}(\xi(t_6), N^+) = \delta_N. \end{cases} \tag{2.10}$$

Observe that

$$|\xi'(t)| \ge \left|\frac{d}{dt}\big(\mathrm{d}(\xi(t))\big)\right| \quad \forall t \in \mathbb{R}$$

where

$$\mathrm{d}(\xi(t)) = \begin{cases} \mathrm{d}(\xi(t), N^-) & \text{if } \mathrm{d}(\xi(t), N^-) \le \frac{d_N}{2}, \\ \mathrm{d}(\xi(t), N^+) & \text{if } \mathrm{d}(\xi(t), N^+) \le \frac{d_N}{2}. \end{cases}$$

Applying both the Hölder inequality and Federer's co-area formula, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \big(|\xi'(t)|^2 + F(\xi(t))\big) dt \\ &\ge 2 \int_{\mathbb{R}} \sqrt{f\big(\mathrm{d}^2(\xi(t))\big)}|\xi'(t)| dt \\ &\quad \ge 2\left\{\int_{-\infty}^{t_1} + \int_{t_2}^{t_3} + \int_{t_4}^{t_5} + \int_{t_6}^{+\infty}\right\} \sqrt{f\big(\mathrm{d}^2(\xi(t))\big)}|\xi'(t)| dt \\ &\ge 2\left\{\int_{-\infty}^{t_1} + \int_{t_2}^{t_3} + \int_{t_4}^{t_5} + \int_{t_6}^{+\infty}\right\} \sqrt{f\big(\mathrm{d}^2(\xi(t))\big)}\left|\frac{d}{dt}\big(\mathrm{d}(\xi(t))\big)\right| dt \\ &= 2\int_0^{\delta_N} \sqrt{f(\lambda^2)}\, H^0\big(\{t \in (-\infty, t_1] : \mathrm{d}(\xi(t), N^-) = \lambda\}\big) d\lambda \\ &\quad + 2\int_0^{\delta_N} \sqrt{f(\lambda^2)}\, H^0\big(\{t \in [t_6, +\infty) : \mathrm{d}(\xi(t), N^+) = \lambda\}\big) d\lambda \\ &\quad + 2\int_{\delta_N}^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, H^0\big(\{t \in [t_2, t_3] : \mathrm{d}(\xi(t), N^-) = \lambda\}\big) d\lambda \\ &\quad + 2\int_{\delta_N}^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, H^0\big(\{t \in [t_4, t_5] : \mathrm{d}(\xi(t), N^+) = \lambda\}\big) d\lambda \end{aligned}$$

$$\ge 4\left[\int_0^{\delta_N} \sqrt{f(\lambda^2)}\, d\lambda + \int_{\delta_N}^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda\right] = 4\int_0^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda. \tag{2.11}$$

Taking the infimum over all such ξ and all $(p^+, p^-) \in N^+ \times N^-$ yields

$$c_0^F \geq 4 \int_0^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda.$$

To show the other direction of (2.7), let $(p_0^+, p_0^-) \in M^+ \times M^-$ be such that $|p_0^+ - p_0^-| = d_N$. Let $\alpha \in C^\infty(\mathbb{R}, [-\frac{d_N}{2}, \frac{d_N}{2}])$ be a minimizer of (2.1). Define

$$\xi_0(t) = \frac{p_0^+ + p_0^-}{2} + \alpha(t) \frac{p_0^+ - p_0^-}{|p_0^+ - p_0^-|}, \quad t \in \mathbb{R}. \tag{2.12}$$

Then $\xi_0 \in H^1(\mathbb{R}, \mathbb{R}^k)$ satisfies $\xi_0(\pm\infty) = p_0^\pm$. Hence we have

$$c_0^F \leq \int_{\mathbb{R}} \big(|\xi_0'(t)|^2 + F(\xi_0(t))\big) dt.$$

Since

$$|\xi_0'(t)| = |\alpha'(t)|, \quad t \in \mathbb{R},$$

and

$$\mathrm{d}(\xi_0(t)) = \begin{cases} \frac{d_N}{2} + \alpha(t) & \text{if } -\frac{d_N}{2} \leq \alpha(t) \leq 0, \\ \frac{d_N}{2} - \alpha(t) & \text{if } 0 \leq \alpha(t) \leq \frac{d_N}{2}, \end{cases}$$

equation (2.4) implies

$$\begin{aligned} c_0^F \leq \int_{\mathbb{R}} \big(|\xi_0'(t)|^2 + F(\xi_0(t))\big) dt &= \int_{\mathbb{R}} \big(|\alpha'(t)|^2 + \widetilde{F}(\alpha(t))\big) dt \\ &= 4 \int_0^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda. \end{aligned}$$

Hence we prove (i).

(ii) *If part*: For $(p_0^+, p_0^-) \in M^+ \times M^-$ with $|p_0^+ - p_0^-| = d_N$, if

$$\alpha \in H^1\left(\mathbb{R}, \left[-\frac{d_N}{2}, \frac{d_N}{2}\right]\right) \quad \text{with } \alpha(\pm\infty) = \pm\frac{d_N}{2}$$

is a minimizer of (2.1) and $\xi_0 \in H^1(\mathbb{R}, \mathbb{R}^k)$ is given by (2.12), then the second part of the proof of (i) gives

$$c_0^F = c^F(p_0, q_0) = \int_{\mathbb{R}} \big(|\xi_0'(t)|^2 + F(\xi_0(t))\big) dt.$$

Only if part: Suppose $(p^+, p^-) \in N^+ \times N^-$ and $\xi \in H^1(\mathbb{R}, \mathbb{R}^k)$, with $\xi(\pm\infty) = p^\pm$, satisfies (2.8). Let

$$-\infty < t_1 \leq t_2 < t_3 \leq t_4 < t_5 \leq t_6 < +\infty$$

be given by (2.11). Set

$$p_1 = \Pi(\xi(t_1)) \in N^- \quad \text{and} \quad q_1 = \Pi(\xi(t_6)) \in N^+.$$

Denote $\mathrm{d}(\xi) = \mathrm{d}(\xi, N)$ and define $\widetilde{\xi} : \mathbb{R} \to \mathbb{R}^k$ by

$$\widetilde{\xi}(t) = \begin{cases} p_1 + \mathrm{d}(\xi(t))\frac{\xi(t_1)-p_1}{|\xi(t_1)-p_1|} & \text{if } -\infty < t \le t_1, \\ \xi(t) & \text{if } t_1 \le t \le t_6, \\ q_1 + \mathrm{d}(\xi(t))\frac{\xi(t_6)-q_1}{|\xi(t_6)-q_1|} & \text{if } t_6 \le t < +\infty. \end{cases}$$

Then $\widetilde{\xi} \in H^1(\mathbb{R}, \mathbb{R}^k)$ satisfies

$$\widetilde{\xi}(-\infty) = p_1 \in N^- \quad \text{and} \quad \widetilde{\xi}(+\infty) = q_1 \in N^+.$$

Hence

$$\begin{aligned} c_0^F &\le \int_{\mathbb{R}} \left(|\widetilde{\xi}'(t)|^2 + F(\widetilde{\xi}(t))\right) dt \\ &= \int_{-\infty}^{t_1} \left[\left|\frac{d}{dt}\left(\mathrm{d}(\xi(t))\right)\right|^2 + f\left(\mathrm{d}^2(\xi(t))\right)\right] dt \\ &\quad + \int_{t_1}^{t_6} \left[|\xi'(t)|^2 + f\left(\mathrm{d}^2(\xi(t))\right)\right] dt \\ &\quad + \int_{t_6}^{\infty} \left[\left|\frac{d}{dt}\left(\mathrm{d}(\xi(t))\right)\right|^2 + f\left(\mathrm{d}^2(\xi(t))\right)\right] dt. \end{aligned} \tag{2.13}$$

On the other hand, we have that for any $t \in (-\infty, t_1] \cup [t_6, +\infty)$,

$$\begin{aligned} \xi(t) &= \Pi(\xi(t)) + \mathrm{d}(\xi(t))\nu(t), \\ \nu(t) &= \frac{\xi(t) - \Pi(\xi(t))}{|\xi(t) - \Pi(\xi(t))|} \perp T_{\Pi(\xi(t))}N. \end{aligned}$$

Direct calculations then give

$$\xi'(t) = \frac{d}{dt}\left(\Pi(\xi(t))\right) + \frac{d}{dt}(\mathrm{d}(\xi(t))\nu(t) + \mathrm{d}(\xi(t))\nu'(t).$$

Since

$$\left\langle \frac{d}{dt}\left(\Pi(\xi(t))\right), \nu(t)\right\rangle = \langle \nu'(t), \nu(t)\rangle = 0,$$

we have

$$\begin{aligned} |\xi'(t)|^2 &= \left|\frac{d}{dt}\left(\Pi(\xi(t))\right)\right|^2 + \left|\frac{d}{dt}\left(\mathrm{d}(\xi(t))\right)\right|^2 + \mathrm{d}^2(\xi(t))|\nu'(t)|^2 \\ &\quad - 2\mathrm{d}(\xi(t))A\left(\Pi(\xi(t))\right)\left(\frac{d}{dt}\left(\Pi(\xi(t))\right), \frac{d}{dt}\left(\Pi(\xi(t))\right)\right) \\ &\ge \left|\frac{d}{dt}\left(\mathrm{d}(\xi(t))\right)\right|^2 + (1 - C\delta_N)\left|\frac{d}{dt}\left(\Pi(\xi(t))\right)\right|^2 + \mathrm{d}^2(\xi(t))|\nu'(t)|^2 \\ &\ge \left|\frac{d}{dt}\left(\mathrm{d}(\xi(t))\right)\right|^2 + \frac{1}{2}\left|\frac{d}{dt}\left(\Pi(\xi(t))\right)\right|^2 + \mathrm{d}^2(\xi(t))|\nu'(t)|^2 \end{aligned}$$

provided $\delta_N \le \frac{1}{2C}$, where $A(\Pi(\xi(t)))(\cdot,\cdot)$ is the second fundamental form of N at $\Pi(\xi(t))$. Hence we have

$$
\begin{aligned}
c_0^F &= \int_{\mathbb{R}} \big(|\xi'(t)|^2 + F((\xi(t))\big)dt \\
&\ge \left\{\int_{-\infty}^{t_1} + \int_{t_1}^{t_6} + \int_{t_6}^{\infty}\right\}\big(|\xi'(t)|^2 + F(\xi(t))\big)dt \\
&\ge \int_{-\infty}^{t_1}\left[\left|\frac{d}{dt}\big(\mathrm{d}(\xi(t))\big)\right|^2 + f\big(\mathrm{d}^2(\xi(t))\big)\right]dt \\
&\quad + \int_{t_1}^{t_6}\big[|\xi'(t)|^2 + f\big(\mathrm{d}^2(\xi(t))\big)\big]dt \\
&\quad + \int_{t_6}^{\infty}\left[\left|\frac{d}{dt}\big(\mathrm{d}(\xi(t))\big)\right|^2 + f\big(\mathrm{d}^2(\xi(t))\big)\right]dt \\
&\quad + \left\{\int_{-\infty}^{t_1} + \int_{t_6}^{\infty}\right\}\left(\frac{1}{2}\left|\frac{d}{dt}\big(\Pi(\xi(t))\big)\right|^2 + \mathrm{d}^2(\xi(t))|\nu'(t)|^2\right)dt.
\end{aligned} \tag{2.14}
$$

Combining (2.13) with (2.14), we conclude that

$$\left|\frac{d}{dt}\big(\Pi(\xi(t))\big)\right|^2 = |\nu'(t)|^2 = 0 \quad \text{on } (-\infty, t_1] \cup [t_6, +\infty).$$

Hence $p_1 = p^-$, $q_1 = p^+$, and $\xi((-\infty, t_1]) \subset (N^-)_{\delta_N}$ and $\xi([t_6, +\infty)) \subset (N^+)_{\delta_N}$ are straight-line segments. In particular, we have

$$\xi(t) \equiv \widetilde{\xi}(t) \quad \forall t \in (-\infty, t_1] \cup [t_6, +\infty).$$

Recall that the first part of proof of (i) yields

$$\left\{\int_{-\infty}^{t_1} + \int_{t_6}^{\infty}\right\}\big(|\xi'(t)|^2 + F(\xi(t))\big) \ge 4\int_0^{\delta_N} \sqrt{f(\lambda^2)}\,d\lambda \tag{2.15}$$

and

$$
\begin{aligned}
\int_{t_1}^{t_6}\big(|\xi'(t)|^2 + F(\xi(t))\big)dt &\ge \left\{\int_{t_2}^{t_3} + \int_{t_4}^{t_5}\right\}\big(|\xi'(t)|^2 + f(\mathrm{d}^2(\xi(t)))\big)dt \\
&\ge 2\left\{\int_{t_2}^{t_3} + \int_{t_4}^{t_5}\right\}\sqrt{f\big(\mathrm{d}^2(\xi(t))\big)}|\xi'(t)|dt \\
&\ge 2\left\{\int_{t_2}^{t_3} + \int_{t_4}^{t_5}\right\}\sqrt{f\big(\mathrm{d}^2(\xi(t))\big)}\left|\frac{d}{dt}\big(\mathrm{d}(\xi(t))\big)\right|dt \\
&\ge 4\int_{\delta_N}^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\,d\lambda.
\end{aligned} \tag{2.16}
$$

Combining (2.15) and (2.16) with (2.14) yields

$$t_1 = t_2, \quad t_3 = t_4, \quad t_5 = t_6,$$

and

$$|\xi'(t)| = \left|\frac{d}{dt}\big(\mathrm{d}(\xi(t))\big)\right| \quad \text{on } [t_1, t_3] \cup [t_3, t_6].$$

This is possible only if

$$\begin{cases} \xi(t) - p^- \perp T_{p^-}N^- \text{ and } \mathrm{d}(\xi(t)) = |\xi(t) - p^-| & \text{for } t_1 \le t \le t_3, \\ \xi(t) - p^+ \perp T_{p^+}N^+ \text{ and } \mathrm{d}(\xi(t)) = |\xi(t) - p^+| & \text{for } t_3 \le t \le t_6. \end{cases}$$

Thus $\xi(\mathbb{R})$ is a straight line connecting p^- and p^+. Moreover, since

$$d_N \le |p^+ - p^-| \le |p^- - \xi(t_3)| + |\xi(t_3) - p^+| = \frac{d_N}{2} + \frac{d_N}{2} = d_N,$$

we conclude that $(p^-, p^+) \in M^- \times M^+$ satisfies $|p^- - p^+| = d_N$. This proves (ii).

□

3 Proof of Theorem 1.1

This section is devoted to the proof of the asymptotic formula (1.5). First, we will establish the lower bound. Then we will construct a map whose energy matches the lower bound.

For the part of the lower bound, we have the following:

PROPOSITION 3.1. *Under the assumptions of Theorem* 1.1*, if* $u_\epsilon \in H^1(\Omega, \mathbb{R}^k)$ *satisfies* $u_\epsilon = g_\epsilon$ *on* $\partial\Omega$*, then*

$$\liminf_{\epsilon \downarrow 0} \int_\Omega \left(\epsilon |\nabla u_\epsilon|^2 + \frac{1}{\epsilon} F(u_\epsilon)\right) dx \ge c_0^F H^{n-1}(\Gamma). \tag{3.1}$$

PROOF. Without loss of generality, we may assume that

$$\Lambda := \liminf_{\epsilon \downarrow 0} \int_\Omega \left(\epsilon |\nabla u_\epsilon|^2 + \frac{1}{\epsilon} F(u_\epsilon)\right) dx < +\infty.$$

For any fixed $\delta \in [\frac{\delta_N}{2}, \delta_N]$, define the two disjoint measurable subsets

$$\Omega^\pm_{\epsilon,\delta} = \{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^\pm) < \delta\}$$

and the subset

$$E_{\epsilon,\delta} = \Omega \setminus (\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}) \equiv \{x \in \Omega : \mathrm{d}(u_\epsilon(x), N) \ge \delta\}.$$

By (1.8), there exists $C > 0$ depending on N and F such that

$$F(u_\epsilon(x)) \ge C > 0 \quad \text{for all } x \in E_{\epsilon,\delta}.$$

Hence

$$|E_{\epsilon,\delta}| \le \frac{1}{C}\int_\Omega F(u_\epsilon)dx \le \frac{\Lambda\epsilon}{C} \to 0 \quad \text{as } \epsilon \to 0. \tag{3.2}$$

Now we need three claims.

Claim 1. There exist $\delta_0 \in [\frac{\delta_N}{2}, \delta_N]$ and $C_1 > 0$ such that

$$H^{n-1}(\partial\Omega^+_{\epsilon,\delta_0} \cap \Omega) + H^{n-1}(\partial\Omega^-_{\epsilon,\delta_0} \cap \Omega) \le C_1. \tag{3.3}$$

To prove (3.3), first observe that

$$|\nabla \mathrm{d}(p, N)| = 1 \quad \text{for } p \in N^\pm_{\delta_N}.$$

This, (1.8), the Cauchy-Schwarz inequality, and Federer's co-area formula imply

$$\begin{aligned}
\Lambda &\ge 2 \int_{\substack{\{x\in\Omega:\mathrm{d}(u_\epsilon(x),N^+)\le\delta_N \\ \text{or } \mathrm{d}(u_\epsilon(x),N^-)\le\delta_N\}}} \sqrt{F(u_\epsilon(x))}|\nabla u_\epsilon(x)|dx \\
&\ge C \int_{\substack{\{x\in\Omega:\mathrm{d}(u_\epsilon(x),N^+)\le\delta_N \\ \text{or } \mathrm{d}(u_\epsilon(x),N^-)\le\delta_N\}}} \mathrm{d}(u_\epsilon(x), N)|\nabla u_\epsilon(x)|dx \\
&\ge C\delta_N \int_{\substack{\{x\in\Omega:\frac{\delta_N}{2}\le\mathrm{d}(u_\epsilon(x),N^+)\le\delta_N \\ \text{or } \frac{\delta_N}{2}\le\mathrm{d}(u_\epsilon(x),N^-)\le\delta_N\}}} \big|\nabla(\mathrm{d}(u_\epsilon(x), N))\big|dx \\
&= C\delta_N \int_{\frac{\delta_N}{2}}^{\delta_N} H^{n-1}\big(\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^+) = \delta\} \\
&\qquad\qquad \cup \{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^-) = \delta\}\big)d\delta.
\end{aligned}$$

Thus, by Fubini's theorem, there exists $\delta_0 \in [\frac{\delta_N}{2}, \delta_N]$ such that (3.3) holds and Claim 1 follows.

From (3.2) and (3.3), we may assume that there are two subsets $E^\pm \subset \Omega$ with finite perimeters such that, after passing to possible subsequences,

$$\chi_{\Omega^\pm_{\epsilon,\delta_0}} \to \chi_{E^\pm} \quad \text{as } \epsilon \to 0,$$

weakly in $\mathrm{BV}(\mathbb{R}^n)$ and strongly in $L^1(\mathbb{R}^n)$. Moreover, it is easy to see that

$$|\Omega \setminus (E^+ \cup E^-)| = 0, \quad |E^+ \cap E^-| = 0.$$

For $\epsilon > 0$, let $\phi_\epsilon : \overline{\Omega} \to \mathbb{R}$ be defined by

$$\phi_\epsilon(x) = d^F_{N^-}(u_\epsilon(x)), \quad x \in \overline{\Omega}, \tag{3.4}$$

where $d^F_{N^-}(\cdot)$ is given by (A.2). Then Proposition A.3 implies

$$|\nabla\phi_\epsilon(x)| \le 2\sqrt{F(u_\epsilon(x))}|\nabla u_\epsilon(x)| \quad \text{a.e. } x \in \Omega. \tag{3.5}$$

This and the Cauchy-Schwarz inequality yield

$$\int_\Omega |\nabla\phi_\epsilon| dx \le \int_\Omega \left(\epsilon|\nabla u_\epsilon|^2 + \frac{1}{\epsilon} F(u_\epsilon)\right) dx \le \Lambda + 1. \tag{3.6}$$

Proposition A.2 and (1.2) imply that

$$\phi_\epsilon(x) = 0 \quad \forall x \in \Sigma^-_{\epsilon^\alpha}, \tag{3.7}$$

where $\Sigma^-_{\epsilon^\alpha}$ is defined in Section 1. Since

$$H^{n-1}(\Sigma^-_{\epsilon^\alpha}) \ge C_0 > 0 \quad \text{for all small } \epsilon > 0,$$

from this and (3.7) we can apply the Poincaré inequality to get

$$\int_\Omega |\phi_\epsilon| dx \le C \int_\Omega |\nabla\phi_\epsilon| dx \le C(\Lambda + 1).$$

Hence $\{\phi_\epsilon\} \subseteq \mathrm{BV}(\Omega)$ is bounded, and we may assume that there exists a nonnegative $\phi \in \mathrm{BV}(\Omega)$ such that $\phi_\epsilon \to \phi$ weakly in $\mathrm{BV}(\Omega)$ and strongly in $L^1(\Omega)$. By the lower semicontinuity, we have

$$\begin{aligned} \Lambda &= \liminf_{\epsilon\downarrow 0} \int_\Omega \left(\epsilon|\nabla u_\epsilon|^2 + \frac{1}{\epsilon} F(u_\epsilon)\right) dx \\ &\ge \liminf_{\epsilon\downarrow 0} \int_\Omega |\nabla\phi_\epsilon| dx \ge |D\phi|(\Omega). \end{aligned} \tag{3.8}$$

Now we need the following:

Claim 2. For the function ϕ obtained as above, we have

$$\begin{cases} \phi(x) = 0 & \text{for a.e. } x \in E^-, \\ \phi(x) \ge c_0^F & \text{for a.e. } x \in E^+. \end{cases} \tag{3.9}$$

To prove (3.9), we observe that Proposition A.2 implies

$$d^F_{N^-}\big(\Pi(u_\epsilon(x))\big) = 0 \quad \text{for any } x \in \Omega^-_{\epsilon,\delta_0}.$$

Hence, by Fatou's lemma, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\phi|^2 \chi_{E^-} dx &\le \liminf_{\epsilon\downarrow 0} \int_{\mathbb{R}^n} |\phi_\epsilon|^2 \chi_{\Omega^-_{\epsilon,\delta_0}} dx \\ &= \liminf_{\epsilon\downarrow 0} \int_{\mathbb{R}^n} \left|\phi_\epsilon - d^F_{N^-}(\Pi(u_\epsilon))\right|^2 \chi_{\Omega^-_{\epsilon,\delta_0}} dx \le \end{aligned}$$

$$\le C \left\| \nabla d^F_{N^-}(\cdot) \right\|_{L^\infty(N_{\delta_0})} \liminf_{\epsilon \downarrow 0} \int\limits_{\Omega^-_{\epsilon,\delta_0}} |u_\epsilon - \Pi(u_\epsilon)|^2 \, dx$$

$$\le C \liminf_{\epsilon \downarrow 0} \int\limits_{\Omega} F(u_\epsilon) dx = 0.$$

This implies $\phi(x) = 0$ for a.e. $x \in E^-$.

It is easy to see from the definition of d_0^F and Proposition A.1 that

$$d^F_{N^-}\big(\Pi(u_\epsilon(x))\big) \ge c_0^F \quad \text{for any } x \in \Omega^+_{\epsilon,\delta_0}.$$

Similar to the above argument, we also have that for any $0 < \eta < c_0^F$,

$$\eta \left| \{ x \in \Omega^+_{\epsilon,\delta_0} : \phi_\epsilon(x) \le c_0^F - \eta \} \right|$$

$$\le \int\limits_{\mathbb{R}^n} \left| d^F_{N^-}(u_\epsilon(x)) - d^F_{N^-}\big(\Pi(u_\epsilon(x))\big) \right| \chi_{\Omega^+_{\epsilon,\delta_0}} \, dx$$

$$\le C \left\| \nabla d^F_{N^-}(\cdot) \right\|_{L^\infty(N_{\delta_0})} \int\limits_{\mathbb{R}^n} |u_\epsilon(x) - \Pi(u_\epsilon(x))| \chi_{\Omega^+_{\epsilon,\delta_0}} \, dx$$

$$\le C \int\limits_{\Omega} \sqrt{F(u_\epsilon)} \, dx \le C \epsilon^{1/2}.$$

After sending ϵ to 0, this implies $\phi(x) \ge c_0^F$ for a.e. $x \in E^+$. So Claim 2 is true.

It follows from Claim 2 that the set E^+ can be represented as

$$E^+ = \{ x \in \Omega : \phi(x) > t \} \quad \forall t \in \big[0, c_0^F\big). \tag{3.10}$$

In the following, for a set $E \subset \mathbb{R}^n$ of finite perimeter, we also view E as a n-dimensional current in $\mathbb{R}^n$ and ∂E denotes the boundary of the current E. Then we have the following:

Claim 3. If we view both $\partial(\partial E^+ \llcorner \Omega)$ and Σ as $(n-2)$-dimensional currents, we have

$$\partial(\partial E^+ \llcorner \Omega) = \Sigma. \tag{3.11}$$

In fact, (3.11) follows directly from the fact that

$$\partial(\partial \Omega^+_{\epsilon,\delta_0} \llcorner \Omega) = \{ x \in \partial\Omega : \mathrm{d}(g_\epsilon(x), N^+) = \delta_0 \} \rightharpoonup \Sigma \quad \text{as } \epsilon \to 0$$

as weak convergence of currents, which follows from the properties of g_ϵ given in Section 1.

PROOF. Now we prove (3.1). It follows from (3.10), (3.11), and the area minimality of Γ that

$$H^{n-1}(\partial^* \{ x \in \Omega : \phi(x) > t \} \llcorner \Omega) \ge H^{n-1}(\Gamma) \quad \forall t \in \big[0, c_0^F\big),$$

where $\partial^* E$ denotes the reduced boundary of E. Thus, using the co-area formula for BV functions [6] and (3.8), we have

$$\Lambda \geq |D\phi|(\Omega) = \int_{\mathbb{R}_+} H^{n-1}(\partial^*\{x \in \mathbb{R}^n : \phi > t\} \llcorner \Omega)dt$$
$$\geq \int_0^{c_0^F} H^{n-1}(\partial^*\{x \in \mathbb{R}^n : \phi > t\} \llcorner \Omega)dt \geq c_0^F H^{n-1}(\Gamma).$$

So (3.1) is true. This completes the proof of Proposition 3.1. □

The next step is to show by explicit constructions that the lower bound obtained in Proposition 3.1 is also a upper bound for $\epsilon\mathbf{E}(\epsilon)$.

To do it, we first introduce some additional notation. For small $\eta > 0$, set

$$\Omega_\eta = \{x \in \Omega : \mathrm{d}(x, \partial\Omega) > \eta\}, \quad \Omega_\eta^\pm = \Omega_\eta \cap \Omega^\pm,$$

and

$$O_\eta = \Gamma_\eta \cap (\Omega \setminus \Omega_\eta), \quad U_\eta^\pm = (\Omega^\pm \setminus \Omega_\eta^\pm) \setminus O_\eta, \quad U_\eta = U_\eta^+ \cup U_\eta^-.$$

Now we have the following:

PROPOSITION 3.2. *Under the same assumptions as in Theorem* 1.1*, we have*

$$\lim_{\epsilon \downarrow 0} \epsilon\mathbf{E}(\epsilon) \leq c_0^F H^{n-1}(\Gamma). \tag{3.12}$$

PROOF.

Step 1. Let $(p^+, p^-) \in M^+ \times M^-$ be such that $|p^+ - p^-| = d_N$. For $\beta \in (\frac{1}{2}, 1)$, it is easy to see that we can identify $U_{\epsilon^\beta}^\pm$ by $\Sigma_{\epsilon^\beta}^\pm \times [0, \epsilon^\beta]$ via the differomorphisms

$$T_\epsilon^\pm : (\partial U_{\epsilon^\beta}^\pm \cap \partial\Omega) \cup (\partial U_{\epsilon^\beta}^\pm \cap \partial\Omega_{\epsilon^\beta}) \to (\Sigma_{\epsilon^\beta}^\pm \times \{0\}) \cup (\Sigma_{\epsilon^\beta}^\pm \times \{\epsilon^\beta\})$$

such that

$$\frac{1}{2} \leq \|\nabla T_\epsilon^\pm(x) - x\|_{L^\infty(U_{\epsilon^\beta}^\pm)} \leq 2.$$

Hence condition (1.3) implies that there are extension maps

$$G_\epsilon^\pm : U_{\epsilon^\beta}^\pm \to N^\pm$$

such that

$$G_\epsilon^\pm|_{\partial U_{\epsilon^\beta}^\pm \cap \partial\Omega} = g_\epsilon, \quad G_\epsilon^\pm|_{\partial U_{\epsilon^\beta}^\pm \cap \partial\Omega_{\epsilon^\beta}} = p^\pm, \tag{3.13}$$

and

$$\int_{U_{\epsilon^\beta}^\pm} |\nabla G_\epsilon^\pm|^2 \, dx \leq C\Big[\epsilon^\beta \int_{\partial U_{\epsilon^\beta}^\pm \cap \partial\Omega} |\nabla_\tau g_\epsilon|^2 \, dH^{n-1} + \frac{1}{\epsilon^\beta} \int_{\partial U_{\epsilon^\beta}^\pm \cap \partial\Omega} |g_\epsilon - p^\pm|^2\Big] dH^{n-1}. \tag{3.14}$$

By Fubini's theorem, we can assume that there exists $\theta_0 \in (1,2)$, depending on ϵ and β, such that

$$\begin{aligned}\int_{\{x\in U^{\pm}_{\epsilon^\beta}:\mathrm{d}_\Gamma(x)=\pm\theta_0\epsilon^\beta\}} &|\nabla G^{\pm}_\epsilon|^2\, dH^{n-1}\\ &\le \frac{2}{\epsilon^\beta}\int_{U^{\pm}_{\epsilon^\beta}} |\nabla G^{\pm}_\epsilon|^2\, dx \\ &\le C\Bigg[\int_{\partial U^{\pm}_{\epsilon^\beta}\cap\partial\Omega} |\nabla_\tau g_\epsilon|^2\, dH^{n-1} + \frac{1}{\epsilon^{2\beta}}\int_{\partial U^{\pm}_{\epsilon^\beta}\cap\partial\Omega} |g_\epsilon - p^{\pm}|^2\, dH^{n-1}\Bigg].\end{aligned} \tag{3.15}$$

Set $G_\epsilon :\ U_{\epsilon^\beta}\setminus\Gamma_{\theta_0\epsilon^\beta}\to\mathbb{R}^k$ by letting

$$G_\epsilon = G^{\pm}_\epsilon \quad \text{on } U^{\pm}_{\epsilon^\beta}\setminus\Gamma_{\theta_0\epsilon^\beta}.$$

Denote by $\mathcal{S}\subset\Gamma$ the singular set of Γ. By both the boundary regularity theorem of Hardt and Simon [12, p. 440] and the interior partial regularity theorem of Federer [8, p. 796, theorem 1], we may assume that

$$\Gamma\in C^\infty(\overline{\Omega}\setminus\Omega_{\epsilon^\beta}),\quad \mathcal{S}\subset\Omega_{\epsilon^\beta},\quad \text{and}\quad \dim_{\mathbb{H}}(\mathcal{S})\le n-7.$$

Hence we have

$$H^n(\mathcal{S}_{\epsilon^\beta})\le C\epsilon^{6\beta}, \tag{3.16}$$

where

$$\mathcal{S}_{\epsilon^\beta} = \{x\in\Omega : \mathrm{d}(x,\mathcal{S})\le 2\epsilon^\beta\}$$

is the $2\epsilon^\beta$-neighborhood of $\mathcal{S}$.

Step 2. Now we construct a comparison map $v_\epsilon \in H^1_{g_\epsilon}(\Omega,\mathbb{R}^k)$ as follows. Applying Proposition 7.4 in Section 7, with $L=\epsilon^{\beta-1}$, we obtain a map $\widetilde{\alpha}_\epsilon$ such that

$$\widetilde{\alpha}_\epsilon \in H^1\bigg([-\theta_0\epsilon^{\beta-1},\theta_0\epsilon^{\beta-1}],\Big[-\frac{d_N}{2},\frac{d_N}{2}\Big]\bigg),$$
$$\widetilde{\alpha}_\epsilon = \begin{cases} -\frac{d_N}{2} & \text{on } [-\theta_0\epsilon^{\beta-1},-\epsilon^{\beta-1}],\\ \alpha_{\epsilon^{\beta-1}} & \text{on } [-\epsilon^{\beta-1},\epsilon^{\beta-1}],\\ \frac{d_N}{2} & \text{on } [\epsilon^{\beta-1},\theta_0\epsilon^{\beta-1}].\end{cases}$$

Then $\widetilde{\alpha}_\epsilon$ satisfies

$$\int_{\theta_0\epsilon^{\beta-1}}^{\theta_0\epsilon^{\beta-1}} \big(|\widetilde{\alpha}'_\epsilon(t)|^2 + \widetilde{F}(\widetilde{\alpha}_\epsilon(t))\big)dt \le c_0^F + c_2\exp(-c_1\epsilon^{\beta-1}). \tag{3.17}$$

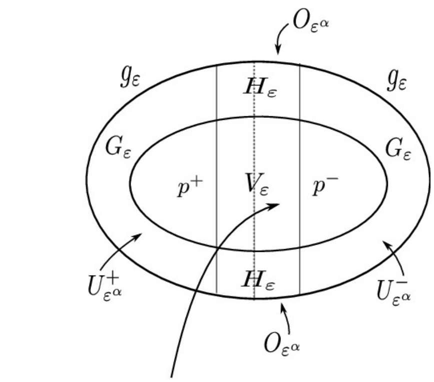

FIGURE 3.1. Comparison maps ($\alpha = \beta$).

The value of v_ϵ on Ω_{ϵ^β} is defined by

$$v_\epsilon(x) := \begin{cases} p^- & \text{if } x \in \Omega^-_{\epsilon^\beta} \setminus \Gamma_{\theta_0 \epsilon^\beta}, \\ \frac{p^+ + p^-}{2} + \widetilde{\alpha}_\epsilon\big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\big)\frac{p^+ - p^-}{|p^+ - p^-|} & \text{if } x \in \Gamma_{\theta_0 \epsilon^\beta}, \\ p^+ & \text{if } x \in \Omega^+_{\epsilon^\beta} \setminus \Gamma_{\theta_0 \epsilon^\beta}. \end{cases}$$

Let

$$H_\epsilon : (\Omega \setminus \Omega_{\epsilon^\beta}) \cap \Gamma_{\theta_0 \epsilon^\beta} \to \mathbb{R}^k$$

be the harmonic function with the Dirichlet boundary data:

$$H_\epsilon(x) = \begin{cases} g_\epsilon(x) & \text{if } x \in \partial\Omega \cap \Gamma_{\theta_0 \epsilon^\beta}, \\ G_\epsilon(x) & \text{if } x \in \partial(\Gamma_{\theta_0 \epsilon^\beta}) \cap (\Omega \setminus \Omega_{\epsilon^\beta}), \\ v_\epsilon(x) & \text{if } x \in \partial\Omega_{\epsilon^\beta} \cap \Gamma_{\theta_0 \epsilon^\beta}. \end{cases}$$

Then we define the value of v_ϵ on $\Omega \setminus \Omega_{\epsilon^\beta}$ by

$$v_\epsilon(x) = \begin{cases} G_\epsilon(x) & \text{if } x \in (\Omega \setminus \Omega_{\epsilon^\beta}) \setminus \Gamma_{\theta_0 \epsilon^\beta}, \\ H_\epsilon(x) & \text{if } x \in (\Omega \setminus \Omega_{\epsilon^\beta}) \cap \Gamma_{\theta_0 \epsilon^\beta}. \end{cases}$$

One can see Figure 3.1 for the construction.

Step 3. Now we estimate the energy of v_ϵ on Ω as follows:

$$\begin{aligned} \int_\Omega \Big(\epsilon |\nabla v_\epsilon|^2 + \frac{1}{\epsilon} F(v_\epsilon)\Big) dx &= \Big\{ \int_{\Omega_{\epsilon^\beta}} + \int_{\Omega \setminus \Omega_{\epsilon^\beta}} \Big\} \Big(\epsilon |\nabla v_\epsilon|^2 + \frac{1}{\epsilon} F(v_\epsilon)\Big) dx \\ &= \mathrm{I}_\epsilon + \mathrm{II}_\epsilon. \end{aligned}$$

We first estimate I_ϵ. We have

$$\begin{aligned}
\mathrm{I}_\epsilon &= \int\limits_{\Omega_{\epsilon^\beta}\cap\Gamma_{\theta_0\epsilon^\beta}} \left(\epsilon\left|\nabla\left(\widetilde{\alpha}_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right)\right|^2 + \frac{1}{\epsilon}\,\widetilde{F}\left(\widetilde{\alpha}_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right)\right)dx \\
&= \frac{1}{\epsilon}\int\limits_{\Omega_{\epsilon^\beta}\cap\Gamma_{\theta_0\epsilon^\beta}} \left(\left(\widetilde{\alpha}'_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right)^2|\nabla \mathrm{d}_\Gamma(x)|^2 + \widetilde{F}(\widetilde{\alpha}_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right))\right)dx \\
&\le \frac{1}{\epsilon}\int\limits_{\Omega_{\epsilon^\beta}\cap\mathcal{S}_{\epsilon^\beta}} \left(\left(\widetilde{\alpha}'_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right)^2|\nabla \mathrm{d}_\Gamma(x)|^2 + \widetilde{F}\left(\widetilde{\alpha}_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right)\right)dx \\
&\quad + \frac{1}{\epsilon}\int\limits_{\Omega_{\epsilon^\beta}\cap(\Gamma_{\theta_0\epsilon^\beta}\setminus\mathcal{S}_{\epsilon^\beta})} \left(\left(\widetilde{\alpha}'_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right)^2|\nabla \mathrm{d}_\Gamma(x)|^2 + \widetilde{F}\left(\widetilde{\alpha}_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right)\right)dx \\
&= \mathrm{III}_\epsilon + \mathrm{IV}_\epsilon.
\end{aligned}$$

For the term III_ϵ, we use the fact that $|\nabla \mathrm{d}_\Gamma(x)| \le 1$ and Proposition A.4 to get

$$\max_{|t|\le\theta_0\epsilon^{\beta-1}} \left(|\widetilde{\alpha}'_\epsilon(t)|^2 + \widetilde{F}(\widetilde{\alpha}_\epsilon(t))\right) \le C.$$

We have, by (3.16),

$$\mathrm{III}_\epsilon \le \frac{C}{\epsilon}\,H^n(\Omega_{\epsilon^\beta}\cap\mathcal{S}_{\epsilon^\beta}) \le C\epsilon^{6\beta-1} \le C\epsilon^2.$$

For the term IV_ϵ, since

$$|\nabla \mathrm{d}_\Gamma(x)| = 1 \quad \text{for } x\in\Omega_{\epsilon^\beta}\cap(\Gamma_{\theta_0\epsilon^\beta}\setminus\mathcal{S}_{\epsilon^\beta}),$$

we have, by both the co-area formula and the formula of change of variables,

$$\begin{aligned}
\mathrm{IV}_\epsilon = \int_{-\theta_0\epsilon^{\beta-1}}^{\theta_0\epsilon^{\beta-1}} &\left[|\widetilde{\alpha}'_\epsilon(s)|^2 + \widetilde{F}(\widetilde{\alpha}_\epsilon(s))\right] \\
&\times H^{n-1}\left(\{x\in\Omega_{\epsilon^\beta}\cap(\Gamma_{\theta_0\epsilon^\beta}\setminus\mathcal{S}_{\epsilon^\beta}) : \mathrm{d}_\Gamma(x) = \epsilon s\}\right)ds.
\end{aligned}$$

Since $\Gamma\setminus\mathcal{S}_{\theta_0\epsilon^\beta}$ is a smooth hypersurface, we have that

$$\{x\in\Omega\cap(\Gamma_{\theta_0\epsilon^\beta}\setminus\mathcal{S}_{\epsilon^\beta}) : \mathrm{d}_\Gamma(x) = \tau\}$$

is a family of smooth hypersurfaces, which converges to $\Gamma\setminus\mathcal{S}$ locally in $C^2(\Omega\setminus\mathcal{S})$ uniformly with respect to $\tau\in[-\theta_0\epsilon^\beta, \theta_0\epsilon^\beta]$. This, combined with the boundary regularity of Γ and (3.16), implies that for any $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that for any $\epsilon \le \epsilon(\delta)$,

$$\max_{|\tau|\le\theta_0\epsilon^\beta} H^{n-1}\left(\{x\in\Omega_{\epsilon^\beta}\cap(\Gamma_{\theta_0\epsilon^\beta}\setminus\mathcal{S}_{\epsilon^\beta}) : \mathrm{d}_\Gamma(x) = \tau\}\right) \le H^{n-1}(\Gamma)+\delta. \tag{3.18}$$

This and (3.17) imply

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\mathrm{IV}_\epsilon &\le (H^{n-1}(\Gamma)+\delta)\lim_{\epsilon\downarrow 0}\int_{-\theta_0\epsilon^{\beta-1}}^{\theta_0\epsilon^{\beta-1}}\left(|\tilde{\alpha}_\epsilon'(s)|^2+\tilde{F}(\tilde{\alpha}_\epsilon(s))\right)ds\\ &\le (H^{n-1}(\Gamma)+\delta)c_0^F. \end{aligned}\tag{3.19}$$

Putting the estimates of III_ϵ and IV_ϵ into I_ϵ gives

$$\lim_{\epsilon\downarrow 0}\mathrm{I}_\epsilon \le (H^{n-1}(\Gamma)+\delta)c_0^F. \tag{3.20}$$

Now we estimate II_ϵ. We have

$$\begin{aligned}\mathrm{II}_\epsilon &= \left\{\int_{(\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta}}+\int_{(\Omega\setminus\Omega_{\epsilon^\beta})\setminus\Gamma_{\theta_0\epsilon^\beta}}\right\}\left(\epsilon|\nabla v_\epsilon|^2+\frac{1}{\epsilon}F(v_\epsilon)\right)dx\\ &= \mathrm{V}_\epsilon+\mathrm{VI}_\epsilon.\end{aligned}$$

For the term VI_ϵ, we use (3.14) to get

$$\begin{aligned}\mathrm{VI}_\epsilon &= \int_{(\Omega\setminus\Omega_{\epsilon^\beta})\setminus\Gamma_{\theta_0\epsilon^\beta}}\epsilon|\nabla G_\epsilon|^2\,dx\\ &\le C\left[\epsilon^\beta\int_{\partial\Omega}\epsilon|\nabla_\tau g_\epsilon|^2\,dH^{n-1}+\epsilon^{1-\beta}\right]\le C(\epsilon^\beta+\epsilon^{1-\beta}).\end{aligned}\tag{3.21}$$

For the term V_ϵ, since $v_\epsilon = H_\epsilon$ is harmonic on $(\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta}$ and $g_\epsilon(\partial\Omega)\subset B_R^k$ by (1.2), the maximum principle implies

$$\|v_\epsilon\|_{L^\infty((\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta})}\le\|H_\epsilon\|_{L^\infty(\partial((\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta}))}\le C$$

for some $C>0$ depending only on N and R. Thus we have

$$\int_{(\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta}}\frac{1}{\epsilon}F(v_\epsilon)dx\le\frac{C}{\epsilon}H^n((\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta})\le C\epsilon^{2\beta-1}. \tag{3.22}$$

For the Dirichlet energy of v_ϵ on $(\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta}$, we have

$$\begin{aligned}&\int_{(\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta}}|\nabla v_\epsilon|^2\,dx\\ &\le C\epsilon^\beta\int_{\partial((\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta})}|\nabla_\tau H_\epsilon|^2\,dH^{n-1}=\end{aligned}$$

$$= C\epsilon^{\beta}\Bigg[\int\limits_{\partial\Omega\cap\Gamma_{\theta_0\epsilon^\beta}} |\nabla_\tau g_\epsilon|^2\, dH^{n-1} + \int\limits_{\partial\Omega_{\epsilon^\beta}\cap\Gamma_{\theta_0\epsilon^\beta}} |\nabla_\tau v_\epsilon|^2\, dH^{n-1}$$

$$+ \int\limits_{\partial(\Gamma_{\theta_0\epsilon^\beta})\cap(\Omega\setminus\Omega_{\epsilon^\beta})} |\nabla_\tau G_\epsilon|^2\, dH^{n-1}\Bigg]. \tag{3.23}$$

Now we estimate each term in the right-hand side of (3.23). Condition (1.2) implies

$$\int\limits_{\partial\Omega\cap\Gamma_{\theta_0\epsilon^\beta}} |\nabla_\tau g_\epsilon|^2\, dH^{n-1} \le \frac{L}{\epsilon},$$

inequality (3.15) implies

$$\int\limits_{\partial(\Gamma_{\theta_0\epsilon^\beta})\cap(\Omega\setminus\Omega_{\epsilon^\beta})} |\nabla_\tau G_\epsilon|^2\, dH^{n-1} \le C\Bigg[\int\limits_{\partial\Omega\setminus\Gamma_{\theta_0\epsilon^\beta}} |\nabla_\tau g_\epsilon|^2\, dH^{n-1} + \epsilon^{-2\beta}\Bigg]$$

$$\le C(\epsilon^{-1} + \epsilon^{-2\beta}),$$

and a direct calculation, using the formula of v_ϵ on $\partial\Omega_{\epsilon^\beta}\cap\Gamma_{\theta_0\epsilon^\beta}$, gives

$$\int\limits_{\partial\Omega_{\epsilon^\beta}\cap\Gamma_{\theta_0\epsilon^\beta}} |\nabla_\tau v_\epsilon|^2\, dH^{n-1} \le \frac{1}{\epsilon^2}\int\limits_{\partial\Omega_{\epsilon^\beta}\cap\Gamma_{\theta_0\epsilon^\beta}} \left|\tilde{\alpha}'_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right|^2 dH^{n-1}$$

$$\le \frac{C}{\epsilon^2} H^{n-1}(\partial\Omega_{\epsilon^\beta}\cap\Gamma_{\theta_0\epsilon^\beta}) \le C\epsilon^{\beta-2}.$$

Putting all these estimates into (3.23), we obtain

$$\int\limits_{(\Omega\setminus\Omega_{\epsilon^\beta})\cap\Gamma_{\theta_0\epsilon^\beta}} \epsilon|\nabla v_\epsilon|^2\, dx \le C(\epsilon^\beta + \epsilon^{1-\beta} + \epsilon^{2\beta-1}). \tag{3.24}$$

Now we combine (3.22) and (3.24) to get an estimate of the term V_ϵ:

$$\mathrm{V}_\epsilon \le C(\epsilon^\beta + \epsilon^{1-\beta} + \epsilon^{2\beta-1}). \tag{3.25}$$

Combining (3.21) and (3.25) we obtain

$$\mathrm{II}_\epsilon \le (\epsilon^\beta + \epsilon^{1-\beta} + \epsilon^{2\beta-1}). \tag{3.26}$$

Finally, we use (3.20) and (3.26) to obtain

$$\lim_{\epsilon\downarrow 0}\int\limits_{\Omega}\left(\epsilon|\nabla v_\epsilon|^2 + \frac{1}{\epsilon}F(v_\epsilon)\right)dx \le c_0^F H^{n-1}(\Gamma) + C\delta;$$

this implies (3.12), since $\delta > 0$ is arbitrary. The proof of Proposition 3.2 is complete. □

It is clear that Theorem 1.1 follows directly from the above two propositions. □

4 Refined Lower Bound Estimates of $\mathbf{E}(\epsilon)$

This section is devoted to the refined estimate of lower bounds of $\mathbf{E}(\epsilon)$. To state it, we need to introduce some additional notation.

For $t \in [-\epsilon^\beta, \epsilon^\beta]$, define

$$\Sigma(t) = \{x \in \partial\Omega : \mathrm{d}_\Gamma(x) = t\} \quad (= \partial\Gamma(t)).$$

In particular, $\Sigma(0) = \Sigma$. For any such t, let $\mathcal{S}(t) \subset \Omega$ be a minimal hypersurface spanned by $\Sigma(t)$:

$$H^{n-1}(\mathcal{S}(t)) = \min\{H^{n-1}(S) : S \text{ is an integral } (n-1) \text{ current in } \Omega,\ \partial S = \Sigma(t)\}.$$

In particular, we may assume $\mathcal{S}(0) = \Gamma$. The aim of this section is to prove the following lemma:

LEMMA 4.1. *Assume Γ satisfies* (A1), $g_\epsilon : \partial\Omega \to \mathbb{R}^k$ *satisfies* (1.17), *and* $u_\epsilon \in H^1_{g_\epsilon}(\Omega, \mathbb{R}^k)$. *Then there exists $C = C(N) > 0$ such that for any $0 < \delta \le \delta_N$, it holds that*

$$\begin{aligned}\int_\Omega \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx &\ge (1 - C\delta) \int_{\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}} |\nabla(\Pi(u_\epsilon))|^2\, dx \\ &\quad + \frac{2}{\epsilon} \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\big) \sqrt{\widetilde{F}(\lambda)}\, d\lambda.\end{aligned} \tag{4.1}$$

PROOF. For $0 < \delta \le \delta_N$, recall that, for $x \in \Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}$,

$$\begin{aligned} u_\epsilon(x) &= \Pi(u_\epsilon(x)) + \mathrm{d}(u_\epsilon(x))\nu_\epsilon(x), \\ \nu_\epsilon(x) &= \frac{u_\epsilon(x) - \Pi(u_\epsilon(x))}{|u_\epsilon(x) - \Pi(u_\epsilon(x))|}, \end{aligned}$$

Calculations similar to those for Theorem 2.1 imply that, for $x \in \Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}$,

$$\begin{aligned} |\nabla u_\epsilon(x)|^2 &= \big|\nabla\big(\Pi(u_\epsilon(x))\big)\big|^2 + \big|\nabla(\mathrm{d}(u_\epsilon(x))\big|^2 + \mathrm{d}^2(u_\epsilon(x))\big|\nabla(\nu_\epsilon(x))\big|^2 \\ &\quad - 2\mathrm{d}(u_\epsilon(x))A(\Pi(u_\epsilon(x))\big(\nabla\big(\Pi(u_\epsilon(x))\big), \nabla\big(\Pi(u_\epsilon(x))\big)\big) \\ &\ge \big|\nabla\big(\mathrm{d}(u_\epsilon(x))\big)\big|^2 + (1 - C\delta)\big|\nabla\big(\Pi(u_\epsilon(x))\big)\big|^2, \end{aligned}$$

where $A(\cdot)(\cdot,\cdot)$ is the second fundamental form of N and $C = 2\|A(\cdot)\|_{L^\infty(N)}$. Thus we obtain

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx \\
&\geq (1 - C\delta) \int_{\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}} \left|\nabla(\Pi(u_\epsilon))\right|^2 dx \\
&\quad + \int_{\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}} \left(|\nabla(\mathrm{d}(u_\epsilon))|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx \\
&\quad + \int_{\Omega \setminus (\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta})} \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx.
\end{aligned} \tag{4.2}$$

By the Cauchy-Schwarz inequality and Federer's co-area formula, we have

$$\begin{aligned}
&\int_{\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}} \left(|\nabla(\mathrm{d}(u_\epsilon))|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx \\
&\geq \frac{2}{\epsilon} \int_{\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}} \sqrt{f(\mathrm{d}^2(u_\epsilon))} |\nabla(\mathrm{d}(u_\epsilon))| dx \\
&= \frac{2}{\epsilon} \int_0^\delta H^{n-1}\big(\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^\pm) = \lambda\}\big) \sqrt{f(\lambda^2)}\, d\lambda.
\end{aligned} \tag{4.3}$$

Define

$$H^\pm_{\epsilon,\delta} = \left\{x \in \Omega : \delta \leq \mathrm{d}(u_\epsilon(x), N^\pm) \leq \frac{d_N}{2}\right\}.$$

Since

$$\mathrm{d}(u_\epsilon(x)) = \mathrm{d}(u_\epsilon(x), N^\pm) \quad \text{and} \quad \left|\nabla\big(\mathrm{d}(u_\epsilon(x))\big)\right| \leq |\nabla u_\epsilon(x)| \quad \text{for } x \in H^\pm_{\epsilon,\delta},$$

we have

$$\begin{aligned}
&\int_{\Omega \setminus (\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta})} \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx \\
&\geq \int_{H^+_{\epsilon,\delta} \cup H^-_{\epsilon,\delta}} \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx \geq
\end{aligned}$$

$$\geq \frac{2}{\epsilon} \int_{H^+_{\epsilon,\delta} \cup H^-_{\epsilon,\delta}} \sqrt{f(\mathrm{d}^2(u_\epsilon))} |\nabla u_\epsilon| dx$$

$$\geq \frac{2}{\epsilon} \int_{H^+_{\epsilon,\delta} \cup H^-_{\epsilon,\delta}} \sqrt{f(\mathrm{d}^2(u_\epsilon))} \big|\nabla(\mathrm{d}(u_\epsilon))\big| dx$$

$$= \frac{2}{\epsilon} \int_{\delta}^{\frac{d_N}{2}} H^{n-1}\big(\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^\pm) = \lambda\}\big) \sqrt{f(\lambda^2)}\, d\lambda. \tag{4.4}$$

Putting (4.3) and (4.4) together, we have

$$\begin{aligned} &\int_{\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}} \left(|\nabla(\mathrm{d}(u_\epsilon))|^2 + \frac{1}{\epsilon^2} F(u_\epsilon) \right) dx \\ &+ \int_{\Omega \setminus (\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta})} \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon) \right) dx \\ &\geq \frac{2}{\epsilon} \int_0^{\frac{d_N}{2}} H^{n-1}\big(\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^\pm) = \lambda\}\big) \sqrt{f(\lambda^2)}\, d\lambda. \end{aligned} \tag{4.5}$$

For $0 < \lambda < \frac{d_N}{2}$, since

$$\partial\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^\pm) = \lambda\} = \{x \in \partial\Omega : \mathrm{d}(g_\epsilon(x), N^\pm) = \lambda\},$$

(1.17) implies

$$\begin{aligned} \partial\left\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^+) = \frac{d_N}{2} - \lambda\right\} &= \{x \in \partial\Omega : \mathrm{d}_\Gamma(x) = \epsilon\alpha_\epsilon^{-1}(\lambda)\} \\ &= \Sigma(\epsilon\alpha_\epsilon^{-1}(\lambda)), \\ \partial\left\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^-) = \frac{d_N}{2} - \lambda\right\} &= \{x \in \partial\Omega : \mathrm{d}_\Gamma(x) = \epsilon\alpha_\epsilon^{-1}(-\lambda)\} \\ &= \Sigma(\epsilon\alpha_\epsilon^{-1}(-\lambda)). \end{aligned}$$

Hence

$$H^{n-1}\left(\left\{x \in \Omega : \mathrm{d}(u_\epsilon(x), N^\pm) = \frac{d_N}{2} - \lambda\right\}\right) \geq H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\pm\lambda))\big) \quad \forall \lambda \in \left(0, \frac{d_N}{2}\right).$$

Thus we have

$$\begin{aligned}
&\int_{\Omega^+_{\epsilon,\delta}\cup\Omega^-_{\epsilon,\delta}} \Big(\big|\nabla(\mathrm{d}(u_\epsilon))\big|^2 + \frac{1}{\epsilon^2}F(u_\epsilon)\Big)dx \\
&\quad + \int_{\Omega\setminus(\Omega^+_{\epsilon,\delta}\cup\Omega^-_{\epsilon,\delta})} \Big(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2}F(u_\epsilon)\Big)dx \\
&\quad \geq \frac{2}{\epsilon}\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\big)\sqrt{\widetilde{F}(\lambda)}\,d\lambda.
\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.2) yields (4.1). The proof is complete. □

As an immediate consequence of (4.1), we have the following estimate in dimension 1:

COROLLARY 4.2. *Suppose $u_\epsilon \in H^1([a,b],\mathbb{R}^k)$, with $u_\epsilon(a) \in N^-$ and $u_\epsilon(b) \in N^+$. Then there exists $C = C(N) > 0$ such that for any $0 < \delta \leq \delta_N$, we have*

$$\begin{aligned}
&\int_{[a,b]} \Big(|u_\epsilon'(t)|^2 + \frac{1}{\epsilon^2}F(u_\epsilon(t))\Big)dt \geq \\
&\qquad (1 - C\delta)\int_{[a,b]^+_{\epsilon,\delta}\cup[a,b]^-_{\epsilon,\delta}} \left|\frac{d}{dt}\big(\Pi(u_\epsilon(t))\big)\right|^2 + \frac{c_0^F}{\epsilon}.
\end{aligned} \tag{4.7}$$

PROOF. For any $0 \leq \lambda \leq \frac{d_N}{2}$, there exists $t_\lambda^\pm \in [a,b]$ such that

$$\mathrm{d}(u_\epsilon(t_\lambda^\pm), N^\pm) = \lambda.$$

Hence

$$H^0\big(\{t \in [a,b] : \mathrm{d}(u_\epsilon(t), N^\pm) = \lambda\}\big) \geq 1 \quad \forall\lambda \in \left[0, \frac{d_N}{2}\right].$$

This and (4.1) imply (4.7). □

For higher dimensions, we have the following crude estimate:

COROLLARY 4.3. *Under the same assumptions as in Lemma* 4.1*, there exists* $C = C(N) > 0$ *such that for any* $0 < \delta \leq \delta_N$*, it holds that*

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx \\
&\geq (1 - C\delta) \int_{\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}} \left|\nabla(\Pi(u_\epsilon))\right|^2 dx + \frac{c_0^F}{\epsilon} H^{n-1}(\Gamma) \\
&\quad - C \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha^{-1}(\lambda)| \sqrt{\widetilde{F}(\lambda)}\, d\lambda + o(1),
\end{aligned} \tag{4.8}$$

where $\alpha \in H^1(\mathbb{R})$ *with* $\alpha(\pm\infty) = \pm\frac{d_N}{2}$ *is a minimizer of* (2.1).

PROOF. Define the hypersurface $\Lambda(t)$ by

$$\Lambda(t) = \begin{cases} \{x \in \partial\Omega : t \leq \mathrm{d}_\Gamma(x) \leq 0\} & \text{if } -\epsilon^\beta \leq t \leq 0, \\ \{x \in \partial\Omega : 0 \leq \mathrm{d}_\Gamma(x) \leq t\} & \text{if } 0 \leq t \leq \epsilon^\beta. \end{cases}$$

Then we have

$$\partial(\mathcal{S}(t) \cup \Lambda(t)) = \partial\Gamma \quad \forall t \in [-\epsilon^\beta, \epsilon^\beta].$$

Hence the area minimality of Γ implies that for all $t \in [-\epsilon^\beta, \epsilon^\beta]$,

$$\begin{aligned}
H^{n-1}(\Gamma) \leq H^{n-1}(\Lambda(t) \cup \mathcal{S}(t)) &= H^{n-1}(\Lambda(t)) + H^{n-1}(\mathcal{S}(t)) \\
&\leq H^{n-1}(\mathcal{S}(t)) + C|t|.
\end{aligned} \tag{4.9}$$

Therefore we have

$$\begin{aligned}
&\frac{2}{\epsilon} \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\big) \sqrt{\widetilde{F}(\lambda)}\, d\lambda \\
&\geq \frac{2}{\epsilon} \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} \big(H^{n-1}(\Gamma) - C\epsilon|\alpha_\epsilon^{-1}(\lambda)|\big) \sqrt{\widetilde{F}(\lambda)}\, d\lambda \\
&= \frac{4H^{n-1}(\Gamma)}{\epsilon} \int_0^{\frac{d_N}{2}} \sqrt{\widetilde{F}(\lambda)}\, d\lambda - C \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha_\epsilon^{-1}(\lambda)| \sqrt{\widetilde{F}(\lambda)}\, d\lambda.
\end{aligned}$$

This, (4.1), and (A.11) imply (4.8). □

5 Refined Upper Bound Estimates of $\mathbf{E}(\epsilon)$

In this section, we will prove an optimal estimate for the upper bound of $\mathbf{E}(\epsilon)$ by constructing a comparison map that is approximately a minimal connecting orbit in the transition region near Γ and an approximate minimizing harmonic map associated with $\mathbf{D}$ outside the transition region. More precisely, we have the following:

LEMMA 5.1. *Assume* Γ *satisfies* (A1), $g_\epsilon : \partial\Omega \to \mathbb{R}^k$ *satisfies* (1.17), *and there exists a minimizing harmonic map* $v \in \mathbf{A}$ *such that* $v(x^\pm) \in H^1(\Gamma, M^\pm)$. *Then*

$$\mathbf{E}(\epsilon) \leq \frac{2}{\epsilon} \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\Gamma(\epsilon \alpha_\epsilon^{-1}(\lambda))\big) \sqrt{\widetilde{F}(\lambda)}\, d\lambda + \mathbf{D} + o(1). \tag{5.1}$$

PROOF. For $\epsilon > 0$, define the comparison map $u_\epsilon \in H^1_{g_\epsilon}(\Omega, \mathbb{R}^k)$ by

$$u_\epsilon(x) = \begin{cases} v(\mathbf{\Psi}_\epsilon(x)) & \text{if } x \in \Omega^\pm \setminus \Gamma_{\epsilon^\beta}, \\ \frac{v(T(x)^+)+v(T(x)^-)}{2} + \alpha_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right) \frac{v(T(x)^+)-v(T(x)^-)}{|v(T(x)^+)-v(T(x)^-)|} & \text{if } x \in \Gamma_{\epsilon^\beta}, \end{cases} \tag{5.2}$$

where $T(x) \in \Gamma$ is the unique solution of

$$\mathbf{\Phi}(T(x), \mathrm{d}_\Gamma(x)) = x \quad \text{for } x \in \Gamma_{\epsilon^\beta}.$$

Since $v\big|_{\Sigma^\pm} = g$, it follows from (5.2) and (1.17) that $u_\epsilon\big|_{\partial\Omega} = g_\epsilon$. Now we want to estimate the energy of u_ϵ:

$$\begin{aligned} \int_\Omega \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx &= \left(\int_{\Omega^+\setminus\Gamma_{\epsilon^\beta}} + \int_{\Omega^-\setminus\Gamma_{\epsilon^\beta}}\right) |\nabla u_\epsilon|^2\, dx \\ &\quad + \int_{\Gamma_{\epsilon^\beta}} \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon)\right) dx \\ &= \mathrm{I} + \mathrm{II}, \end{aligned}$$

where we have used the fact that $F(u_\epsilon) = 0$ on $\Omega^\pm \setminus \Gamma_{\epsilon^\beta}$, since $u_\epsilon(\Omega^\pm \setminus \Gamma_{\epsilon^\beta}) \subset N^\pm$.

For I, since $u_\epsilon(x) = v(\mathbf{\Psi}_\epsilon(x))$ and $|\nabla \mathbf{\Psi}_\epsilon(x) - \mathrm{Id}| \leq C\epsilon^\beta$ for $x \in \Omega^\pm \setminus \Gamma_{\epsilon^\beta}$ (see its definition in Section 1), by a simple change of variables we have

$$\left| \mathrm{I} - \left(\int_{\Omega^+} |\nabla v|^2\, dx + \int_{\Omega^-} |\nabla v|^2\, dx \right) \right| \leq C\epsilon^{2\beta} \leq C\epsilon, \tag{5.3}$$

where $C > 0$ depends only on the value $\mathbf{D}$.

For II, since $|v(T(x)^+) - v(T(x)^-)| = d_N$ for $x \in \Gamma_{\epsilon^\beta}$, we see that

$$F(u_\epsilon(x)) = \widetilde{F}\left(\alpha_\epsilon\left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\right)\right), \quad x \in \Gamma_{\epsilon^\beta}. \tag{5.4}$$

Moreover, for $x \in \Gamma_{\epsilon^\beta}$,

$$\begin{aligned}\nabla u_\epsilon(x) = \nabla\Big(\frac{v(T(x)^+) + v(T(x)^-)}{2}\Big) \\ + \nabla\Big(\alpha_\epsilon\Big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\Big)\Big)\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|} \\ + \alpha_\epsilon\Big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\Big)\nabla\Big(\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|}\Big).\end{aligned}$$

Observe that

$$\begin{aligned}\nabla\Big(\frac{v(T(x)^+) + v(T(x)^-)}{2}\Big) = \\ \nabla\Big(v(T(x)^-)\Big) + \frac{d_N}{2}\nabla\Big(\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|}\Big).\end{aligned}$$

Since

$$\nabla\big(v(T(x)^-)\big) \in T_{v(T(x)^-)}N^-,$$
$$\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|} \perp T_{v(T(x)^-)}N^-, \quad \Big|\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|}\Big| = 1,$$

we have

$$\Big\langle\nabla\Big(\frac{v(T(x)^+) + v(T(x)^-)}{2}\Big),\ \nabla\Big(\alpha_\epsilon\big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\big)\Big)\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|}\Big\rangle = 0$$

and

$$\Big\langle\nabla\Big(\alpha_\epsilon\Big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\Big)\Big)\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|},\ \alpha_\epsilon\Big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\Big)\nabla\Big(\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|}\Big)\Big\rangle = 0.$$

Hence we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\Big|\nabla u_\epsilon(x)\Big|^2 \le \Big|\nabla\Big(\alpha_\epsilon\Big(\frac{\mathrm{d}_\Gamma(x)}{\epsilon}\Big)\Big)\Big|^2 + 2\Big|\nabla\Big(\frac{v(T(x)^+) + v(T(x)^-)}{2}\Big)\Big|^2 \\ + \frac{d_N^2}{2}\Big|\nabla\Big(\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|}\Big)\Big|^2.\end{aligned} \tag{5.5}$$

Since

$$\mathbf{\Phi}(T(x), \mathrm{d}_\Gamma(x)) = x \quad \text{for } x \in \Gamma_{\epsilon^\beta},$$

it is not hard to see that

$$|\nabla T(x)| \le C(\|\mathbf{\Phi}\|_{C^1}, \|\mathbf{\Phi}^{-1}\|_{C^1}) \quad \forall x \in \Gamma_{\epsilon^\beta}. \tag{5.6}$$

Putting (5.4) and (5.5) together, we have

$$
\begin{aligned}
\text{II} \le & \int\limits_{\Gamma_{\epsilon\beta}} \left\{ \left| \nabla \left(\alpha_\epsilon \left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon} \right) \right) \right|^2 + \frac{1}{\epsilon^2} \widetilde{F} \left(\alpha_\epsilon \left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon} \right) \right) \right\} dx \\
& + \int\limits_{\Gamma_{\epsilon\beta}} \left\{ 2 \left| \nabla \left(\frac{v(T(x)^+) + v(T(x)^-)}{2} \right) \right|^2 \right. \\
& \qquad \left. + \frac{d_N^2}{2} \left| \nabla \left(\frac{v(T(x)^+) - v(T(x)^-)}{|v(T(x)^+) - v(T(x)^-)|} \right) \right|^2 \right\} dx \\
= & \ \text{III} + \text{IV}.
\end{aligned}
\tag{5.7}
$$

Now we estimate the terms III and IV. Direct calculations employing (5.6) imply

$$
|\text{IV}| \le C\epsilon^\beta \left(\int\limits_\Gamma |\nabla_\tau v(x^+)|^2 \, dH^{n-1}x + \int\limits_\Gamma |\nabla_\tau v(x^-)|^2 \, dH^{n-1}x \right)
\tag{5.8}
$$

for some $C > 0$ depending only on d_N and $\mathbf{\Phi}$.

For III, since $|\nabla \mathrm{d}_\Gamma(x)| = 1$ for $x \in \Gamma_{\epsilon^\beta}$, Federer's co-area formula implies

$$
\begin{aligned}
\text{III} &= \frac{1}{\epsilon^2} \int\limits_{\Gamma_{\epsilon\beta}} \left\{ \left| \alpha_\epsilon' \left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon} \right) \right|^2 + \frac{1}{\epsilon^2} \widetilde{F} \left(\alpha_\epsilon \left(\frac{\mathrm{d}_\Gamma(x)}{\epsilon} \right) \right) \right\} dx \\
&= \frac{1}{\epsilon} \int_{-\epsilon^{\beta-1}}^{\epsilon^{\beta-1}} H^{n-1}(\Gamma(\epsilon\lambda)) \cdot \left[\alpha_\epsilon'(\lambda)^2 + \widetilde{F}(\alpha_\epsilon(\lambda)) \right] d\lambda
\end{aligned}
$$

Since (A.10) gives

$$
\left| \alpha_\epsilon'(\lambda)^2 - \widetilde{F}(\alpha_\epsilon(\lambda)) \right| \le C_2 e^{-C_1 \epsilon^{\beta-1}} \quad \forall \lambda \in [-\epsilon^{\beta-1}, \epsilon^{\beta-1}],
$$

it follows that

$$
\left| \alpha_\epsilon'(\lambda)^2 + \widetilde{F}(\alpha_\epsilon(\lambda)) - 2|\alpha_\epsilon'(\lambda)| \sqrt{\widetilde{F}(\alpha_\epsilon(\lambda))} \right| \le C_2 e^{-C_1 \epsilon^{\beta-1}} \qquad \forall \lambda \in [-\epsilon^{\beta-1}, \epsilon^{\beta-1}].
$$

Therefore we have

$$
\begin{aligned}
\text{III} &\le \frac{2}{\epsilon} \int_{-\epsilon^{\beta-1}}^{\epsilon^{\beta-1}} H^{n-1}(\Gamma(\epsilon\lambda)) |\alpha_\epsilon'(\lambda)| \sqrt{\widetilde{F}(\alpha_\epsilon(\lambda))} \, d\lambda + C\epsilon^{-2} e^{-C_1 \epsilon^{\beta-1}} \\
&\le \frac{2}{\epsilon} \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\Gamma(\epsilon \alpha_\epsilon^{-1}(\lambda))\big) \sqrt{\widetilde{F}(\lambda)} \, d\lambda + C\epsilon^{-2} e^{-C_1 \epsilon^{\beta-1}}.
\end{aligned}
\tag{5.9}
$$

Combining (5.3), (5.7), and (5.9) with (5.8) yields (5.1). The proof is now complete. □

An immediate consequence of Lemma 5.1 is the following:

COROLLARY 5.2. *Under the same assumptions as in Lemma* 5.1*, there exists* $C > 0$ *such that*

$$\mathbf{E}(\epsilon) \le \frac{c_0^F}{\epsilon} H^{n-1}(\Gamma) + \mathbf{D} + C \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha^{-1}(\lambda)| \sqrt{\widetilde{F}(\lambda)}\, d\lambda + o(1). \tag{5.10}$$

PROOF. Since Γ is smooth, it is easy to see that

$$H^{n-1}(\Gamma(t)) \le H^{n-1}(\Gamma) + C|t|, \quad t \in [-\epsilon^\beta, \epsilon^\beta].$$

Thus we have

$$\begin{aligned}
&\frac{2}{\epsilon}\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\Gamma(\epsilon\alpha_\epsilon^{-1}(\lambda))\big)\sqrt{\widetilde{F}(\lambda)}\, d\lambda \\
&\quad \le \frac{2}{\epsilon}\bigg(\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} \sqrt{\widetilde{F}(\lambda)}\, d\lambda\bigg) H^{n-1}(\Gamma) + C\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha_\epsilon^{-1}(\lambda)|\sqrt{\widetilde{F}(\lambda)}\, d\lambda \\
&\quad = \frac{c_0^F}{\epsilon} H^{n-1}(\Gamma) + C\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha_\epsilon^{-1}(\lambda)|\sqrt{\widetilde{F}(\lambda)}\, d\lambda.
\end{aligned}$$

This, (5.1), and (A.11) imply (5.10). □

By comparing Corollary 4.3 with Corollary 5.2, we find that the expansions of both the lower bound and the upper bound of $\mathbf{E}(\epsilon)$ have the same leading-order term

$$\frac{c_0^F}{\epsilon} H^{n-1}(\Gamma).$$

Based on this, we have the following:

COROLLARY 5.3. *Under the assumptions as in Lemma* 5.1*, if* $u_\epsilon \in H^1_{g_\epsilon}(\Omega, \mathbb{R}^k)$ *is a minimizer of* $\mathbf{E}(\epsilon)$*, then, for any small* $\delta > 0$*, there exists* $C > 0$*, independent of* ϵ*, such that the map* $\widetilde{u}_\epsilon$ *defined by*

$$\widetilde{u}_\epsilon(x) = \begin{cases} \Pi(u_\epsilon(x)) & \text{if } x \in \Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}, \\ u_\epsilon(x) & \text{if } x \in \Omega \setminus (\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}), \end{cases} \tag{5.11}$$

satisfies

$$\int_\Omega |\nabla \widetilde{u}_\epsilon| \le C. \tag{5.12}$$

PROOF. First observe that Corollaries 4.3 and 5.2 imply that

$$\int_{\Omega^+_{\epsilon,\delta}\cup\Omega^-_{\epsilon,\delta}} \big|\nabla\big(\Pi(u_\epsilon(x))\big)\big|^2\, dx \le \mathbf{D} + C\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha^{-1}(\lambda)|\sqrt{\widetilde{F}(\lambda)}\, d\lambda + o(1). \tag{5.13}$$

On the other hand, there exist $C > 0$ independent of ϵ and $c = c(\delta) > 0$ such that

$$\mathbf{E}(\epsilon) \le \frac{C}{\epsilon}$$

and

$$F(u_\epsilon(x)) \ge c \quad \text{for all } x \in \Omega \setminus (\Omega^+_{\epsilon,\delta} \cup \Omega^-_{\epsilon,\delta}).$$

So we have

$$\begin{aligned} \int\limits_{\Omega\setminus(\Omega^+_{\epsilon,\delta}\cup\Omega^-_{\epsilon,\delta})} |\nabla u_\epsilon| dx &\le \frac{1}{c} \int\limits_{\Omega\setminus(\Omega^+_{\epsilon,\delta}\cup\Omega^-_{\epsilon,\delta})} \sqrt{F(u_\epsilon)}|\nabla u_\epsilon| dx \\ &\le \frac{\epsilon}{c} \int\limits_{\Omega} \left(|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2} F(u_\epsilon) \right) dx \le C. \end{aligned} \tag{5.14}$$

It is clear that (5.12) follows from (5.13) and (5.14). □

6 Proof of Theorem 1.3

Preliminaries

This section is devoted to the proof of Theorem 1.3. First, we will show that the sequence of maps

$$\Pi(u_\epsilon)\chi_{\Omega^+_{\epsilon,\delta}\cup\Omega^-_{\epsilon,\delta}}$$

converges to a map $v \in \mathbf{A}$ weakly in $\mathrm{SBV}(\mathbb{R}^n)$. Then we will show that v is a minimizing harmonic map in $\mathbf{A}$, under the assumption (A2) that Γ is strictly stable.

Recall from [2] that $\mathrm{SBV}(\Omega)$, a subspace of $\mathrm{BV}(\Omega)$, consists of all functions of bounded variations such that the Cantor part of the distributional derivatives is 0, i.e.,

$$u \in \mathrm{SBV}(\Omega) \Longleftrightarrow Du = \nabla u \, dx + (u^+ - u^-) H^{n-1} \llcorner J_u,$$

where J_u denotes the set of jump discontinuity of u. Set

$$\mathrm{SBV}(\Omega, \mathbb{R}^k) = \{u = (u^1, \dots, u^k) : u^i \in \mathrm{SBV}(\Omega) \text{ for } i = 1, \dots, k\}.$$

The crucial step to prove Theorem 1.3 is the following:

LEMMA 6.1. *Assume Γ satisfies* (A1)*, $g_\epsilon : \partial\Omega \to \mathbb{R}^k$ satisfies* (1.17)*, and $\mathbf{A} \neq \varnothing$, and there exists at least a minimizing harmonic map $u \in \mathbf{A}$ such that $u(x^\pm) \in H^1(\Gamma, M^\pm)$. Then there exist $\delta_i \to 0$, $\epsilon_i = \epsilon_i(\delta_i) \to 0$, and $v \in \mathrm{SBV}(\Omega, N)$ such that after passing to subsequences, the maps v_i defined by*

$$v_i \equiv \Pi(u_{\epsilon_i})\chi_{\Omega^+_{\epsilon_i,\delta_i}\cup\Omega^-_{\epsilon_i,\delta_i}}$$

converge to v in $L^1(\Omega,\mathbb{R}^k)$, $\nabla v_i \to \nabla v$ weakly in $L^1(\Omega,\mathbb{R}^{nk})$. Moreover, the following statements hold:

$$\begin{gathered} v \in H^1(\Omega^+,N^+)\cap H^1(\Omega^-,N^-), \quad v\big|_{\partial\Omega} = g, \\ |v(x^+) - v(x^-)| = d_N \quad \text{for } H^{n-1} \text{ a.e. } x\in\Gamma, \end{gathered} \tag{6.1}$$

and

$$\int_{\Omega^+} |\nabla v|^2\,dx + \int_{\Omega^-} |\nabla v|^2\,dx \le \liminf_{i\to\infty} \int_{\Omega^+_{\epsilon_i,\delta_i}\cup\Omega^-_{\epsilon_i,\delta_i}} |\nabla(\Pi(u_{\epsilon_i}))|^2\,dx. \tag{6.2}$$

Furthermore, $u_{\epsilon_i}\to v$ in $L^1(\Omega,\mathbb{R}^k)$.

PROOF. The proof of the lemma consists of six claims.

Claim 1. There exist $\delta_i\to 0$ and $\epsilon_i(\le \delta_i^{4/\alpha})\to 0$ such that

$$\begin{aligned} H^{n-1}(\Gamma)-\epsilon_i^\alpha \le H^{n-1}(\partial\Omega^+_{\epsilon_i,\delta_i}\cap\Omega) \le H^{n-1}(\Gamma)+\epsilon_i^{\alpha/2}, \\ H^{n-1}(\Gamma)-\epsilon_i^\alpha \le H^{n-1}(\partial\Omega^-_{\epsilon_i,\delta_i}\cap\Omega) \le H^{n-1}(\Gamma)+\epsilon_i^{\alpha/2}. \end{aligned} \tag{6.3}$$

To prove this claim, observe that Corollary 5.2 and the proof of Lemma 4.1 imply that there exists $C>0$, independent of ϵ, such that

$$\int_0^{\frac{d_N}{2}} \big(H^{n-1}(\partial\Omega^+_{\epsilon,\lambda}\cap\Omega)+H^{n-1}(\partial\Omega^-_{\epsilon,\lambda}\cap\Omega)\big)\big(2\sqrt{f(\lambda^2)}\big)d\lambda \le c_0^F H^{n-1}(\Gamma)+C\epsilon.$$

It follows from the area minimality of Γ that (see (4.9) in the proof of Corollary 4.3)

$$H^{n-1}(\partial\Omega^\pm_{\epsilon,\lambda}\cap\Omega)\ge H^{n-1}(\Gamma)-C\epsilon^\alpha \quad \forall\lambda\in\left[0,\frac{d_N}{2}\right]. \tag{6.4}$$

Hence, for any $\delta>0$, we have

$$\begin{aligned} &\int_{\frac{\delta}{2}}^{\delta} \big(H^{n-1}(\partial\Omega^+_{\epsilon,\lambda}\cap\Omega)+H^{n-1}(\partial\Omega^-_{\epsilon,\lambda}\cap\Omega)\big)\big(2\sqrt{f(\lambda^2)}\big)d\lambda \\ &\quad\le c_0^F H^{n-1}(\Gamma)+C\epsilon \\ &\qquad -2\left\{\int_0^{\frac{\delta}{2}}+\int_\delta^{\frac{d_N}{2}}\right\}H^{n-1}\big((\partial\Omega^+_{\epsilon,\lambda}\cap\Omega)\cup(\partial\Omega^-_{\epsilon,\lambda}\cap\Omega)\big)\sqrt{f(\lambda^2)}\,d\lambda \\ &\quad\le c_0^F H^{n-1}(\Gamma)+C\epsilon-4[(H^{n-1}(\Gamma)-C\epsilon^\alpha]\left[\left\{\int_0^{\frac{\delta}{2}}+\int_\delta^{\frac{d_N}{2}}\right\}\sqrt{f(\lambda^2)}\,d\lambda\right] \\ &\quad\le 4H^{n-1}(\Gamma)\int_{\frac{\delta}{2}}^{\delta}\sqrt{f(\lambda^2)}\,d\lambda + C\epsilon^\alpha, \end{aligned}$$

where we have used (2.7) in the last step. This, combined with Fubini's theorem, implies that there exists $\widetilde{\delta} \in (\frac{\delta}{2}, \delta)$ such that

$$H^{n-1}(\partial\Omega^+_{\epsilon,\widetilde{\delta}} \cap \Omega) + H^{n-1}(\partial\Omega^-_{\epsilon,\widetilde{\delta}} \cap \Omega) \le 2H^{n-1}(\Gamma) + C\frac{\epsilon^\alpha}{\delta^2}. \tag{6.5}$$

Hence (6.3) follows from (6.4) and (6.5) provided

$$\epsilon_i \approx \delta_i^{4/\alpha}, \quad \delta_i \to 0.$$

Claim 2. We have

$$\chi_{\Omega^+_{\epsilon_i,\delta_i}} \rightharpoonup \chi_{\Omega^+}, \quad \chi_{\Omega^-_{\epsilon_i,\delta_i}} \rightharpoonup \chi_{\Omega^-} \text{ weakly in BV}(\Omega) \text{ as } i \to \infty.$$

It follows from (6.3) that there exist two sets $E^\pm \subset \Omega$ with finite perimeters such that, after taking subsequences,

$$\Omega^+_{\epsilon_i,\delta_i} \rightharpoonup E^+, \quad \Omega^-_{\epsilon_i,\delta_i} \rightharpoonup E^-,$$

as weak convergence of sets of finite perimeters. In particular,

$$\partial\Omega^\pm_{\epsilon_i,\delta_i} \llcorner \Omega \rightharpoonup \partial E^\pm \llcorner \Omega$$

as weak convergence of currents. By the lower semicontinuity and (6.3), we have

$$H^{n-1}(\partial^* E^\pm \llcorner \Omega) \le H^{n-1}(\Gamma). \tag{6.6}$$

On the other hand, since

$$\partial\big(\partial\Omega^\pm_{\epsilon_i,\delta_i} \llcorner \Omega\big) = \{x \in \partial\Omega : \mathrm{d}(g_\epsilon(x), N^\pm) = \delta_i\} \rightharpoonup \Sigma$$

as weak convergence of currents, we have

$$\partial(\partial E^\pm \llcorner \Omega) = \Sigma.$$

Hence the area minimality of Γ implies

$$H^{n-1}(\partial^* E^\pm \llcorner \Omega) \ge H^{n-1}(\Gamma). \tag{6.7}$$

(6.6) and (6.7) imply that $\partial E^\pm \llcorner \Omega$ is also area minimizing with $\partial(\partial E^\pm \llcorner \Omega) = \Sigma$. By the uniqueness assumption (A1), we conclude

$$\partial E^\pm \llcorner \Omega = \Gamma.$$

On the other hand, since

$$\partial\Omega^\pm_{\epsilon_i,\delta_i} \llcorner (\mathbb{R}^n \setminus \Omega) \to \Sigma^\pm,$$

as weak convergence of currents, we conclude

$$\partial E^\pm \llcorner (\mathbb{R}^n \setminus \Omega) = \Sigma^\pm.$$

Therefore $E^\pm = \Omega^\pm$ and Claim 2 is proved.

Claim 3. For any $\eta > 0$, it holds

$$\lim_{i\to\infty} H^{n-1}\big(\partial\Omega^\pm_{\epsilon_i,\delta_i} \cap \{x \in \Omega : \mathrm{d}_\Gamma(x) \ge \eta\}\big) = 0. \tag{6.8}$$

To prove it, we denote

$$\mu_i^{\pm} = H^{n-1} \llcorner \left(\partial\Omega^{\pm}_{\epsilon_i,\delta_i} \cap \Omega\right).$$

Then we have

$$\mu_i^{\pm} \rightharpoonup H^{n-1} \llcorner \Gamma$$

as weak convergence of Radon measures. By the lower semicontinuity of Radon measures, we have

$$\begin{aligned} H^{n-1}(\Gamma) &= H^{n-1}(\Gamma \cap \{x \in \Omega : \mathrm{d}_\Gamma(x) < \eta\}) \\ &\le \liminf_{i\to\infty} \mu_i^{\pm}(\{x \in \Omega : \mathrm{d}_\Gamma(x) < \eta\}). \end{aligned}$$

Since (6.3) implies

$$\lim_{i\to\infty} \mu_i^{\pm}(\Omega) = H^{n-1}(\Gamma),$$

(6.8) follows and Claim 3 is proved.

Claim 4. For the sequence $\{v_i\}$ given above, there exists $v \in \mathrm{SBV}(\Omega, \mathbb{R}^k)$ such that, after taking possible subsequences,

$$v_i \to v \ \text{ in } L^1(\Omega, \mathbb{R}^k) \quad \text{and} \quad \nabla v_i \to \nabla v \ \text{ weakly in } L^1(\Omega, \mathbb{R}^{nk}). \tag{6.9}$$

It follows from the definition of v_i that the absolute continuous part of its distributional derivative is

$$\nabla v_i = \nabla(\Pi(u_{\epsilon_i}))\chi_{\Omega^+_{\epsilon_i,\delta_i} \cup \Omega^-_{\epsilon_i,\delta_i}},$$

so that (5.12) implies

$$\int_\Omega |\nabla v_i|^2\, dx \le \int_{\Omega^+_{\epsilon_i,\delta_i} \cup \Omega^-_{\epsilon_i,\delta_i}} |\nabla(\Pi(u_{\epsilon_i}))|^2\, dx \le C. \tag{6.10}$$

The set of jump discontinuities of v_i satisfies

$$J_{v_i} \subset \partial\left(\Omega^+_{\epsilon_i,\delta_i} \cup \Omega^-_{\epsilon_i,\delta_i}\right) \cap \Omega,$$

so

$$\begin{aligned} H^{n-1}(J_{v_i}) &\le H^{n-1}\left(\partial\Omega^+_{\epsilon_i,\delta_i} \cap \Omega\right) + H^{n-1}\left(\partial\Omega^-_{\epsilon_i,\delta_i} \cap \Omega\right) \\ &\le 2H^{n-1}(\Gamma) + 1. \end{aligned} \tag{6.11}$$

On the other hand, it is easy to see that

$$\|v_i\|_{L^\infty(\Omega)} \le C. \tag{6.12}$$

It follows from (6.10), (6.11), and (6.12) that $\{v_i\} \subset \mathrm{SBV}(\Omega, \mathbb{R}^k)$ is a weakly compact sequence (see [2, p. 128]). Hence we may assume that there exists $v \in \mathrm{SBV}(\Omega, \mathbb{R}^k)$ such that (6.9) holds. Thus Claim 4 is proved.

Claim 5. The map v obtained in Claim 4 has all properties stated in Lemma 6.1.

Since $v_i(x) \in N^\pm$ for a.e. $x \in \Omega^\pm_{\epsilon_i,\delta_i}$ and $v_i \to v$ in $L^1(\Omega, \mathbb{R}^k)$, we have

$$v(x) \in N^\pm \quad \text{for a.e. } x \in \Omega^\pm. \tag{6.13}$$

By lower semicontinuity, we have

$$\int_{\Omega^\pm} |\nabla v|^2\, dx \le \liminf_{i\to\infty} \int_{\Omega^\pm} |\nabla v_i|^2\, dx = \liminf_{i\to\infty} \int_{\Omega^\pm_{\epsilon_i,\delta_i}} \left|\nabla(\Pi(u_{\epsilon_i}))\right|^2 dx. \tag{6.14}$$

This implies (6.2).

Now we want to show that the set of jump discontinuities of v satisfies

$$H^{n-1}(J_v \cap \Omega^\pm) = 0. \tag{6.15}$$

For any $\eta > 0$, by lower semicontinuity and (6.8), we have

$$\begin{aligned}
&H^{n-1}(J_v \cap \{x \in \Omega^\pm : \mathrm{d}_\Gamma(x) > \eta\}) \\
&\quad \le \liminf_{i\to\infty} H^{n-1}(J_{v_i} \cap \{x \in \Omega^\pm : \mathrm{d}_\Gamma(x) > \eta\}) \\
&\quad \le \liminf_{i\to\infty} H^{n-1}\big(\partial\Omega^\pm_{\epsilon_i,\delta_i} \cap \{x \in \Omega : \mathrm{d}_\Gamma(x) \ge \eta\}\big) = 0.
\end{aligned}$$

Sending $\eta \to 0$ in the above inequalities yields (6.15). It follows from (6.13), (6.14), and (6.15) that $v \in H^1(\Omega^\pm, N^\pm)$. It is easy to see that $v = g$ on $\partial\Omega$, and the traces $v(x^+) \in N^+$ and $v(x^-) \in N^-$ exist for H^{n-1} a.e. $x \in \Gamma$.

Next we need to show that

$$|v(x^+) - v(x^-)| = d_N \quad \text{for } H^{n-1}\text{-a.e. } x \in \Gamma. \tag{6.16}$$

Suppose that (6.16) were false. Then there exist $\eta_0 > 0$, and an H^{n-1}-measurable set $E \subset \Gamma \cap \Omega_{\eta_0}$, with $H^{n-1}(E) \ge \eta_0$, such that

$$|v(x^+) - v(x^-)| \ge d_N + \eta_0 \quad \forall x \in E. \tag{6.17}$$

From the definition of one-sided traces of v on Γ and L^1-convergence of v_i to v on generic slices $\{x \in \Omega : \mathrm{d}_\Gamma(x) = t\}$ ($t > 0$), we may assume that there exists $t_i \downarrow 0$, with $\frac{t_i}{\epsilon_i} \to \infty$, such that

$$\{x \pm t_i \nu_\Gamma(x) : x \in E\} \subset \Omega^\pm_{\epsilon_i,\delta_i}, \tag{6.18}$$

where ν_Γ is the unit normal vector field of Γ, pointed inward to Ω^+, and

$$|v_i(x + t_i\nu_\Gamma(x)) - v_i(x - t_i\nu_\Gamma(x))| \ge d_N + \frac{\eta_0}{2} \quad \forall x \in E. \tag{6.19}$$

(6.18) and the definition of v_i imply that (6.19) holds with v_i replaced by $\Pi(u_{\epsilon_i})$ and $\frac{\eta_0}{2}$ replaced by $\frac{\eta_0}{4}$. In fact, since

$$|u_{\epsilon_i} - \Pi(u_{\epsilon_i})| \le \delta_i \quad \text{on } \Omega^\pm_{\epsilon_i,\delta_i},$$

we have

$$|u_{\epsilon_i}(x + t_i\nu_\Gamma(x)) - u_{\epsilon_i}(x - t_i\nu_\Gamma(x))| \ge d_N + \frac{\eta_0}{4} \quad \forall x \in E. \tag{6.20}$$

We will show that (6.18) and (6.20) give a desired contradiction. To see it, we need the next claim:

Claim 6. For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$c^F(p^+, p^-) \le c_0^F + \delta \quad \text{for } p^\pm \in N^\pm \Rightarrow |p^+ - p^-| \le d_N + \epsilon. \tag{6.21}$$

Suppose that (6.21) were false. Then there exist $\epsilon_0 > 0$ and $p_i^\pm \in N^\pm$ such that

$$(c_0^F \le)\, c^F(p_i^+, p_i^-) \le c_0^F + \frac{1}{i}, \quad \text{but } |p_i^+ - p_i^-| \ge d_N + \epsilon_0 \;\; \forall i.$$

We may assume that there are $p_0^\pm \in N^\pm$ such that $p_i^\pm \to p_0^\pm$. Then $|p_0^+ - p_0^-| \ge d_N + \epsilon_0$. Since $c^F(\cdot,\cdot)$ is Lipschitz-continuous, we have $c^F(p_0^+, p_0^-) = c_0^F$. Hence Theorem 2.1 implies $|p_0^+ - p_0^-| = d_N$. We get a contradiction.

Let E_i be the subset of Γ_{t_i} defined by

$$E_i = \{x + t\nu_\Gamma(x) : x \in E,\ -t_i \le t \le t_i\}.$$

Since $\Gamma \in C^\infty$ and t_i is sufficiently small, by a simple change of variables we have

$$\begin{aligned}
&\int_{E_i} \left(|\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i^2} F(u_{\epsilon_i}) \right) dx \\
&\ge (1 - o(1)) \int_E \int_{[-t_i, t_i]} \Big(\big|\nabla_t (u_{\epsilon_i}(x + t\nu_\Gamma(x)))\big|^2 \\
&\qquad\qquad + \frac{1}{\epsilon_i^2} F(u_{\epsilon_i}(x + t\nu_\Gamma(x))) \Big) dt\, dH^{n-1}(x) \\
&= \frac{1 - o(1)}{\epsilon_i} \int_E \int_{\left[-\frac{t_i}{\epsilon_i}, \frac{t_i}{\epsilon_i}\right]} \Big(\big|\nabla_t (u_{\epsilon_i}(x + \epsilon_i t\nu_\Gamma(x)))\big|^2 \\
&\qquad\qquad + F(u_{\epsilon_i}(x + t\nu_\Gamma(x))) \Big) dt\, dH^{n-1}(x) \\
&\ge \frac{1 - o(1)}{\epsilon_i} \int_E \alpha_i(x) dH^{n-1}(x),
\end{aligned}$$

where $\alpha_i(x)$, $x \in E$, is given by

$$\begin{aligned}
\alpha_i(x) := \inf \Big\{ &\int_{-\frac{t_i}{\epsilon_i}}^{\frac{t_i}{\epsilon_i}} (|X'|^2 + F(X)) dt : \\
&X \in H^1\left(\left[-\frac{t_i}{\epsilon_i}, \frac{t_i}{\epsilon_i} \right], \mathbb{R}^k \right),\ X\left(\pm \frac{t_i}{\epsilon_i} \right) = u_{\epsilon_i}(x \pm t_i \nu_\Gamma(x)) \Big\}.
\end{aligned}$$

It follows from (6.18) and (6.20) that for $x \in E$, there exist $p^{\pm}(x) \in N^{\pm}$ such that

$$u_{\epsilon_i}(x \pm t_i \nu_\Gamma(x)) \to p^{\pm}(x) \quad \text{and} \quad |p^+(x) - p^-(x)| \geq d_N + \frac{\eta_0}{4}.$$

This and (6.21) imply that there exists $\eta_1 > 0$ such that

$$c^F(p^+(x), p^-(x)) \geq c_0^F + \eta_1, \quad x \in E.$$

It is not hard to see from the definition of $\alpha_i(x)$ and $c^F(\cdot, \cdot)$ that we have

$$\alpha_i(x) \geq c^F(p^+(x), p^-(x)) - o(1), \quad x \in E.$$

Therefore we obtain

$$\begin{aligned} &\int_{E_i} \left(|\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i^2} F(u_{\epsilon_i}) \right) dx \\ &\quad \geq \frac{1 - o(1)}{\epsilon_i} \int_E (c^F(p^+(x), p^-(x)) - o(1)) dH^{n-1}(x) \\ &\quad \geq \frac{c_0^F + \eta_1 - o(1)}{\epsilon_i} H^{n-1}(E). \end{aligned} \tag{6.22}$$

Notice that arguments similar to those for Corollary 4.2 imply that

$$\int_{\Omega \setminus E_i} \left(|\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i^2} F(u_{\epsilon_i}) \right) dx \geq \frac{c_0^F}{\epsilon_i} H^{n-1}(\Gamma \setminus E) - C. \tag{6.23}$$

Combining (6.22) with (6.23), we obtain that for ϵ_i sufficiently small,

$$\int_{\Omega} \left(|\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i^2} F(u_{\epsilon_i}) \right) dx \geq \frac{c_0^F}{\epsilon_i} H^{n-1}(\Gamma) + \frac{\eta_1}{2\epsilon_i} H^{n-1}(E) - C.$$

This contradicts the upper bound obtained by Corollary 5.2. Thus (6.16) holds and hence Claim 5 is proved.

To show u_{ϵ_i} converges to v in $L^1(\Omega, \mathbb{R}^k)$, let $\widetilde{u}_{\epsilon_i}$ be given by (5.11). Then by Corollary 5.3, we have

$$\int_{\Omega} |\nabla \widetilde{u}_{\epsilon_i}| dx \leq C.$$

Since $\widetilde{u}_{\epsilon_i}\big|_{\partial\Omega} = g_{\epsilon_i}$ and g_{ϵ_i} is uniformly bounded on $\partial\Omega$, it follows from the Sobolev and Poincaré inequalities that

$$\int_{\Omega} |\widetilde{u}_{\epsilon_i}|^{\frac{n}{n-1}} dx \leq C.$$

Since $\widetilde{u}_{\epsilon_i} = u_{\epsilon_i}$ on $\Omega \setminus (\Omega^+_{\epsilon_i,\delta_i} \cup \Omega^-_{\epsilon_i,\delta_i})$,

$$\left|\Omega \setminus \left(\Omega^+_{\epsilon_i,\delta_i} \cup \Omega^-_{\epsilon_i,\delta_i}\right)\right| \le C\frac{\epsilon_i}{\delta_i^2} \le C\delta_i^{\frac{4}{\alpha}-2} \to 0;$$

we have

$$\int_{\Omega\setminus(\Omega^+_{\epsilon_i,\delta_i}\cup\Omega^-_{\epsilon_i,\delta_i})} |u_{\epsilon_i}|dx \le \|\widetilde{u}_{\epsilon_i}\|_{L^{\frac{n}{n-1}}(\Omega)} |\Omega \setminus \left(\Omega^+_{\epsilon_i,\delta_i} \cup \Omega^-_{\epsilon_i,\delta_i}\right)|^{\frac{1}{n}} \to 0.$$

On the other hand,

$$\begin{aligned}\int_{\Omega^{\pm}_{\epsilon_i,\delta_i}} |u_{\epsilon_i} - \Pi(u_{\epsilon_i})|dx &\le C\int_{\Omega^{\pm}_{\epsilon_i,\delta_i}} d(u_{\epsilon_i}, N)dx \\ &\le C\int_{\Omega^{\pm}_{\epsilon_i,\delta_i}} \sqrt{F(u_{\epsilon_i})}\,dx \le C\sqrt{\epsilon_i} \to 0.\end{aligned}$$

These two inequalities, combined with

$$\Pi(u_{\epsilon_i})\chi_{\Omega^+_{\epsilon_i,\delta_i}\cup\Omega^-_{\epsilon_i,\delta_i}} \to v \quad \text{in } L^1(\Omega),$$

imply $u_{\epsilon_i} \to v$ in $L^1(\Omega)$. Thus we prove all the conclusions of Lemma 6.1. □

Remark 6.2. We would like to remark that without the strict stability assumption (A2) on Γ, Lemma 5.1 and Lemma 6.1 imply that the following weaker version of Theorem 1.3 holds:

$$\left|\mathbf{E}(\epsilon) - \frac{c_0^F}{\epsilon} H^{n-1}(\Gamma) - \mathbf{D}\right| \le C\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha^{-1}(\lambda)|\sqrt{\widetilde{F}(\lambda)}\,d\lambda + o(1), \tag{6.24}$$

where $\alpha \in H^1(\mathbb{R})$, with $\alpha(\pm\infty) = \pm\frac{d_N}{2}$, is given by (1.13). In other words, modulo the correction term

$$O\left(\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha^{-1}(\lambda)|\sqrt{\widetilde{F}(\lambda)}\,d\lambda\right),$$

$\mathbf{D}$ is the zeroth-order term in the expansion of $\mathbf{E}(\epsilon)$ with respect to $\frac{1}{\epsilon}$.

Completion of Proof of Theorem 1.3

With the help of Lemmas 4.1, 5.1, and 6.1, there are two more steps needed to prove Theorem 1.3. The first step is to show the leading-order term in the lower-bound estimate (4.1) matches that in the upper-bound estimate (5.1), i.e.,

$$\left|\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} \left[H^{n-1}\left(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\right) - H^{n-1}\left(\Gamma(\epsilon\alpha_\epsilon^{-1}(\lambda))\right)\right]\sqrt{\widetilde{F}(\lambda)}\,d\lambda\right| = o(\epsilon). \tag{6.25}$$

To show (6.25), we need the following:

Claim 7. Under condition (A2), there exist $\eta_0 > 0$ and $C_0 > 0$ depending only on Γ and Ω such that

$$H^{n-1}(\Gamma(\lambda)) \leq H^{n-1}(\mathcal{S}(\lambda)) + C\lambda^2 \quad \forall \lambda \in [-\eta_0, \eta_0]. \tag{6.26}$$

It suffices to show (6.26) for $\lambda \in [0, \eta_0]$. First observe that since Γ is the unique minimal hypersurface spanned by Σ and $\Sigma(\lambda) \to \Sigma$ in the $C^{2,\alpha}$-norm for any $\alpha \in (0,1)$ as $\lambda \to 0^+$, we have

$$\mathcal{S}(\lambda) \rightharpoonup \mathcal{S}(0) \equiv \Gamma \quad \text{as } \lambda \to 0^+$$

as weak convergence of currents. Also, since

$$\lim_{\lambda \to 0^+} H^{n-1}(\mathcal{S}(\lambda)) = H^{n-1}(\Gamma)$$

and Γ is smooth up to its boundary, the standard interior regularity and boundary regularity of area-minimal hypersurfaces (cf. [24, chap. 7] and [12, p. 440]) imply that there exists η_0 such that $\mathcal{S}(\lambda), \lambda \in [0, \eta_0]$, is a family of minimal hypersurfaces smooth up to the boundaries, and $\mathcal{S}(\lambda)$ converges to Γ both in the Gromov-Hausdorff distance and the $C^{2,\alpha}$-topology as $\lambda \to 0^+$. Notice also that the $\mathcal{S}(\lambda)$ all lie in Ω^+, one side of Γ for all $0 \leq \lambda \leq \eta_0$. Thus we may assume that there exists a family of smooth unit vector fields on Γ, $V(\lambda, x) \in C^\infty([0, \eta_0] \times \Gamma, S^{n-1})$, with the property that

$$\lim_{\lambda \downarrow 0+} \|V(\lambda, \cdot) - \nu_\Gamma(\cdot)\|_{C^{2,\alpha}(\Gamma)} = 0, \tag{6.27}$$

and a family of smooth functions on Γ, $\phi : [0, \eta_0] \times \Gamma \to \mathbb{R}_+$, with the property

$$\|\phi(\lambda, \cdot)\|_{C^{2,\alpha}(\partial\Gamma)} \leq C\lambda \quad \forall \lambda \in [0, \eta_0], \tag{6.28}$$

such that

$$\mathcal{S}(\lambda) = \{x + \phi(\lambda, x)V(\lambda, x) : x \in \Gamma\} \quad \forall 0 \leq \lambda \leq \eta_0.$$

In order to complete the proof of Claim 7, now we need to show the following claim.

Claim 8. There exists $C_0 > 0$ depending only on Γ and Ω such that

$$\|\phi(\lambda, \cdot)\|_{C^0(\Gamma)} \leq C_0 \lambda \quad \forall \lambda \in [0, \eta_0]. \tag{6.29}$$

For otherwise there exists $\lambda_i \to 0^+$ such that

$$\|\phi(\lambda_i, \cdot)\|_{C^{2,\alpha}(\Gamma)} \to 0, \quad \text{but} \quad \frac{1}{|\lambda_i|}\|\phi(\lambda_i, \cdot)\|_{C^0(\Gamma)} = k_i \to +\infty.$$

Set $\Phi(\lambda, x) = \phi(\lambda, x)V(\lambda, x)$ for $(\lambda, x) \in [0, \eta_0] \times \Gamma$. Consider the functions

$$\Phi_i(x) = \frac{\Phi(\lambda_i, x)}{\|\phi(\lambda_i, \cdot)\|_{C^0(\Gamma)}}, \quad x \in \Gamma.$$

Then we have

$$\|\Phi_i\|_{C^0(\Gamma)} = 1, \quad \|\Phi_i\|_{C^{2,\alpha}(\partial\Gamma)} \leq \frac{C_0}{k_i} \to 0 \quad \text{as } i \to \infty.$$

Let $\mathcal{M}$ denote the minimal surface operator. Since Γ and $\mathcal{S}(\lambda_i)$ are minimal surfaces, we have

$$\mathcal{M}[x] = \mathcal{M}[x + \Phi(\lambda_i, x)] = 0, \quad x \in \Sigma.$$

It is easy to check that $\Phi_i(\cdot)$ satisfies an almost Jacobi field equation:[2]

$$\mathcal{L}_x^i[\Phi_i] = 0 \quad \text{on } \Gamma,$$

where

$$\mathcal{L}_x^i = \int_0^1 (D_x\mathcal{M})[x + t\Phi(\lambda_i, x)]dt.$$

Notice that $\mathcal{L}_x^i$ is a family of second-order uniformly elliptic equations whose coefficients are uniformly bounded in $C^{2,\alpha}$. Thus the regularity theorem for linear elliptic equations implies that

$$\sup_i \|\Phi_i\|_{C^{2,\alpha}(\Gamma)} \leq C.$$

Hence, after taking possible subsequences, we can assume that $\Phi_i \to \Phi$ in $C^2(\Gamma)$ as $i \to \infty$ so that $\Phi \in C^2(\Gamma)$ is a normal vector field on Γ, which satisfies the Jacobi field equation

$$\mathcal{L}_x[\Phi] = 0 \quad \text{on } \Gamma,$$

where $\mathcal{L}_x$ is the Jacobi field operator on Γ, and

$$\|\Phi\|_{C^0(\Gamma)} = 1, \quad \Phi\big|_{\partial\Gamma} = 0.$$

In particular, Φ is a nontrivial Jacobi field on Γ. This contradicts the strict stability assumption (A2) for Γ. This proves Claim 8.

Now we continue to prove Claim 7. For simplicity, we may assume that there exists $\psi \in C^\infty([0, \eta_0] \times \Gamma)$ with

$$\|\psi(\lambda, \cdot)\|_{C^0(\Gamma)} \leq C\lambda \quad \forall \lambda \in [0, \eta_0]$$

such that

$$\Gamma(\lambda) = \{x + \psi(\lambda, x)V(\lambda, x) : x \in \Gamma\}.$$

Then by (6.29) we have

$$\|\phi(\lambda, \cdot)\|_{C^0(\Gamma)} + \|\psi(\lambda, \cdot)\|_{C^0(\Gamma)} \leq C\lambda \quad \forall \lambda \in [0, \eta_0].$$

Consider the area function

$$\mathcal{A}_\lambda(t) = H^{n-1}\big(\{x + (t\psi(\lambda, x) + (1-t)\phi(\lambda, x))V(\lambda, x) : x \in \Gamma\}\big), \quad t \in \left[-\frac{1}{2}, 1\right].$$

Since $\mathcal{S}(\lambda)$ is area minimizing, and

$$\partial\big(\{x + (t\psi(\lambda, x) + (1-t)\phi(\lambda, x))V(\lambda, x) : x \in \Gamma\}\big) = \partial\mathcal{S}(\lambda) \quad \forall t \in \left[-\frac{1}{2}, 1\right],$$

[2] This type of argument is standard in the theory of minimal hypersurfaces; the interested reader can refer to [25] for more details.

we have $\mathcal{A}_\lambda(0)(= H^{n-1}(\mathcal{S}(\lambda))) \le \mathcal{A}_\lambda(t)$ for $t \in [-\frac{1}{2}, 1]$. Hence

$$\mathcal{A}_\lambda'(0) = 0.$$

Moreover, by the inequality (6.29) and direct calculations, we have that there exists $C > 0$ independent of $\lambda \in [0, \eta_0]$ such that

$$|\mathcal{A}_\lambda''(t)| \le C \|\phi(\lambda, x) - \psi(\lambda, x)\|^2_{C^0(\Gamma)} \le \lambda^2 \quad \text{for } t \in [0, 1].$$

Hence by Taylor's expansion, we have

$$H^{n-1}(\Gamma(\lambda)) = \mathcal{A}_\lambda(1) \le \mathcal{A}_\lambda(0) + C\lambda^2 = H^{n-1}(\mathcal{S}(\lambda)) + C\lambda^2 \quad \forall \lambda \in [0, \eta_0].$$

This proves Claim 7.

Now we continue to prove Theorem 1.3. As an immediate consequence of Claim 7, we obtain

$$\begin{aligned}
&\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\big)\sqrt{\widetilde{F}(\lambda)}\,d\lambda \\
&\quad \le \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\Gamma(\epsilon\alpha_\epsilon^{-1}(\lambda))\big)\sqrt{\widetilde{F}(\lambda)}\,d\lambda \\
&\quad \le \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\big)\sqrt{\widetilde{F}(\lambda)}\,d\lambda \\
&\qquad + C\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} (\epsilon\alpha_\epsilon^{-1}(\lambda))^2\sqrt{\widetilde{F}(\lambda)}\,d\lambda \\
&\quad = \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\big)\sqrt{\widetilde{F}(\lambda)}\,d\lambda \\
&\qquad + C\epsilon^2\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} (\alpha_\epsilon^{-1}(\lambda))^2\sqrt{\widetilde{F}(\lambda)}\,d\lambda \\
&\quad \le \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\mathcal{S}(\epsilon\alpha_\epsilon^{-1}(\lambda))\big)\sqrt{\widetilde{F}(\lambda)}\,d\lambda + C\epsilon^2, \qquad (6.30)
\end{aligned}$$

where we have used in the last step the following estimate:

$$\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} (\alpha_\epsilon^{-1}(\lambda))^2\sqrt{\widetilde{F}(\lambda)}\,d\lambda \le C.$$

Thus we prove (6.25).

It is clear that Lemma 4.1, Lemma 5.1, and (6.25) imply, after taking ϵ to 0, that

$$\int_{\Omega^+} |\nabla v|^2\,dx + \int_{\Omega^-} |\nabla v|^2\,dx \le \mathbf{D}.$$

On the other hand, by Lemma 6.1 we have $v \in \mathbf{A}$ so that

$$\int_{\Omega^+} |\nabla v|^2 \, dx + \int_{\Omega^-} |\nabla v|^2 \, dx \geq \mathbf{D}.$$

Hence v is a minimizing harmonic map in the class $\mathbf{A}$, and

$$\mathbf{E}(\epsilon) = \frac{2}{\epsilon} \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\Gamma(\epsilon\alpha_\epsilon^{-1}(\lambda))\big) \sqrt{\widetilde{F}(\lambda)} \, d\lambda + \mathbf{D} + o(1). \tag{6.31}$$

To show (1.22), first observe that, by the definition of α_ϵ (see (A.7)), a change of variables, and Proposition A.4, we have

$$\begin{aligned}
&\frac{2}{\epsilon} \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} H^{n-1}\big(\Gamma(\epsilon\alpha_\epsilon^{-1}(\lambda))\big) \sqrt{\widetilde{F}(\lambda)} \, d\lambda \\
&\quad = \frac{2}{\epsilon} \int_{-\frac{\epsilon^{\beta-1}}{2}}^{\frac{\epsilon^{\beta-1}}{2}} H^{n-1}(\Gamma(\epsilon t)) \sqrt{\widetilde{F}(\alpha(t))} \, \alpha'(t) dt + o(1) \\
&\quad = \frac{2H^{n-1}(\Gamma)}{\epsilon} \int_{-\frac{\epsilon^{\beta-1}}{2}}^{\frac{\epsilon^{\beta-1}}{2}} \sqrt{\widetilde{F}(\alpha(t))} \, \alpha'(t) dt \\
&\qquad + 2 \int_{-\frac{\epsilon^{\beta-1}}{2}}^{\frac{\epsilon^{\beta-1}}{2}} \left(\frac{H^{n-1}(\Gamma(\epsilon t)) - H^{n-1}(\Gamma)}{\epsilon t} \right) t \sqrt{\widetilde{F}(\alpha(t))} \, \alpha'(t) dt + o(1) \\
&\quad = \mathrm{I} + \mathrm{II} + o(1).
\end{aligned} \tag{6.32}$$

For the term I, we observe that by Proposition A.4 and (2.4), we have

$$\mathrm{I} = \frac{2H^{n-1}(\Gamma)}{\epsilon} \int_{\mathbb{R}} \sqrt{\widetilde{F}(\alpha(t))} \, \alpha'(t) dt + o(1) = \frac{c_0^F}{\epsilon} H^{n-1}(\Gamma) + o(1).$$

To estimate II, observe that since there exists $\eta_0 > 0$ such that

$$\mathcal{B}(\lambda) = H^{n-1}(\Gamma(\lambda)) \in C^1(-\eta_0, \eta_0),$$

we have

$$\mathcal{B}(\lambda) = \mathcal{B}(0) + \mathcal{B}'(0)\lambda + o(\lambda), \quad \lambda \in (-\eta_0, \eta_0).$$

This implies

$$\begin{aligned}
\frac{H^{n-1}(\Gamma(\epsilon t)) - H^{n-1}(\Gamma)}{\epsilon t} &= \frac{\mathcal{B}(\epsilon t) - \mathcal{B}(0)}{\epsilon t} \\
&= \mathcal{B}'(0) + o(1) \quad \forall t \in \left(-\frac{\epsilon^{\beta-1}}{2}, \frac{\epsilon^{\beta-1}}{2} \right).
\end{aligned}$$

Therefore

$$\left| \mathrm{II} - 2\mathcal{B}'(0) \int_{-\frac{\epsilon^{\beta-1}}{2}}^{\frac{\epsilon^{\beta-1}}{2}} t \sqrt{\widetilde{F}(\alpha(t))} \, \alpha'(t) dt \right| \leq o(1) \int_{-\frac{\epsilon^{\beta-1}}{2}}^{\frac{\epsilon^{\beta-1}}{2}} |t| \sqrt{\widetilde{F}(\alpha(t))} \, \alpha'(t) dt.$$

Since $t\sqrt{\widetilde{F}(\alpha(t))}\,\alpha'(t)$ is an odd function of t, we have

$$2\mathcal{B}'(0)\int_{-\frac{\epsilon^{\beta-1}}{2}}^{\frac{\epsilon^{\beta-1}}{2}} t\sqrt{\widetilde{F}(\alpha(t))}\,\alpha'(t)dt = 0.$$

Moreover, by Proposition A.4 we have

$$\int_{-\frac{\epsilon^{\beta-1}}{2}}^{\frac{\epsilon^{\beta-1}}{2}} |t|\sqrt{\widetilde{F}(\alpha(t))}\alpha'(t)dt \le C.$$

Hence

$$\mathrm{II} = o(1).$$

Putting these estimates into (6.32) and (6.31) yields (1.22). The proof of Theorem 1.3 is now complete.

Appendix A: The Semidistance Function

In this appendix, we will first discuss some properties of the semidistance function $d^F(\cdot,\cdot)$ on $\mathbb{R}^k$ induced by any potential function F satisfying (1.1), which is needed for the proof of lower-bound estimates of $\epsilon\mathbf{E}(\epsilon)$ in Theorem 1.1.

The semidistance function $d^F(\cdot,\cdot)$ on $\mathbb{R}^k$ induced by the potential function F is defined as follows: For $p^\pm \in \mathbb{R}^k$,

$$d^F(p^-,p^+) = \inf\left\{2\int_{-1}^{1}\sqrt{F(\xi(t))}|\xi'(t)v|dt : \xi \in H^1([-1,1],\mathbb{R}^k),\ \xi(\pm 1) = p^\pm\right\}. \tag{A.1}$$

Since the integral in (A.1) is invariant under parametrization, we have that for any $L > 0$,

$$\begin{aligned} d^F(p^-,p^+) &= \inf\left\{2\int_{-L}^{L}\sqrt{F(\xi(t))}|\xi'(t)|dt : \xi \in H^1([-L,L],\mathbb{R}^k),\ \xi(\pm L) = p^\pm\right\} \\ &= \inf\left\{2\int_{\mathbb{R}}\sqrt{F(\xi(t))}|\xi'(t)|dt : \xi \in H^1(\mathbb{R},\mathbb{R}^k),\ \xi(\pm\infty) = p^\pm\right\}. \end{aligned}$$

It is easy to check that $d^F(p^-,p^+) = 0$ if and only if either $p^+ = p^-$ or $p^\pm \in N^-$ or $p^\pm \in N^+$. Using $d^F(\cdot,\cdot)$, we also define

$$d^F_{N^-}(q) = \inf_{p\in N^-} d^F(p,q) \quad \forall q \in \mathbb{R}^k, \tag{A.2}$$

$$d^F_0 = \inf\{d^F_{N^-}(q) : q \in N^+\}. \tag{A.3}$$

The following relation between c^F_0 and d^F_0 follows directly from the proof of Theorem 2.1.

PROPOSITION A.1. *Let c_0^F and d_0^F be defined by* (1.5) *and* (A.3)*, respectively. Then it holds that*

$$d_0^F = c_0^F \quad \left(= 4\int_0^{\frac{d_N}{2}} \sqrt{f(\lambda^2)}\, d\lambda \right). \tag{A.4}$$

Now we prove several important properties of $d_{N^-}^F(\cdot)$ and d_0^F.

PROPOSITION A.2. $d_{N^-}^F(q) = 0$ *if and only if* $q \in N^-$.

PROOF. If $q \in N^-$, then for any $p \in N^-$ we choose $\gamma_0 \in H^1([-1,1], N^-)$ such that $\gamma_0(-1) = p$ and $\gamma_0(1) = q$. Since $F(\gamma_0(t)) = 0$ for $t \in [-1,1]$, we have $d^F(p,q) = 0$. Hence $d_{N^-}^F(q) = 0$.

If $q \notin N^-$, then $\mathrm{d}(q, N^-) = \delta_0 > 0$. For any $p \in N^-$ and $\xi \in H^1([-1,1], \mathbb{R}^k)$ with $\xi(-1) = p$ and $\xi(1) = q$, there exists $-1 < t_0 < t_1 \le 1$ such that

$$\frac{\delta_0}{2} \le \mathrm{d}(\xi(t), N^-) \le \delta_0 \quad \forall t \in [t_0, t_1],$$
$$\mathrm{d}(\xi(t_0), N^-) = \frac{\delta_0}{2}, \quad \mathrm{d}(\xi(t_1), N^-) = \delta_0.$$

Since F satisfies (1.1), there exists $C_0 > 0$ depending on f and δ_0 such that

$$\inf_{t \in [t_0, t_1]} F(\xi(t)) \ge C_0^2.$$

Hence

$$\begin{aligned} 2\int_{-1}^{1} \sqrt{F(\xi(t))}|\xi'(t)|dt &\ge 2\int_{t_0}^{t_1} \sqrt{F(\xi(t))}|\xi'(t)|dt \\ &\ge 2C_0 \int_{t_0}^{t_1} |\xi'(t)|dt \ge 2C_0|\xi(t_1) - \xi(t_0)) \ge \delta_0 C_0. \end{aligned}$$

Taking the infimum over all such paths ξ yields

$$d^F(p,q) \ge \delta_0 C_0;$$

this implies $d_{N^-}^F(q) > 0$. □

PROPOSITION A.3. $d_{N^-}^F(\cdot) \in W_{\mathrm{loc}}^{1,\infty}(\mathbb{R}^k)$ *and*

$$|\nabla d_{N^-}^F(q)| = 2\sqrt{F(q)} \quad \text{a.e. } q \in \mathbb{R}^k. \tag{A.5}$$

PROOF. The proof is similar to [26, lemma 11], and we sketch it here.

To see that $d_{N^-}^F(\cdot)$ is locally Lipschitz, we pick any two points $y_1, y_2 \in \mathbb{R}^k$ and let

$$\gamma(t) = \frac{1+t}{2} y_2 + \frac{1-t}{2} y_1, \quad -1 \le t \le 1,$$

be the line segment joining y_1 to y_2. For any $p \in N^-$, it is easy to see that

$$d^F_{N^-}(y_2) \le d^F(p, y_2) \le d^F(p, y_1) + 2\int_{-1}^{1} \sqrt{F(\gamma(t))}|\gamma'(t)|dt$$
$$\le d^F(p, y_1) + 2\big(\max_{-1\le t\le 1} \sqrt{F(\gamma(t))}\big)|y_2 - y_1|.$$

Taking the infimum over $p \in N^1$, we have

$$d^F_{N^-}(y_2) \le d^F_{N^-}(y_1) + 2\big(\max_{y\in B_{y_1}(|y_2-y_1|)} \sqrt{F(y)}\big)|y_2 - y_1|.$$

Interchanging y_1 and y_2 implies

$$\big|d^F_{N^-}(y_1) - d^F_{N^-}(y_2)\big| \le 2\big(\max_{y\in B_{y_1}(|y_2-y_1|)} \sqrt{F(y)}\big)|y_2 - y_1|. \tag{A.6}$$

Hence $d^F_{N^-}(\cdot)$ is locally Lipschitz on $\mathbb{R}^k$.

Applying (A.6) with $y_1 = y$ and $y_2 = y_i \to y$, we have

$$\lim_{y_i\to y} \frac{\big|d^F_{N^-}(y_i) - d^F_{N^-}(y)\big|}{|y_i - y|} \le 2\sqrt{F(y)}.$$

Hence, by Rademacher's theorem (see, for example, the book [6]) we have

$$\big|\nabla d^F_{N^-}(y)\big| \le 2\sqrt{F(y)} \quad \text{a.e. } y \in \mathbb{R}^k.$$

To prove the other direction of (A.5), let $y \in \mathbb{R}^k$ be a differentiable point of $d^F_{N^-}(\cdot)$ and $p_0 \in N^-$ be such that

$$d^F_{N^-}(y) = d^F(p_0, y).$$

For any $\delta > 0$, let $\xi \in H^1([-1, 1], \mathbb{R}^k)$ be such that $\xi(-1) = p_0$, $\xi(1) = y$, and

$$\delta + d^F_{N^-}(y) \ge 2\int_{-1}^{1} \sqrt{F(\xi(t))}|\xi'(t)|dt.$$

For any small $0 < r < |y - p_0|$, let $0 < t_r < 1$ be such that $\xi(t) \in B_r(y)$ for $t_r < t \le 1$ and $|\xi(t_r) - y| = r$. Then we have

$$\begin{aligned}
&\delta + d^F_{N^-}(y) - d^F_{N^-}(\xi(t_r))\\
&\quad\ge \delta + d^F(p_0, y) - d^F(p_0, \xi(t_r))\\
&\quad\ge 2\Big[\int_{-1}^{1} \sqrt{F(\xi(t))}|\xi'(t)|dt - \int_{-1}^{t_r} \sqrt{F(\xi(t))}|\xi'(t)|dt\Big]\\
&\quad= 2\int_{t_r}^{1} \sqrt{F(\xi(t))}|\xi'(t)|dt\\
&\quad\ge 2\big(\inf_{z\in B_r(y)} \sqrt{F(z)}\big)\int_{t_r}^{1} |\xi'(t)|dt \ge 2\big(\inf_{z\in B_r(y)} \sqrt{F(z)}\big)r.
\end{aligned}$$

Therefore, after sending $\delta \downarrow 0$ and $r \downarrow 0$, we obtain

$$\left|\nabla d^F_{N^-}(y)\right| \geq \limsup_{r\downarrow 0} \frac{\left|d^F_{N^-}(y) - d^F_{N^-}(\xi(t_r))\right|}{r} \geq 2\sqrt{F(y)}.$$

This completes the proof. □

In order to obtain the upper-bound estimate of $\mathbf{E}(\epsilon)$ in Theorem 1.3 for potential functions F satisfying (1.1), we need to introduce the approximate minimal connecting orbit $\alpha_{2L} : [-2L, 2L] \to \mathbb{R}$, $L > 0$, given by

$$\alpha_{2L}(t) = \begin{cases} \frac{t+2L}{L}\alpha(-L) + \frac{t+L}{L}\frac{d_N}{2} & \text{if } -2L \leq t \leq -L, \\ \alpha(t) & \text{if } -L \leq t \leq L, \\ \frac{2L-t}{L}\alpha(L) + \frac{t-L}{L}\frac{d_N}{2} & \text{if } L \leq t \leq 2L, \end{cases} \tag{A.7}$$

where $\alpha \in H^1(\mathbb{R})$, with $\alpha(\pm\infty) = \pm\frac{d_N}{2}$, is a minimizer of (2.1). For α_{2L}, we have the following:

PROPOSITION A.4.

(i) α_{2L} *is monotonically increasing,*

$$-\frac{d_N}{2} < \alpha_{2L}(t) < \frac{d_N}{2}, \quad \alpha_{2L}(\pm 2L) = \pm\frac{d_N}{2},$$

and there exists $C > 0$ *such that*

$$\max_{|t|\leq 2L} \left(|\alpha'_{2L}(t)|^2 + \widetilde{F}(\alpha_{2L}(t))\right) \leq C. \tag{A.8}$$

(ii) *There exist* $L_0 > 0$ *and* $c_1, c_2 > 0$ *such that for any* $L \geq L_0$*, we have*

$$\int_{-2L}^{2L} \left(|\alpha'_{2L}(t)|^2 + \widetilde{F}(\alpha_{2L}(t))\right)dt \leq c_0^F + c_2 e^{-c_1 L}, \tag{A.9}$$

$$\max_{|t|\leq 2L} \left||\alpha'_{2L}(t)|^2 - \widetilde{F}(\alpha_{2L}(t))\right| \leq c_2 e^{-c_1 L}, \tag{A.10}$$

$$\begin{aligned} &\left|\int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha_{2L}^{-1}(\lambda)| \sqrt{\widetilde{F}(\alpha_{2L}(t))}\, d\lambda - \int_{-\frac{d_N}{2}}^{\frac{d_N}{2}} |\alpha^{-1}(\lambda)| \sqrt{\widetilde{F}(\alpha(t))}\, d\lambda\right| \\ &\quad \leq C_2 e^{-c_1 L}. \end{aligned} \tag{A.11}$$

PROOF. All the above properties of α_{2L} follow directly from the following property of α: $\exists L_0 > 0$ such that

$$\begin{cases} |\alpha'(t)| + \left|\alpha(t) - \frac{d_N}{2}\right| \leq C_1 e^{-C_2 t} & \text{if } t \geq L_0, \\ |\alpha'(t)| + \left|\alpha(t) + \frac{d_N}{2}\right| \leq C_1 e^{C_2 t} & \text{if } t \leq -L_0. \end{cases}$$

Consequently, we only sketch the proof of (A.9). Direct calculations imply that for $L \geq L_0$,

$$
\begin{aligned}
&\left\{ \int_{-2L}^{-L} + \int_{L}^{2L} \right\} \big(|\alpha_{2L}'(t)|^2 + \widetilde{F}(\alpha_{2L}(t)) \big) dt \\
&\quad \leq C \left(\frac{1}{L} \left| \alpha(-L) + \frac{d_N}{2} \right|^2 + L \left| \alpha(-L) + \frac{d_N}{2} \right|^2 \right) \\
&\qquad + C \left(\frac{1}{L} \left| \alpha(L) - \frac{d_N}{2} \right|^2 + L \left| \alpha(L) - \frac{d_N}{2} \right|^2 \right) \\
&\quad \leq C_1 e^{-C_2 L}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_{-2L}^{2L} \big(|\alpha_{2L}'(t)|^2 + \widetilde{F}(\alpha_{2L}(t)) \big) dt \\
&\quad = \int_{-L}^{L} \big(|\alpha'(t)|^2 + \widetilde{F}(\alpha(t)) \big) dt \\
&\qquad + \left\{ \int_{-2L}^{-L} + \int_{L}^{2L} \right\} \big(|\alpha_{2L}'(t)|^2 + \widetilde{F}(\alpha_{2L}(t)) \big) dt \\
&\quad \leq c_0^F + C_1 e^{-C_2 L}.
\end{aligned}
$$

This completes the proof. □

Appendix B: List of Notations

Ω: bounded smooth domain in $\mathbb{R}^n$
Σ: smooth $(n-2)$-dimensional closed submanifold of $\partial\Omega$
Γ: area-minimizing hypersurface in Ω with $\partial\Gamma = \Sigma \subset \partial\Omega$
ν_Γ: unit normal vector field on Γ
$\mathcal{L}_x$: Jacobi field operator on Γ
$\Sigma^\pm$: two connected components of $\partial\Omega$ with $\partial\Sigma = \Sigma$
∇_τ: tangential derivative on hypersurfaces in $\mathbb{R}^n$
$\Omega^\pm$: two disjoint subdomains of Ω with $\partial\Omega^\pm = \Sigma^\pm \cup \Gamma$
$N^\pm$: two disjoint compact Riemannian manifolds without boundary in $\mathbb{R}^k$
$d(\cdot, N)$: distance function to $N = N^+ \cup N^-$
d_N: euclidean distance between N^+ and N^-
$M^\pm$: pair of sets in $N^\pm$ of minimal distance d_N
N_δ: δ-neighborhood of $N = N^+ \cup N^-$
Π: nearest-point projection from N_δ to N
F: potential function of high-dimensional wells
$\widetilde{F}$: scalar-valued potential function induced by F
B_R^k: ball of radius R in $\mathbb{R}^k$
Σ^η: η-neighborhood of Σ

Σ_η: complement of Σ^η in $\Sigma^\pm$
$\Gamma(t)$: hypersurface in Ω whose distance to Γ equals t
Γ_t: t-neighborhood of Γ in Ω
g_ϵ: Dirichlet boundary data
g: Dirichlet boundary data obtained as limit of g_ϵ
$g(x^\pm)$: one-sided trace value of g on Σ for $x \in \Sigma$
$\mathbf{E}(\epsilon)$: minimal energy of the singular perturbation functional with boundary data g_ϵ
u_ϵ: minimizer of $\mathbf{E}(\epsilon)$
$c^F(p^+, p^-)$: minimal connecting energy between a pair of points $p^\pm \in N^\pm$
c_0^F: minimal connecting energy between $N^\pm$
H^{n-1}: $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$
$\mathbf{D}(\epsilon)$: second term in the expansion of $\mathbf{E}(\epsilon)$ with respect to $\frac{1}{\epsilon}$
$\dim_{\mathbb{H}}$: Hausdorff dimension of sets in $\mathbb{R}^n$
$d_\Gamma(x)$: signed distance function to Γ
Φ: diffeomorphism map from $\Gamma \times [-\delta_0, \delta_0]$ to Γ_{δ_0}
Ψ_ϵ: diffeomorphism map from $\Omega^\pm \setminus \Gamma_{\epsilon^\alpha}$ to $\Omega^\pm$
α: minimal connecting orbit for the double-well potential functions $\widetilde{F}$
α_ϵ: truncated version of α in $[-\epsilon^{\alpha-1}, \epsilon^{\alpha-1}]$
$\mathbf{A}$: Sobolev space with the Dirichlet and partially constrained boundary condition
$\mathbf{D}$: value of Dirichlet energy of minimizing harmonic maps in $\mathbf{A}$
$v(x^\pm)$: one-sided trace value of $v \in \mathbf{A}$ on $x \in \Gamma$
$\Omega^\pm_{\epsilon,\delta}$: subset where u_ϵ is in the δ-neighborhood of $N^\pm$
$E_{\epsilon,\delta}$: subset where u_ϵ is outside the δ-neighborhood of N
Ω_η: subdomain of Ω that has distance to $\partial\Omega$ larger than η
$\Omega^\pm_\eta$: intersection of Ω_η and $\Omega^\pm$
O_η: intersection of Γ_η and $(\Omega \setminus \Omega_\eta)$
$U^\pm_\eta$: complement of O_η in $(\Omega^\pm \setminus \Omega^\pm_\eta)$
U_η: union of $U^\pm_\eta$
$\Sigma(t)$: intersection of $\Gamma(t)$ and $\partial\Omega$
$\mathcal{S}(t)$: area-minimal hypersurface in Ω with $\partial\mathcal{S}(t) = \Sigma(t)$
$\mathrm{BV}(\Omega)$: space of functions with bounded variations in Ω
$\mathrm{SBV}(\Omega)$: space of special BV functions in Ω
$\partial^* E$: reduced boundary of set E of finite perimeters
χ_E: characteristic function of a set E

Acknowledgment. The first author is partially supported by National Science Foundation Grant 0700517. The second author is supported by National Natural Science Foundation of China Grant No. 10871071 and National Basic Research

Program of China Grant No. 2006CB805902. The third author is partially supported by National Science Foundation Grant 1001115 and an IMA New Directions Research Professorship award, and he thanks Ryan Walker for the help in generating the figures. The authors would also like to thank the referee for his/her careful reading and helpful comments.

Bibliography

[1] Ambrosio, L. Metric space valued functions of bounded variation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **17** (1990), no. 3, 439–478.

[2] Ambrosio, L. A new proof of the SBV compactness theorem. *Calc. Var. Partial Differential Equations* **3** (1995), no. 1, 127–137.

[3] André, N.; Shafrir, I. On a vector-valued singular perturbation problem on the sphere. *Recent advances in nonlinear analysis*, 11–42. World Scientific, Hackensack, N.J., 2008.

[4] Bethuel, F.; Brezis, H.; Hélein, F. *Ginzburg-Landau vortices*. Progress in Nonlinear Differential Equations and Their Applications, 13. Birkhäuser Boston, Boston, 1994.

[5] Bronsard, L.; Stoth, B. The singular limit of a vector-valued reaction-diffusion process. *Trans. Amer. Math. Soc.* **350** (1998), no. 12, 4931–4953.

[6] Evans, L. C.; Gariepy, R. E. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC, Boca Raton, Fla., 1992.

[7] Evans, L. C.; Soner, H. M.; Souganidis, P. E. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* **45** (1992), no. 9, 1097–1123.

[8] Federer, H. The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bull. Amer. Math. Soc.* **76** (1970) 767–771.

[9] Fonseca, I.; Tartar, L. The gradient theory of phase transitions for systems with two potential wells. *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), no. 1-2, 89–102.

[10] Gurtin, M. E. On a theory of phase transitions with interfacial energy. *Arch. Rational Mech. Anal.* **87** (1985), no. 3, 187–212.

[11] Hardt, R.; Lin, F.-H. Mappings minimizing the L^p-norm of the gradient. *Comm. Pure Appl. Math.* **40** (1987), no. 5, 555–588.

[12] Hardt, R.; Simon, L. Boundary regularity and embedded solutions for the oriented Plateau problem. *Ann. of Math. (2)* **110** (3) (1979), no. 3, 439–486.

[13] Ilmanen, T. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.* **38** (1993), no. 2, 417–461.

[14] Kohn, R.; Sternberg, P. Local minimisers and singular perturbations. *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), no. 1-2, 69–84.

[15] Lin, F. H.; Pan, X. B.; Wang, C. Y. Boundary regularity for a new class of minimizing harmonic maps. In preparation.

[16] Luckhaus, S. Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold. *Indiana Univ. Math. J.* **37** (1988), no. 2, 349–367.

[17] Luckhaus, S.; Modica, L. The Gibbs-Thompson relation within the gradient theory of phase transitions. *Arch. Rational Mech. Anal.* **107** (1989), no. 1, 71–83.

[18] Modica, L. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.* **98** (1987), no. 2, 123–142.

[19] Modica, L.; Mortola, S. Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. *Boll. Un. Mat. Ital. A (5)* **14** (1977), no., 3, 526–529.

[20] Owen, N. C.; Rubinstein, J.; Sternberg, P. Minimizers and gradient flows for singularly perturbed bi-stable potentials with a Dirichlet condition. *Proc. Roy. Soc. London Ser. A* **429** (1990), no. 1877, 505–532.

[21] Rubinstein, J.; Sternberg, P.; Keller, J. Fast reaction, slow diffusion, and curve shortening. *SIAM J. Appl. Math.* **49** (1989), no. 1, 116–133.
[22] Rubinstein, J.; Sternberg P.; Keller, J. Reaction-diffusion processes and evolution to harmonic maps. *SIAM J. Appl. Math.* **49** (1989), no. 6, 1722–1733.
[23] Schoen, R.; Uhlenbeck, K. A regularity theory for harmonic maps. *J. Differential Geom.* **17** (1982), no. 2, 307–335.
[24] Simon, L. *Lectures on geometric measure theory*. Proceedings of the Centre for Mathematical Analysis, Australian National University, 3. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
[25] Simon, L. Cylindrical tangent cones and the singular set of minimal submanifolds. *J. Differential Geom.* **38** (1993), no. 3, 585–652.
[26] Sternberg, P. The effect of a singular perturbation on nonconvex variational problems. *Arch. Rational Mech. Anal.* **101** (1988), no. 3, 209–260.
[27] Sternberg, P. Vector-valued local minimizers of nonconvex variational problems. Current directions in nonlinear partial differential equations (Provo, UT, 1987). *Rocky Mountain J. Math.* **21** (1991), no. 2, 799–807.

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Received April 2011.