Solutions to Final Exam Review

1)
$$\left[\begin{array}{cccc|c} 1 & 4 & -2 & 8 & 12 \\ & 1 & -7 & 2 & -4 \\ & & 5 & -1 & 7 \\ & & 1 & 3 & -5 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 4 & -2 & 8 & 12 \\ & 1 & -7 & 2 & -4 \\ & & 1 & 3 & -5 \\ & & 0 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 4 & -2 & 8 & 12 \\ & 1 & -7 & 2 & -4 \\ & & 1 & 0 & 1 \\ & & 0 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 4 & -2 & 8 & 12 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\rightsquigarrow \begin{cases} x_1 = 12 - 4\cdot 7 + 2\cdot 1 - 8(-2) = 2 \\ x_2 = 7 \\ x_3 = 1 \\ x_4 = -2 \end{cases}$$

□

2).
$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & -2 & -6 & 11 \\ 0 & \boxed{3} & 7 & -5 \\ 0 & 0 & \boxed{11} & -2 \end{array}\right]$$

$\rightsquigarrow$ The matrix has 3 pivot positions, so it has full rank. Hence b is in the span of a_1, a_2, a_3.

3).
$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & \boxed{1} & 4 \\ 0 & 0 & \boxed{13} \end{bmatrix}$$

$\rightsquigarrow$ YES, it is linearly independent.

4) $$A^{-1} = \begin{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} & & 0 \\ & -\frac{1}{2} & \\ 0 & & \begin{bmatrix} 10 & 7 \\ 7 & 5 \end{bmatrix}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix},$$

$$\begin{bmatrix} 10 & 7 \\ 7 & 5 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 5 & -7 \\ -7 & 10 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -7 & 10 \end{bmatrix}.$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -3 & 2 & 0 & 0 & 0 \\ 5 & -3 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & -7 & 10 \end{bmatrix}$$

5). $$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} \boxed{1} & -2 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 5/3 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right]$$

The matrix only has 2 pivot position,
So it is not invertible.

6). $A \sim \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & -12 & 20 & -7 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 \\ 3 \\ -1 \\ -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -7 \end{bmatrix}$ $\qquad \begin{bmatrix} -3 \\ 0 \\ -12 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

$= U \qquad [1]$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix}$$

$\underline{A = LU}$ $\qquad L(Ux) = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$ $\qquad Ly = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$, $\quad Ux = y$

$$\Rightarrow \begin{cases} y_1 = 1 \\ y_2 = 3 \\ y_3 = 1 \\ y_4 = 3 + 3y_1 - 4y_2 + 2y_3 = 3 + 3 - 12 + 2 = -4 \end{cases}$$

$$Ux = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -4 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 38 \\ x_2 = 16 \\ x_3 = 1 - 4(-4)/2 = 17/2 \\ x_4 = -4 \end{cases}$$

□

7. $(I - C)x = d$

$$\Rightarrow \begin{bmatrix} 0.9 & -0.2 & 0 \\ -0.3 & 0.6 & -0.2 \\ -0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 9 & -2 & 0 & 400 \\ -3 & 6 & -2 & 600 \\ -1 & 0 & 8 & 800 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ -3 & 6 & -2 & 600 \\ 9 & -2 & 0 & 400 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 6 & -26 & -1800 \\ 0 & -2 & 72 & 7600 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 1 & -36 & -3800 \\ 0 & 6 & -26 & -1800 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 1 & -36 & -3800 \\ 0 & 0 & 190 & 21000 \end{array}\right]$$

$$\sim \begin{cases} x_1 = 8\cdot\frac{2100}{19} - 800 \\ x_2 = 36\cdot\frac{2100}{19} - 3800 \\ x_3 = \frac{2100}{19} \end{cases}$$

8) $A \sim \begin{bmatrix} 1 & 3 & 2 & -6 \\ 0 & 0 & -5 & 23 \\ 0 & 0 & -5 & 21 \\ 0 & 0 & -10 & 44 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 2 & -6 \\ 0 & 0 & -5 & 23 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -6 \\ 0 & 0 & -5 & 23 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $A = 3$, $\dim \mathrm{Nul}\, A = 1$ $\square$

9) a) False The number of L.I. vectors can't extend the dimension $(=4)$.

b) False: $A=0$, $B\neq C$ arbitary $\Rightarrow AB=AC=0$

c) ~~False:~~ Rank A + dim Nul $A = 5+3 \neq$ ~~#~~ Colum $A = 9$. This contradicts the Rank Theorem.

d) False: $\det(A)\neq 0$ if the col(A) is n.

e) YES:

10) $\det A = 6\begin{vmatrix} 7 & 2 & -5 \\ -6 & -7 & 5 \\ 1 & 4 & 3 \end{vmatrix} - 5\begin{vmatrix} 1 & 7 & 2 \\ 2 & -6 & -7 \\ 5 & 1 & 4 \end{vmatrix}$

$= 6(-147+10+120-35+36-140)$

$-5(-24-245+4+60+7-56)$

$= 6(-156) - 5(-254)$

$= 1270 - 936 = 334$

11).
$$\det A = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & b+a & b^2+ba+a^2 \\ 1 & c+a & c^2+ca+a^2 \\ 1 & d+a & d^2+da+a^2 \end{vmatrix}$$

$= \cdots$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

12)
$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} = 6-1+3-4 = 4$$

$$\det A_1 = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \\ 5 & 1 & 3 \end{vmatrix} = 10+3-9-2 = 2$$

$$\det A_2 = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix} = 18+6-5-9+3-20$$

$= -7$

$$\det A_3 = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 3 \\ 3 & 1 & 5 \end{vmatrix} = 9 - 1 + 5 - 6 = 7$$

$$X_1 = \frac{2}{4} = \frac{1}{2}, \quad X_2 = \frac{-7}{4}, \quad X_3 = \frac{7}{4}.$$

13). $\lambda_1 = 3 \quad \lambda_2 = 2, \quad \lambda_3 = 6, \quad \lambda_4 = -5$

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & +2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$V_1 : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 2 & 2 & 3 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad 6X_1 - X_2 = 0$$
$$X_2 + X_3 = 0$$
$$2X_1 + 2X_2 + 3X_3 - 8X_4 = 0$$

$$\begin{cases} X_1 = 1 \\ X_2 = 6 \\ X_3 = -6 \\ X_4 = \frac{1}{8}\left[2 \cdot 1 + 2 \cdot 6 + 3(-6)\right] \end{cases}$$

$$= \frac{1}{8}\left[-4\right] = -\frac{1}{2}$$

$$V_1 = \begin{bmatrix} 1 \\ 6 \\ -6 \\ -1/2 \end{bmatrix}.$$

$$V_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \\ 2 & 3 & 3 & -7 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{x_1 = 0} \qquad 3x_2 + 4x_3 = 0$$

$$\cancel{2x_1} + 3x_2 + 3x_3 - 7x_4 = 0$$

$$\Rightarrow x_3 = -3 \qquad x_2 = 4$$

$$\Rightarrow x_4 = \frac{1}{7}\left(3\cdot(4) + 3(-3)\right) = \frac{3}{7}$$

$$V_2 = \begin{bmatrix} 0 \\ 4 \\ -3 \\ 3/7 \end{bmatrix}$$

$$V_3 : \begin{bmatrix} -3 & 0 & 0 & 0 \\ 6 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 3 & 3 & -11 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \underline{x_1 = 0} \\ \underline{x_2 = 0} \\ x_3 = 11 \\ x_4 = 3 \end{cases}$$

$$V_3 = \begin{bmatrix} 0 \\ 0 \\ 11 \\ 3 \end{bmatrix}$$

$$V_4 = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 0 & 3 & 11 & 0 \\ 2 & 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 1 \end{cases} \qquad V_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ -6 & -3 & 11 & 0 \\ -1/2 & 3/7 & 3 & 1 \end{bmatrix}.$$

$\square$

14. $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)(\lambda^2 + 4\lambda + 4) = 0 \Rightarrow \lambda_1 = 1$

$\lambda_2 = -2 \ (m = 2)$

$\underline{v_1 =}$ $\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\underline{v_2 =}$ $\begin{bmatrix} 4 & 4 & 3 \\ \cancel{-4} & \cancel{-4} & \cancel{-3} \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{cases} 4x_1 + 4x_2 + 3x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$ $\boxed{x_3 = 0}$

$x_1 = -x_2$

only 1 L.I. eigenvector associated with $\lambda_2 = -2$

So it can't be diagonalizable.

15). $\begin{vmatrix} -11-\lambda & -4 \\ 20 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-5)(\lambda+11) + 80 = 0$

$\Rightarrow \lambda^2 + 6\lambda + 25 = 0$

$\Rightarrow (\lambda+3)^2 + 16 = 0$

$\boxed{\lambda = -3 \pm 4i}$

$\underline{\lambda = -3+4i}$

$$\begin{bmatrix} -8-4i & -4 \\ 20 & 8-4i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad (2+i)\, x_1 + x_2 = 0$$

$$\Rightarrow \quad x_1 = 1 \qquad x_2 = -2-i$$

$$\underset{\sim}{V} = \begin{bmatrix} 1 \\ -2-i \end{bmatrix}.$$

$\square$