Review Problems for Midterm II, MA 213, Fall 2013

Exame Date: Wednesday November 13 2013

- The second midterm exam coveres Chapter 14.5-14.7, and Chapater 15.1-15.5.
- Time and Place: 2:00-2:50 pm, CB 106.

Here is a set of review problems.

1. Find an equation of the tangent plane to the surface $x^2 + z^2e^{y-x} = 13$ at the point $P = \big(2, 3, \frac{3}{\sqrt{e}}\big)$.

<u>Solution</u>: Set $F(x, y, z) = x^2 + z^2e^{y-x} - 13$. Then the surface is $F(x, y, z) = 0$ and

$$DF = (2x - z^2e^{y-x},\ z^2e^{y-x},\ 2ze^{y-x}),$$

and

$$DF(2, 3, \frac{3}{\sqrt{e}}) = (-5, 9, 6\sqrt{e}).$$

An equation for the tangent plane at P is given by

$$(x - 2, y - 3, z - \frac{3}{\sqrt{e}}) \cdot DF(2, 3, \frac{3}{\sqrt{e}}) = 0,$$

or

$$-5x + 3y + \frac{3}{\sqrt{e}}z = 19 + 9e^{-1}.$$

2. Calculate the directional derivative in the direction $\mathbf{v}$ at the given point P for $f(x, y, z) = x\ln(y + z)$, $\mathbf{v} = (2, -1, 1)$, and $P = (2, e, e)$.

<u>Solution</u>: First we need to normalize v to a unit vector $\bar{v} = \frac{1}{\sqrt{6}}(2, -1, 1)$. Then we have

$$D_{\bar{v}}f(P) = Df(p) \cdot \bar{v} = \Big(\ln(y + z), \frac{x}{y + z}, \frac{x}{y + z}\Big)\Big|_{(2,e,e)} \cdot \frac{1}{\sqrt{6}}(2, -1, 1) = \frac{2}{\sqrt{6}}(1 + \ln 2).$$

3. Use the chain rule to calculate the partial derivatives: $\frac{\partial h}{\partial q}$ at $(q, r) = (3, 2)$, where $h(u, v) = ue^v$, $u = q^3$, $v = qr^2$.

<u>Solution:</u>

$$\begin{aligned}\frac{\partial h}{\partial q} &= \frac{\partial h}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial q} \\ &= e^v(3q^2) + ue^v(r^2) \\ &= e^{qr^2}(3q^2 + q^3r^2)\end{aligned}$$

so that

$$\frac{\partial h}{\partial q}(3, 2) = 135e^{12}.$$

4. Use implicit differentiation to calculate the partial derivative: $\frac{\partial w}{\partial z}$, where $x^2w+w^3+wz^2+3yz=0$.

Solution: $w_z = -\frac{3y+2zw}{x^2+z^2+3w^2}$.

5. Find the critical points of the function, Then use the Second Derivative Test to determine whether they are local minima, local maxima, or saddle points: $f(x,y)=x^3+y^4-6x-2y^2$; $g(x,y)=\ln x+2\ln y-x-4y$.

Solution:
a)

$$f_x = 3x^2-6,\;\; f_y = 4y(y^2-1).$$

So the critical points are $(\pm\sqrt{2},0)$, $(\pm\sqrt{2},\pm 1)$.

$$f_{xx}=6x,\;\; f_{xy}=0,\;\; f_{yy}=12y^2-4.$$

At $(\sqrt{2},0)$, $D=-24\sqrt{2}<0$, $(\sqrt{2},0)$ is saddle point.
At $(-\sqrt{2},0)$, $D=24\sqrt{2}>0$. Since $f_{xx}(-\sqrt{2},0)=-6\sqrt{2}<0$, $(-\sqrt{2},0)$ is a local maxima point.
At $(\sqrt{2},\pm 1)$, $D=48\sqrt{2}>0$. Since $f_{xx}(\sqrt{2},\pm 1)=6\sqrt{2}>0$, $(\sqrt{2},\pm 1)$ is a local minima point.
At $(-\sqrt{2},\pm 1)$, $D=-48\sqrt{2}<0$, so $(-\sqrt{2},\pm 1)$ is a saddle point.
b)

$$g_x=\frac{1}{x}-1;\;\; g_y=\frac{2}{y}-4.$$

So the critical point is $P=(1,\frac{1}{2})$.

$$g_{xx}=-\frac{1}{x^2},\;\; g_{xy}=0,\;\; g_{yy}=-\frac{2}{y^2}.$$

So that

$$g_{xx}(1,\frac{1}{2})=-1, g_{xy}(1,\frac{1}{2})=0,\;\; g_{yy}(1,\frac{1}{2})=-8, D(1,\frac{1}{2})=8>0.$$

Thus $(1,\frac{1}{2})$ is a local maxima point.

6. Determine the global extreme values of the function on the given domain: $f(x,y)=(4y^2-x^2)e^{-x^2-y^2}$, $x^2+y^2\le 2$.

Solution:
a) Interior critical points:

$$f_x=-2x(1+4y^2-x^2)e^{-(x^2+y^2)}, f_y=-2y(-4+4y^2-x^2)e^{-(x^2+y^2)}.$$

$f_x(x,y)=f_y(x,y)=0$ iff $(x,y)=(0,0),(0,\pm 1),(\pm 1,0)$. Hence the interior critical values are $f(0,0)=0$, $f(0,\pm 1)=4e^{-1}$, and $f(\pm 1,0)=-e^{-1}$.
b) Boundary extreme values: Since the boundary is $x^2+y^2=2$, we have $f(x,y)=e^{-2}(5y^2-2)$. It is easy to see that the maximum value is when $y^2=2$ so that $f(x,y)=8e^{-2}$, and the minimum value is when $y=0$ so that $f(x,y)=-2e^{-2}$.
c) The global maximum value is $f(0,\pm 1)=4e^{-1}$, and the global minimum value is $f(\pm 1,0)=-e^{-1}$.

7. Calculate the double integral

$$\int\int_R (xy^2 + \frac{y}{x})\, dA$$

where

$$R = \{(x, y)|2 \le x \le 3, -1 \le y \le 0\}.$$

Solution:

$$\begin{aligned}
&= \int_{-1}^{0}\int_{2}^{3} (xy^2 + \frac{y}{x})\, dx\, dy \\
&= \int_{-1}^{0} (\frac{x^2y^2}{2} + y\ln x)\Big|_2^3\, dy \\
&= \int_{-1}^{0} \frac{5y^2}{2} + y\ln(\frac{3}{2})\, dy \\
&= \frac{5y^3}{6} + \ln(3/2)y^2/2\Big|_{-1}^{0} = \frac{5}{6} - 1/2\ln(3/2).
\end{aligned}$$

8. Use the polar coordinate to calculate the double integral

$$\int_0^1 \int_y^{\sqrt{1-y^2}} \frac{1}{3 + x^2 + y^2}\, dx\, dy.$$

Solution: The domain is

$$0 \le y \le 1,\ y \le x \le \sqrt{1-y^2}.$$

In the polar coordinates, it can be written by

$$0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{4}.$$

The integral equals to

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \int_0^1 \frac{1}{3+r^2} r\, dr d\theta \\
&= \frac{\pi}{8}\ln(3+r^2)\Big|_{r=0}^{1} = \frac{\pi}{8}\ln(\frac{4}{3}).
\end{aligned}$$

9. Evaluate the double integral

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\, dxdy.$$

Solution:

$$\begin{aligned}
&= \int_0^2 \int_0^{\pi} r^4\cos^2\theta\sin^2\theta\ r\, dr\, d\theta \\
&= (\frac{1}{24}r^6\Big|_0^2)\cdot \int_0^{\pi} \sin^2(2\theta)\, d\theta \\
&= \frac{8}{3}\int_0^{\pi} \frac{1-\cos(4\theta)}{2}\, d\theta = \frac{4\pi}{3}.
\end{aligned}$$

10. Calculate the volume of the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

Solution: The region is bounded by

$$x^2 + y^2 \le \frac{1}{2}, \ \sqrt{x^2 + y^2} \le z \le \sqrt{1 - (x^2 + y^2)}.$$

Hence the volume is given by

$$\begin{aligned} &= \int_{x^2+y^2 \le \frac{1}{2}} (\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2})\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}/2} (\sqrt{1 - r^2} - r)\ r\, dr\, d\theta \\ &= \frac{4 - \sqrt{2}}{3} \pi \end{aligned}$$

11. Find the mass of the region D that is enclosed by the cardioid $r = 1 + \cos\theta$ with density $\rho(x, y) = \sqrt{x^2 + y^2}$.

Solution:

$$\begin{aligned} &= 2 \int_0^{\pi} \int_0^{1+\cos\theta} \int r^2\, dr\, d\theta = \frac{2}{3} \int_0^{\pi} (1 + \cos\theta)^3\, d\theta \\ &= \frac{2}{3} \int_0^{\pi} (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\, d\theta \\ &= \frac{2\pi}{3} + \pi = \frac{5}{3}\pi. \end{aligned}$$

12. Use the Fubini's theorem (or equivalently, the iterated integration) to evulate the triple integral

$$\int\int\int_E yz\cos(x^5)\, dV,$$

where

$$E = \Big\{(x, y, z) \mid 0 \le 1 \le 1,\ 0 \le y \le x,\ 0 \le z \le 2x\Big\}.$$

Solution: The integral equals

$$\begin{aligned} &= \int_0^1 \cos(x^5)(\int_0^x y\, dy)(\int_0^{2x} z\, dz)\, dx \\ &= \int_0^1 x^4 \cos(x^5)\, dx = \frac{1}{5}. \end{aligned}$$

13. Use the spherical coordinates to calculate

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2}\, dzdxdy.$$

Solution:

$$\begin{aligned}
&= \int_0^{\pi}\int_0^{\pi}\int_0^2 \rho^3 \sin^2\phi \sin^2\theta \; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \Big(\frac{\rho^6}{6}\Big|_0^2\Big)\Big(\int_0^{\pi} \sin^2\theta \, d\theta\Big)\Big(\int_0^{\pi} \sin^3\phi \, d\phi\Big) \\
&= \frac{32}{3}\frac{\pi}{2}\Big[\frac{\cos^3\phi}{3} - \cos\phi\Big]\Big|_0^{\pi} = \frac{64}{9}\pi.
\end{aligned}$$

14. Find the center of mass for the lamina that occupies the region D and has the given density function ρ: D is the triangular region with vertices $(0,0)$, $(2,1)$, $(0,3)$; $\rho(x,y) = x + y$.

Solution: The domain is $0 \le x \le 1$, $\frac{1}{2}x \le y \le 3 - x$. First we calculate the mass

$$\begin{aligned}
m &= \int_0^1 \int_{\frac{1}{2}x}^{3-x} (x+y) \, dy \, dx \\
&= \int_0^1 \Big(\frac{9}{2} - \frac{9}{8}x^2\Big) \, dx = \frac{33}{8}.
\end{aligned}$$

Next we calculate the center of mass

$$\int_0^1 \int_{\frac{1}{2}x}^{3-x} x(x+y) \, dy \, dx = \int_0^1 \Big(\frac{9}{2}x - \frac{9}{8}x^3\Big) \, dx = \frac{63}{32},$$

and

$$\int_0^1 \int_{\frac{1}{2}x}^{3-x} y(x+y) \, dy \, dx = \int_0^1 \frac{9}{2}x - 3x^2 + \frac{x^3}{3} - \frac{(x-3)^3}{3} \, dx = 6\frac{3}{4}.$$

Hence

$$\bar{x} = \frac{63}{132}, \quad \bar{y} = \frac{54}{33}.$$

15. Evaluate the double integral by making an appropriate change of variables

$$\int\int_{\mathbf{R}} \frac{x+2y}{\cos(x-y)} \, dxdy,$$

where $\mathbf{R}$ is the parallelogram bounded by the lines $y = x$, $y = x - 14$, $x + 2y = 0$, $x + 2y = 2$.

Solution: Set

$$u = x + 2y, \; v = x - y.$$

Then we have

$$0 \le u \le 2, \; 0 \le v \le 14.$$

Solving x, y in terms of u, v, we have

$$x = \frac{u+2v}{3}, \; y = \frac{u-v}{3}$$

so that

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}.$$

By the formula of change of variables, we have the integral equals to

$$\begin{aligned} &= \frac{1}{3}\int_0^2\int_0^{14}\frac{u}{\cos v}\,dudv = \frac{1}{3}\Big(\int_0^2 u\,du\Big)\Big(\int_0^{14}\sec v\,dv\Big) \\ &= \frac{4}{3}\ln|\sec v+\tan v|\Big|_0^{14} = \frac{4}{3}\ln(\sec(14)+\tan(14)). \end{aligned}$$

16. Use the map

$$G(u,v) = \Big(\frac{u+v}{2},\ \frac{u-v}{2}\Big)$$

to compute

$$\int\int_{\mathcal{R}}((x-y)\sin(x+y))^2\,dxdy,$$

where $\mathcal{R}$ is the square with vertices $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$, and $(0,\pi)$.

Solution: Set $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$. Then we have

$$u = x+y,\ \ v = x-y.$$

The square is given by

$$\pi\le x+y\le 3\pi;\ \ -\pi\le x-y\le\pi.$$

It is easy to see

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

so that the integral equals to

$$\frac{1}{2}\int_{\{\pi\le u\le 3\pi,\ -\pi\le v\le\pi\}} v^2\sin^2 u\,dudv = \frac{1}{2}\Big(\int_{-\pi}^{\pi} v^2\,dv\Big)\Big(\int_{\pi}^{3\pi}\sin^2 u\,du\Big) = \frac{\pi^4}{3}.$$