

Solution set to Review problems.

$$1. A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 8 \\ 0 & 5 & 8 \end{bmatrix}$$

$$|A_{11}| = 5 - 6 = -1, \quad A_{11} \text{ invertible}$$

$$|A_{22}| = 2(7 \cdot 6 - 5 \cdot 8) = 4, \quad A_{22} \text{ invertible}$$

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}, \quad A_{11}^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$A_{22}^{-1} = \begin{bmatrix} 2^{-1} & 0 \\ 0 & \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -5/2 & 7/2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -5/2 & 7/2 \end{bmatrix}$$



$$2. A \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} [-7]$$

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -7 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow Ly = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$Ux = y$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} y_1 = 1 \\ y_2 = -3 \\ y_3 = 8 \end{cases}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix} \Rightarrow$$

$$\begin{cases} x_1 = -\frac{26}{7} \\ x_2 = \frac{11}{7} \\ x_3 = -\frac{8}{7} \end{cases}$$

$$3. \quad X - CX = d \Rightarrow \begin{bmatrix} 0.8 & -0.2 & 0 \\ -0.3 & 0.9 & -0.3 \\ -0.1 & 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$$

$$\leadsto \left[ \begin{array}{ccc|c} 0.8 & -0.2 & 0 & 40 \\ -0.3 & 0.9 & -0.3 & 60 \\ -0.1 & 0 & -0.8 & 80 \end{array} \right]$$

$$\leadsto \left[ \begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 3 & -9 & 3 & -600 \\ 8 & -2 & 0 & 400 \end{array} \right]$$

$$\leadsto \left[ \begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & -9 & 27 & 1800 \\ 0 & -2 & 64 & 6800 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 1 & -3 & -200 \\ 0 & 1 & -32 & -3400 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 0 & -8 & -800 \\ 0 & 1 & -3 & -200 \\ 0 & 0 & -29 & -3200 \end{array} \right] \sim \begin{cases} x_1 = 8 \cdot \frac{3200}{29} - 800 \\ x_2 = 3 \cdot \frac{3200}{29} - 200 \\ x_3 = \frac{3200}{29} \end{cases}$$

□

$$4. \text{Nul}(A) = \{x \mid Ax=0\}$$

$$\begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \left[ \begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & -8 & 12 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -1 & 5 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\sim \begin{cases} x_1 - x_2 + 5x_3 = 0 \\ 2x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = \frac{3}{2}x_3 - 5x_3 = -\frac{7}{2}x_3 \\ x_2 = \frac{3}{2}x_3 \\ x_3 \text{ free} \end{cases}$$

$$\sim \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} x_3$$

$$\text{Nul}(A) = \left\{ t \begin{bmatrix} -\frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

$$u = 2 \cdot \begin{bmatrix} -\frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \in \text{Nul}(A)$$

$u \notin \text{Col}(A)$ .

$$8) \quad |A| \stackrel{\text{expand by}}{\underset{2^{\text{nd}} \text{ row}}{(-1)^{2+2} \cdot 3}} \begin{vmatrix} 1 & 5 & 2 \\ 2 & -7 & 5 \\ 5 & 4 & 3 \end{vmatrix}$$

$$= 3 [-21 + 125 + 16 + 70 - 20 - 30]$$

$$= 3 [104 + 36] = 3 \cdot 140 = 420.$$

$$9) \quad |A| = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-b)(c+b) \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+b \end{vmatrix}$$

$$= (b-a)(c-a)(c-a).$$

$$10) \quad A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} \quad |A| = 6 - 1 + 3 - 4 = 4$$

$$|A_1(b)| = \begin{vmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{vmatrix} = [-4 + 2 - 6 - 8] = -16$$

$$|A_2(b)| = \begin{vmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{vmatrix} = 12 + 24 + 2 - 6 + 12 + 8 = 60$$

$$|A_3(b)| = \begin{vmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{vmatrix} = 6 - 4 - 2 - 4 = -4$$

$$\begin{cases} x_1 = \frac{-16}{4} = -4 \\ x_2 = \frac{60}{4} = 15 \\ x_3 = \frac{-4}{4} = -1 \end{cases}$$

□

$$11). |A| = 24 + 12 - 28 - 9 = -1$$

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$$

$$C_{12} = - \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = 2$$

$$C_{13} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}$$

$$C_{21} = - \begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} = -3$$

$$= \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

$$C_{22} = \begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = -2$$

□

$$C_{23} = - \begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} = 3$$

$$C_{31} = \begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} = -8$$

$$C_{32} = - \begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} = 3$$

$$C_{33} = \begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} = 6$$