Some Review Problems for Midterm II/ MA 213, Fall 2011

Exam time: Monday, November 22, 1:00-1:50 pm, CB 349

The second midterm covers Chapter 15.1-15.6, 15.8, Chapter 16.1-16.5. Here are some of the review problems.

1. Calculate the double integral

$$\int\int_R (xy^2 + \frac{y}{x})\, dA$$

where

$$R = \{(x, y)|2 \leq x \leq 3, -1 \leq y \leq 0\}.$$

Solution:

$$\begin{aligned}
&= \int_{-1}^{0}\int_{2}^{3}(xy^2 + \frac{y}{x})\, dx\, dy \\
&= \int_{-1}^{0}(\frac{x^2y^2}{2} + y\ln x)\Big|_2^3\, dy \\
&= \int_{-1}^{0}\frac{5y^2}{2} + y\ln(\frac{3}{2})\, dy \\
&= \frac{5y^3}{6} + \ln(3/2)y^2/2\Big|_{-1}^{0} = \frac{5}{6} - 1/2\ln(3/2).
\end{aligned}$$

2. Use the polar coordinate to calculate the double integral

$$\int_0^2 dy \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx.$$

Solution: The domain is : $0 \leq y \leq 2$, $y \leq x \leq \sqrt{4-y^2}$. Changing to polar coordinates, we have $0 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{4}$.

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}}\int_0^2 \frac{1}{1+r^2} r\, dr\, d\theta \\
&= \frac{\pi}{8}\ln(1+r^2)\Big|_{r=0}^{2} = \frac{\pi}{8}\ln 5.
\end{aligned}$$

3. Evaluate the triple integral

$$\int\int\int_E e^x\, dV$$

where

$$E = \{(x, y, z) | 0 \le y \le 1,\ 0 \le x \le y,\ 0 \le z \le e^x + y\}.$$

Solution:

$$\begin{aligned}
&= \int_0^1 \int_0^y \int_0^{e^x+y} e^x \, dz \, dx \, dy \\
&= \int_0^1 \int_0^y (e^{2x} + ye^x) \, dx \, dy \\
&= \int_0^1 (\frac{1}{2} e^{2x}\Big|_0^y + ye^x\Big|_0^y) \, dy \\
&= \int_0^1 (\frac{1}{2} e^{2y} - \frac{1}{2} + ye^y - y) \, dy \\
&= \frac{1}{4} e^{2y}\Big|_0^1 - \frac{1}{2} + ye^y\Big|_0^1 - e^y\Big|_0^1 - \frac{1}{2} \\
&= \frac{1}{4} e^2 - \frac{3}{4} + e - e + 1 - \frac{1}{2} = \frac{1}{4}(e^2 - 1).
\end{aligned}$$

4. Evaluate

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 \, dx dy$$

Solution: Applying the polar coordinate, we have $0 \le r \le 2, 0 \le \theta \le \pi$.

$$\begin{aligned}
&= \int_0^2 \int_0^\pi r^4 \cos^2\theta \sin^2\theta \; r \, dr \, d\theta \\
&= (\frac{1}{24} r^6\Big|_0^2) \cdot \int_0^\pi \sin^2(2\theta) \, d\theta \\
&= \frac{8}{3} \int_0^\pi \frac{1 - \cos(4\theta)}{2} \, d\theta = \frac{4\pi}{3}.
\end{aligned}$$

5. Calculate the volume of the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

Solution: The region is bounded by

$$x^2 + y^2 \le \frac{1}{2},\ \sqrt{x^2 + y^2} \le z \le \sqrt{1 - (x^2 + y^2)}.$$

Hence the volume is given by

$$= \int_{x^2+y^2 \le \frac{1}{2}} (\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}) \, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}/2} (\sqrt{1-r^2}-r)\; r\,dr\,d\theta$$
$$= \frac{4-\sqrt{2}}{3}\pi$$

6. Find the mass of the region D that is enclosed by the cardioid $r = 1+\cos\theta$ with density $\rho(x,y) = \sqrt{x^2+y^2}$.

Solution

$$= 2\int_0^{\pi}\int_0^{1+\cos\theta}\int r^2\,dr\,d\theta = \frac{2}{3}\int_0^{\pi}(1+\cos\theta)^3\,d\theta$$
$$= \frac{2}{3}\int_0^{\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\,d\theta$$
$$= \frac{2\pi}{3}+\pi = \frac{5}{3}\pi.$$

7. Use the spherical coordinates to calculate

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dzdxdy.$$

Solution

$$= \int_0^{\pi}\int_0^{\pi}\int_0^{2}\rho^3\sin^2\phi\sin^2\theta\;\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \Big(\frac{\rho^6}{6}\Big|_0^2\Big)\Big(\int_0^{\pi}\sin^2\theta\,d\theta\Big)\Big(\int_0^{\pi}\sin^3\phi\,d\phi\Big)$$
$$= \frac{32}{3}\frac{\pi}{2}\Big[\frac{\cos^3\phi}{3}-\cos\phi\Big]\Big|_0^{\pi} = \frac{64}{9}\pi.$$

8. The joint density function for random variables X, Y, and Z is

$$f(x,y,z) = Cxyz \;\text{ if }\; 0\le x\le 2, 0\le y\le 2, 0\le z\le 2; \;\; = 0 \;\text{ otherwise.}$$

(a) Find the value of the constant C.
(b) Find $P(X\le 1, Y\le 1, Z\le 1)$.
(c) Find $P(X+Y+Z\le 1)$.

Solution:
(a)

$$C\int_0^2\int_0^2\int_0^2 xyz\,dz\,dy\,dx = 1$$

implies $C = \frac{1}{8}$.
(b)

$$= \frac{1}{8}\int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx = \frac{1}{8}(\frac{1}{2})^3 = \frac{1}{64}.$$

(c) The domain is

$$0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y.$$

Hence the integral becomes

$$\begin{aligned}
&= \frac{1}{8}\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx \\
&= \frac{1}{16}\int_0^1\int_0^{1-x} xy(1-x-y)^2\,dy\,dx \\
&= \frac{1}{16}\int_0^1 x\Big[\frac{(1-x)^2}{2}y^2 - \frac{2}{3}(1-x)y^3 + \frac{y^4}{4}\Big]\Big|_0^{1-x}\,dx \\
&= \frac{1}{192}\int_0^1 x(1-x)^4\,dx \\
&= \frac{1}{192}\Big[\frac{(x-1)^6}{6} - \frac{(x-1)^5}{5}\Big]\Big|_0^1 = \frac{1}{192\times 30}.
\end{aligned}$$

9 Find the center of mass for the lamina that occupies the region D and has the given density function ρ: D is the triangular region with vertices $(0,0)$, $(2,1)$, $(0,3)$; $\rho(x,y) = x+y$.

Solution: The domain is $0 \le x \le 1,\ \frac{1}{2}x \le y \le 3-x$. First we calculate the mass

$$\begin{aligned}
m &= \int_0^1\int_{\frac{1}{2}x}^{3-x}(x+y)\,dy\,dx \\
&= \int_0^1(\frac{9}{2} - \frac{9}{8}x^2)\,dx = \frac{33}{8}.
\end{aligned}$$

Next we calculate the center of mass

$$\int_0^1\int_{\frac{1}{2}x}^{3-x} x(x+y)\,dy\,dx = \int_0^1(\frac{9}{2}x - \frac{9}{8}x^3)\,dx = \frac{63}{32},$$

and

$$\int_0^1 \int_{\frac{1}{2}x}^{3-x} y(x+y)\,dy\,dx = \int_0^1 \frac{9}{2}x - 3x^2 + \frac{x^3}{3} - \frac{(x-3)^3}{3}\,dx = 6\frac{3}{4}.$$

Hence

$$\bar{x} = \frac{63}{132}, \quad \bar{y} = \frac{54}{33}.$$

10. Evaluate $\int_C x \sin y\,dx + xyz\,dz$, where C is given by $x = t, y = t^2, z = t^3$ for $0 \le t \le 1$.

Solution:

$$= \int_0^1 (t \sin t^2 + 3t^8)\,dt = \frac{1}{2}(1 - \cos 1) + \frac{1}{3}.$$

11. Evaluate $\int_C x^3 y\,dx - x\,dy$, where C is the circle $\{(x,y) | x^2 + y^2 = 1\}$ with counterclockwise orientation.
Solution:

$$= -\int_{x^2+y^2 \le 1} (1 + x^3)\,dA = -\pi.$$

12. Evalulate $\int_C F \cdot dr$, where $F(x,y) = (x^2 y, e^y)$ and $C = \{(t^2, -t^3) | 0 \le t \le 1\}$.

Solution:

$$\begin{aligned} &= \int x^2 y dx + e^y dy \\ &= \int_0^1 (-t^7(2t)\,dt + e^{-t^3}(-3t^2)\,dt) \\ &= -2\frac{t^9}{9}\Big|_0^1 + e^{-t^3}\Big|_0^1 \\ &= -2/9 + e^{-1} - 1. \end{aligned}$$

13. Evaluate $\int_C z e^{-xy}\,ds$, where C has parametric eqyations $x = t,\ y = t^2,\ z = e^{-t},\ 0 \le t \le 1$.

Solution:

$$= \int_0^1 e^{-t-t^3}\sqrt{1+4t^2+e^{-2t}}\,dt.$$

14. Verify that F is a conservative vector field and finds it potential function f (i.e. $F = Df$). Here
(i) $F(x, y) = (\sin y, x\cos y + \sin y)$
(ii) $F(x, y, z) = (2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz)$.

Solution:
(i) $Q_x = \cos y$ and $P_y = \cos y$, so F is conservative. To find its potential function, let $f_x = \sin y$ and $f_y = x\cos y + \sin y$. From this, we obtain

$$f(x, y) = x\sin y + G(y),$$

which implies $G'(y) = \sin y$ or $G(y) = -\cos y$. All together we find the potential function is given by

$$f(x, y) = x\sin y - \cos y + C.$$

(ii): Solving $f_x = 2xy^3 + z^2$ we have $f(x, y, z) = x^2y^3 + xz^2 + H(y, z)$. Since $f_y = 3x^2y^2 + H_y = 2x^2y^2 + 2yz$, we get $H_y = 2yz$ so that $H(y, z) = y^2z + G(z)$ and $f(x, y, z) = 2x^2y^2 + xz^2 + y^2z + G(z)$. Since $f_z = 2xz + y^2 + G'(z) = y^2 + 2xz$ we have $G'(z) = 0$ so that $G(z) = C$. All together we obtain

$$f(x, y, z) = 2x^2y^2 + xz^2 + y^2z + C.$$

15. Use the Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is $\{(x, y)|x^2 + y^2 = 4\}$ with counterclockwise orientation.

Solution:

$$\begin{aligned} &= -\int_{x^2+y^2\le 4} (x^2+y^2)\,dA \\ &= -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi. \end{aligned}$$

16. Use the Green's Theorem to evaluate $\int_C (1 + \tan x)\,dx + (9x^2 + e^y)\,dy$ where C is the positively oriented boundary of the region enclosed by the

curves $y = \sqrt{x}$, $x = 1$, and $y = 0$.

Solution:

$$
\begin{aligned}
&= \int_D (18x - 0)\, dA = 18 \int_0^1 \int_0^{\sqrt{x}} x\, dy\, dx \\
&= 18 \int_0^1 x^{\frac{3}{2}}\, dx = \frac{36}{5}.
\end{aligned}
$$