

# Finite Time Singularity of the Landau-Lifshitz-Gilbert Equation

Shijin Ding<sup>1</sup> and Changyou Wang<sup>2</sup>

<sup>1</sup>School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P. R. China and <sup>2</sup>Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

*Correspondence to be sent to: Changyou Wang, Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA. e-mail: cywang@ms.uky.edu*

We prove that in dimensions three or four, for suitably chosen initial data, the short time smooth solution to the Landau-Lifshitz-Gilbert equation blows up at finite time.

## 1 Introduction

The Landau-Lifshitz-Gilbert equation is the fundamental evolution equation for spin fields in the continuum theory of ferromagnetism, first proposed by Landau and Lifshitz [1] in 1935. In the simplest case, where the energy of spin interactions is modeled by  $E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2$  for magnetic moment  $u : \Omega \subset \mathbf{R}^n \rightarrow \mathbf{S}^2$ , the Landau-Lifshitz-Gilbert equation for  $u : \Omega \times (0, +\infty) \rightarrow \mathbf{S}^2$  is given by

$$\alpha u_t + \beta u \wedge u_t = \Delta u + |\nabla u|^2 u, \quad (1.1)$$

where  $\alpha \geq 0, \beta \in \mathbf{R}, \alpha^2 + \beta^2 = 1$ , and  $\wedge$  is the vector product in  $\mathbf{R}^3$ . Note that (1.1) reduces to the heat flow of harmonic maps to  $\mathbf{S}^2$  for  $\alpha = 1, \beta = 0$ , and to the Schrödinger flow of harmonic maps to  $\mathbf{S}^2$  for  $\alpha = 0, \beta = 1$ . We assume throughout this article that  $0 < \alpha < 1$ ,

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hence (1.1) is the hybrid of both the heat flow and Schrödinger flow of harmonic maps to  $\mathbf{S}^2$ , and is of parabolic type.

Motivated by the study on the heat flow of harmonic maps by Chen [2], Struwe [3], Chen-Struwe [4], Chen-Lin [5], Coron-Ghilagdia [6], Chen-Ding [7] and others, people have recently been interested in the analysis of the Landau-Lifshitz-Gilbert Equation (1.1). For example, Alouges-Soyeur [8] established the existence of global weak solutions of (1.1) under the Neumann boundary condition in any dimensions. Chen-Ding-Guo [9] studied partial regularity of (1.1) in dimension two, Moser [10, 11] proved the partial regularity for suitably weak solutions of (1.1) in dimensions three and four, and Liu [12] considered the partial regularity of (1.1) in general dimensions, analogous to Feldman [13] and Chen-Li-Lin [14] on the heat flow of harmonic maps to spheres. More recently, the existence of partially smooth, global weak solutions of (1.1), similar to [4, 5], has been obtained by Guo-Hong [15] for  $n = 2$ , Melcher [16] for  $n = 3$  and  $\Omega = \mathbf{R}^3$ , and Wang [17] for  $n \leq 4$  and  $\Omega = \mathbf{R}^n$  or  $\Omega \subset \mathbf{R}^n$  bounded domains with Dirichlet boundary conditions. Due to the lack of Struwe's parabolic monotonicity formula (cf. [3, 4]), the above mentioned results by [10, 11, 16, 17] are limited to low dimensions. It remains a very interesting and difficult question to investigate (1.1) for dimensions at least five. Since (1.1) is a strongly parabolic system, it is well-known that there always exists a unique short time smooth solution. Another interesting question is whether the short time smooth solution actually blows up at finite time. Recently, there have been several articles by numerical methods which strongly suggest the appearance of singularities in finite time for the Landau-Lifshitz-Gilbert Equation (1.1) (we refer the interested readers to Bartels-Ko-Prohl [18] and Pistella-Valente [19]). Through the works by [6, 7], it is well-known that a finite time singularity does occur for the heat flow of harmonic maps. We would like to remark that the crucial ingredient in [6, 7] is the  $\epsilon$ -regularity for smooth solutions to the heat flow of harmonic map, which is based on both the Struwe's parabolic energy monotonicity formula and the Bochner identity for the heat flow of harmonic maps (cf. [3, 4, 20]). However, neither Struwe's parabolic energy monotonicity formula nor the Bochner identity are available for the Landau-Lifshitz-Gilbert Equation (1.1). Fortunately, inspired by the works of [10, 16, 17], we are able to employ (i) the slice energy monotonicity formula for (1.1) in low dimensions, (ii) the local Hardy space estimate, and (iii) the duality between Hardy and BMO spaces, to establish an  $\epsilon$ - priori estimate for (1.1), and adapt the construction by [7] on suitable initial data to prove.

**Theorem 1.1.** For  $n = 3, 4$ , let  $(M, g)$  be a  $n$ -dimensional, compact Riemannian manifold without boundary, and let  $i_M > 0$  denote the injectivity radius of  $M$ . Then there exists  $\epsilon = \epsilon(M, \alpha) > 0$  such that if  $u_0 \in C^\infty(M, \mathbf{S}^2)$  satisfies  $E(u_0) \leq \epsilon^2$  and is not homotopic to a constant, then the short time smooth solution  $u$  to

$$\alpha u_t + \beta u \wedge u_t = \Delta_g u + |\nabla u|_g^2 u, \quad x \in M, \quad t > 0, \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad x \in M, \quad (1.3)$$

must blow up before  $T = i_M^2$ , where  $\Delta_g$  is the Laplace–Beltrami operator with respect to  $g$  and  $|\nabla u|_g^2 = \sum_{\alpha, \beta=1}^n g^{\alpha\beta} \langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \rangle$ .  $\square$

**Remark 1.2.**

- (1) There exists initial data  $u_0 \in C^\infty(M, \mathbf{S}^2)$  satisfying the conditions of Theorem 1.1. In fact, if  $\Pi_1(M) = \Pi_2(M) = \{0\}$  and  $[M, \mathbf{S}^2] \neq 0$  (i.e. there are nontrivial free homotopy classes, e.g.  $M = \mathbf{S}^3$ ), then a well-known theorem by White [21] asserts

$$\inf \left\{ \int_M |\nabla u|_g^2 dv_g \mid u \in C^\infty(M, \mathbf{S}^2), [u] = \alpha \in [M, \mathbf{S}^2] \right\} = 0. \quad (1.4)$$

- (2) If  $M = \mathbf{S}^3$ , then  $[\mathbf{S}^3, \mathbf{S}^2] = \mathbf{Z}$ . Let  $H(z, w) = (|z|^2 - |w|^2, 2zw) : \mathbf{S}^3 = \{(z, w) \in \mathbf{C} \times \mathbf{C} : |z|^2 + |w|^2 = 1\} \rightarrow \mathbf{S}^2$  be the Hopf map. Let  $\Phi_\lambda(x) = \lambda x : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  be the dilation map for  $\lambda > 0$ ,  $\Pi : \mathbf{S}^3 \rightarrow \overline{\mathbf{R}^3}$  be the stereographic projection map from the north pole, and  $\Psi_\lambda = \Pi^{-1} \circ \Phi_\lambda \circ \Pi : \mathbf{S}^3 \rightarrow \mathbf{S}^3$ . Then direct calculations imply

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbf{S}^3} |\nabla(H \circ \Psi_\lambda)|^2 = 0, \quad (1.5)$$

and  $H \circ \Psi_\lambda, \lambda > 0$ , are homotopic to  $H$ .

For manifolds with boundaries, we can consider either the Dirichlet boundary value problem or the Neumann boundary value problem of (1.2)–(1.3). Recall that for  $\partial M \neq \emptyset$ ,  $\phi, \psi \in C(M, \mathbf{S}^2)$  are homotopic, relative to  $\partial M$ , if  $\phi = \psi$  on  $\partial M$  and there exists  $H \in C(\overline{M} \times [0, 1], \mathbf{S}^2)$  such that  $H(x, 0) = \phi(x)$  and  $H(x, 1) = \psi(x), x \in M$ , and  $H(x, t) = \phi(x)$  for  $(x, t) \in \partial M \times [0, 1]$ .

For the Dirichlet boundary problem of (1.2)–(1.3), we have

**Theorem 1.3.** For  $n = 3, 4$ , let  $(M, g)$  be a  $n$ -dimensional compact Riemannian manifold with boundary, and let  $i_M > 0$  be the injectivity radius of  $M$ . Then there exists  $\epsilon = \epsilon(M, \alpha) > 0$  such that if  $u_0 \in C^\infty(M, \mathbf{S}^2)$  satisfies  $u_0|_{\partial M} = \text{constant}$ ,  $u_0$  is not homotopic to

a constant relative to  $\partial M$ , and  $E(u_0) \leq \epsilon$ , then the short time smooth solution  $u$  to (1.2), (1.3), and

$$u(x, t) = u_0(x) = \text{constant}, \quad x \in \partial M, \quad t > 0, \quad (1.6)$$

must blow-up before  $T = i_M^2$ .  $\square$

**Remark 1.4.** For  $n \geq 3$ , let  $(M, g)$  be a  $n$ -dimensional manifold with boundary and  $\Pi_1(M) = \Pi_2(M) = 0$ . In [21], White also proved that for any  $u_0 \in C^1(M, \mathbf{S}^2)$ , with  $u_0|_{\partial M} = \text{constant}$ ,

$$\inf\{E(u) \mid u \in C^1(M, \mathbf{S}^2), [u] = [u_0] \text{ rel. } \partial M\} = 0. \quad (1.7)$$

In particular, if  $M = B^n, n \geq 3$ , then we can find  $u_0 \in C^1(B^n, \mathbf{S}^2)$  such that  $u_0|_{\partial B^n} = \text{constant}$ ,  $u_0$  is not homotopic to a constant relative to  $\partial B^n$ , and  $E(u_0)$  is arbitrarily small.

For the Neumann boundary value problem, we have the following.

**Theorem 1.5.** Let  $M = \Omega = \{x \in \mathbf{R}^4 : 1 \leq |x| \leq 2\}$ ,  $g = g_0$  be the Euclidean metric on  $\mathbf{R}^4$ , and  $u_0(x) = (H \circ \Psi_\lambda)(\frac{x}{|x|}) : M \rightarrow \mathbf{S}^2$ , where  $H \circ \Psi_\lambda$  is given by Remark 1.2 (2). Then for any  $T > 0$ , there exists  $\lambda = \lambda(T) > 0$  such that the short time smooth solution  $u$  to (1.2)–(1.3) and

$$\frac{\partial u}{\partial \nu}(x, t) = 0, \quad x \in \partial \Omega, \quad t > 0 \quad (1.8)$$

must blow up before time  $t = T$ .  $\square$

**Remark 1.6.**

- (1) It is unknown whether theorem 1.5 holds in dimension three. Namely in dimension three, we are unable to construct a map  $u_0 \in C^\infty(M, \mathbf{S}^2)$  such that  $E(u_0)$  can be arbitrarily small, and it can't be deformed into a constant map through families of maps  $H \in C^1(\overline{M} \times [0, 1], \mathbf{S}^2)$  with  $\frac{\partial H}{\partial \nu}(x, t) = 0$  for  $(x, t) \in \partial M \times [0, 1]$ . In fact, for  $M = B^3 = \{x \in \mathbf{R}^3 : |x| \leq 1\}$ , it is not difficult to show that for any map  $\phi \in C^\infty(\overline{B^3}, \mathbf{S}^2)$ , with  $\frac{\partial \phi}{\partial \nu} = 0$ , there exists  $\Phi \in C^1(\overline{B^3} \times [0, 1], \mathbf{S}^2)$  such that  $\Phi(\cdot, 0) = \phi$ ,  $\Phi(\cdot, 1) = \text{constant}$ , and  $\frac{\partial \Phi}{\partial \nu} = 0$  on  $\partial B^3 \times [0, 1]$ .
- (2) It is a very important open problem whether the Landau-Lifshitz-Gilbert equation has finite time singularity in dimension two. It is well-known (cf. Chang-Ding-Ye [22]) that there exists finite time singularity for the heat flow of harmonic maps in two dimensions.

The article is organized as follows. In §2, we establish a priori estimates for smooth solutions of (1.2) under a small energy condition and prove Theorem 1.1. In §3, we establish boundary a priori estimates for smooth solutions of (1.1). In §4, we prove both Theorem 1.3 and 1.5.

## 2 Hölder continuity estimate and Proof of Theorem 1.1

In this section, we first establish a priori continuity estimate of smooth solutions to (1.2) under small energy condition, and then give a proof of Theorem 1.1.

**Lemma 2.1** (Energy inequality). For any  $n \geq 1$ ,  $T > 0$ , and  $u_0 \in C^\infty(M, \mathbf{S}^2)$ , let  $u \in C^\infty(M \times [0, T], \mathbf{S}^2)$  solve (1.2)–(1.3). For  $t \in (0, T)$ , denote  $u(t) = u(\cdot, t)$ . Then we have

$$\alpha \int_0^t \int_M |u_t|^2 + E(u(t)) = E(u_0), \quad (2.1)$$

and, for any  $0 \leq s < t < T$  and  $\phi \in C_0^\infty(M)$ ,

$$\alpha \int_s^t \int_M |u_t|^2 \phi^2 + \int_M |\nabla u(t)|^2 \phi^2 \leq \int_M |\nabla u(s)|^2 \phi^2 + \frac{4}{\alpha} \int_s^t \int_M |\nabla u|^2 |\nabla \phi|^2. \quad (2.2) \quad \square$$

*Proof.* Since  $u \wedge u_t \cdot u_t = 0$ , (2.1) follows from multiplying (1.2) by  $u_t$  and integrating the resulting equation over  $M \times [0, t]$ . To see (2.2), we multiply (1.2) by  $u_t \phi^2$  and integrate the resulting equation over  $M \times [s, t]$  to get

$$\alpha \int_s^t \int_M |u_t|^2 \phi^2 + \frac{1}{2} \int_M |\nabla u(t)|^2 \phi^2 = \frac{1}{2} \int_M |\nabla u(s)|^2 \phi^2 - 2 \int_s^t \int_M u_t \cdot \nabla u \phi \nabla \phi.$$

By the Hölder inequality, we have

$$|2 \int_s^t \int_M u_t \cdot \nabla u \phi \nabla \phi| \leq \frac{\alpha}{2} \int_s^t \int_M |u_t|^2 \phi^2 + \frac{2}{\alpha} \int_s^t \int_M |\nabla u|^2 |\nabla \phi|^2.$$

Hence (2.2) follows.

Let  $i_M > 0$  be the injectivity radius of  $M$ . For  $x \in M$ ,  $t > 0$ , and  $0 < r < \min\{i_M, \sqrt{t}\}$ , let  $B_r(x) \subset M$  be the ball with center  $x$  and radius  $r$ , and  $P_r(x, t) = B_r(x) \times (t - r^2, t) \subset M \times (0, +\infty)$  be the parabolic ball with center  $(x, t)$  and radius  $r$ . Now we have the localized energy inequality. ■

**Lemma 2.2.** For any  $n \geq 1$  and  $T > 0$ , let  $u \in C^\infty(M \times (0, T), \mathbf{S}^2)$  solve (1.2). Then for any  $z_0 = (x_0, t_0) \in M \times (0, T)$ ,  $0 < r < \min\{i_M, \sqrt{t_0}\}$ , and  $t \in (t_0 - \frac{r^2}{4}, t_0)$ , there exists  $C_\alpha > 0$  depending only on  $\alpha, M$  such that

$$r^{2-n} \int_{B_{\frac{r}{2}}(x_0)} |\nabla u(t)|^2 + r^{2-n} \int_{P_{\frac{r}{2}}(z_0)} |u_t|^2 \leq C_\alpha r^{-n} \int_{P_r(z_0)} |\nabla u|^2. \quad (2.3)$$

*Proof.* For  $0 < r < \min\{i_M, \sqrt{t_0}\}$ , let  $\phi \in C_0^\infty(B_r(x_0))$  be such that  $0 \leq \phi \leq 1$ ,  $\phi \equiv 1$  on  $B_{\frac{r}{2}}(x_0)$ , and  $|\nabla \phi| \leq 4r^{-1}$ . Let  $s_0 \in (t_0 - r^2, t_0 - \frac{r^2}{4})$  be such that

$$\int_{B_r(x_0)} |\nabla u(s_0)|^2 \leq 2r^{-2} \int_{P_r(z_0)} |\nabla u|^2.$$

Putting  $\phi$  into (2.2), we have

$$\alpha \int_{s_0}^{t_0} \int_{B_{\frac{r}{2}}(x_0)} |u_t|^2 \leq \int_{B_{r_0}(x_0)} |\nabla u(s_0)|^2 + \frac{64}{\alpha r^2} \int_{s_0}^{t_0} \int_{B_{r_0}(x_0)} |\nabla u|^2, \quad (2.4)$$

and, for any  $t \in (t_0 - \frac{r^2}{4}, t_0)$ ,

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla u(t)|^2 \leq \int_{B_r(x_0)} |\nabla u(s_0)|^2 + \frac{64}{\alpha r^2} \int_{s_0}^{t_0} \int_{B_r(x_0)} |\nabla u|^2. \quad (2.5)$$

It is clear that (2.4) and (2.5) imply (2.3).

Now we are ready to prove the following decay estimate. ■

**Lemma 2.3.** For  $1 \leq n \leq 4$ , and any  $T > 0$  and  $\gamma \in (0, 1)$ , there exist  $\epsilon_0 > 0$  and  $C_\alpha > 0$  depending only on  $M, g, \gamma, \alpha$  such that if  $u \in C^\infty(M \times (0, T), \mathbf{S}^2)$  solves (1.2) and satisfies, for  $z_0 = (x_0, t_0) \in M \times (0, T)$  and  $0 < r \leq \min\{i_M, \sqrt{t_0}\}$ ,

$$r^{-n} \int_{P_r(z_0)} |\nabla u|^2 \leq \epsilon_0^2,$$

then  $u \in C^\gamma(P_{\frac{r}{2}}(z_0), \mathbf{S}^2)$ , and

$$[u]_{C^\gamma(P_s(z_0))}^2 \leq C_\alpha r^{-(n+2\gamma)} \int_{P_r(z_0)} |\nabla u|^2, \quad \forall 0 < s \leq \frac{r}{2}. \quad (2.6)$$

In order to prove Lemma 2.3, we first need to recall the following decay Lemma that can be proved by a simple blowing up argument (see e.g. [10] Lemma 3.3 or [17] Lemma 5.10). □

**Lemma 2.4.** There exists a constant  $C_0 > 0$  such that for any  $\theta \in (0, \frac{1}{4})$ , there exists  $\epsilon_1(\theta) > 0$  such that for any solution  $u \in C^\infty(M \times [0, T], \mathbf{S}^2)$  of (1.2),  $z_0 = (x_0, t_0) \in M \times (0, T)$  and  $0 < r \leq \min\{i_M, \sqrt{t_0}\}$ , if  $u$  satisfies

$$r^{-n} \int_{P_r(z_0)} |\nabla u|^2 \leq \epsilon_1^2(\theta),$$

then we have

$$(\theta r)^{-n-2} \int_{P_{\theta r}(z_0)} |u - u_{z_0, \theta r}|^2 \leq C_0 \theta^2 r^{-n} \int_{P_r(z_0)} |\nabla u|^2, \quad (2.7)$$

where  $u_{z_0, \theta r} = \frac{1}{|P_{\theta r}(z_0)|} \int_{P_{\theta r}(z_0)} u$ . □

### 2.1 Proof of Lemma 2.3

For simplicity, we may assume  $n = 4$ . In fact, if  $n \leq 3$ , then we let  $\hat{M} = M \times \mathbf{S}^{4-n}$ ,  $\hat{g}(x, y) = g(x) + h_0(y)$  with  $h_0$  the standard metric on  $\mathbf{S}^{4-n}$ , and define  $\hat{u}(x, y, t) = u(x, t)$  for  $x \in M, y \in \mathbf{S}^{4-n}$ . One can easily check that  $\hat{u} \in C^\infty(\hat{M} \times (0, T), \mathbf{S}^2)$  solves (1.2) and satisfies  $r^{-4} \int_{P_r(z_0)} |\nabla \hat{u}|^2 \leq \epsilon_0^2$ . Hence it suffices to prove (2.6) for  $\hat{u}$ . Since it is a local result, we may further assume that  $M = \mathbf{R}^4$  and  $g$  is the Euclidean metric. One can modify without difficulties the following argument to handle the general case, see [4] for example.

Now we have

**Claim.** For any  $\delta \in (0, 1)$ , there exist  $C(\delta) > 0$  and  $\epsilon_2(\delta) > 0$  such that if

$$r^{-4} \int_{P_r(z_0)} |\nabla u|^2 \leq \epsilon_2^2(\delta),$$

then

$$\left(\frac{r}{8}\right)^{-4} \int_{P_{\frac{r}{8}}(z_0)} |\nabla u|^2 \leq \delta r^{-4} \int_{P_r(z_0)} |\nabla u|^2 + \frac{C(\delta)}{\delta} r^{-6} \int_{P_r(z_0)} |u - u_{P_r(z_0)}|^2. \quad (2.8)$$

First, by considering  $u_r(x, t) = u(z_0 + (rx, r^2t)) : \mathbf{R}^4 \times (-\frac{t_0}{r^2}, 0) \rightarrow \mathbf{S}^2$ , we may assume  $r = 1, z_0 = (0, 0)$ , and  $u \in C^\infty(\mathbf{R}^4 \times (-1, 0], \mathbf{S}^2)$  solves (1.2). Denote  $B_r(0)$  by  $B_r$  and  $P_r(0, 0)$  by  $P_r$ . Now we divide the proof of the claim into three steps.

*Step 1* (slice monotonicity inequality). For any  $t \in (-1, 0], x_0 \in \mathbf{R}^4, 0 < r_1 \leq r_2 < +\infty$ , it holds

$$r_1^{-2} \int_{B_{r_1}(x_0)} |\nabla u(t)|^2 \leq 2r_2^{-2} \int_{B_{r_2}(x_0)} |\nabla u(t)|^2 + 2 \int_{B_{r_2}(x_0)} |u_t|^2. \quad (2.9)$$

It is well-known that (2.9) follows from the standard Pohozaev type argument (see [10, 16, 17] for more details). Here we sketch the proof. Assume  $x_0 = 0$ , let  $R(u)(p) := \alpha p + \beta u \wedge p : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ . Since  $u \in C^\infty(\mathbf{R}^4 \times (-1, 0], \mathbf{S}^2)$ , we multiply (1.2) by  $x \cdot \nabla u(t)$  and integrate it over  $B_r$  to get

$$\begin{aligned} \int_{B_r} R(u)(u_t)x \cdot \nabla u &= \int_{B_r} \Delta u x \cdot \nabla u \\ &= r \int_{\partial B_r} \left| \frac{\partial u}{\partial r} \right|^2 + \int_{B_r} |\nabla u|^2 - \frac{r}{2} \int_{\partial B_r} |\nabla u|^2. \end{aligned} \quad (2.10)$$

This, combined with  $|R(u)(u_t)| = |u_t|$ , yields

$$\begin{aligned} \frac{d}{dr} \left( r^{-2} \int_{B_r} \frac{|\nabla u|^2}{2} \right) &= r^{-2} \int_{\partial B_r} \left| \frac{\partial u}{\partial r} \right|^2 - r^{-3} \int_{B_r} R(u)(u_t)x \cdot \nabla u \\ &\geq r^{-2} \int_{\partial B_r} \left| \frac{\partial u}{\partial r} \right|^2 + \frac{d}{dr} \left( r^{-1} \int_{B_r} |u_t| \left| \frac{\partial u}{\partial r} \right| \right) - r^{-1} \int_{\partial B_r} |u_t| \left| \frac{\partial u}{\partial r} \right|. \end{aligned}$$

Integrating this inequality from  $r_1$  to  $r_2$ , we have

$$\begin{aligned} r_2^{-2} \int_{B_{r_2}} |\nabla u|^2 &\geq r_1^{-2} \int_{B_{r_1}} |\nabla u|^2 + 2 \int_{B_{r_2} \setminus B_{r_1}} r^{-2} \left| \frac{\partial u}{\partial r} \right|^2 \\ &\quad - 2r_1^{-1} \int_{B_{r_1}} |u_t| \left| \frac{\partial u}{\partial r} \right| - 2 \int_{r_1}^{r_2} r^{-1} \int_{\partial B_r} |u_t| \left| \frac{\partial u}{\partial r} \right|. \end{aligned} \quad (2.11)$$

By the Hölder inequality, we have

$$2r_1^{-1} \int_{B_{r_1}} |u_t| \left| \frac{\partial u}{\partial r} \right| \leq \frac{1}{2} r_1^{-2} \int_{B_{r_1}} |\nabla u|^2 + 2 \int_{B_{r_2}} |u_t|^2,$$

and

$$2 \int_{r_1}^{r_2} r^{-1} \int_{\partial B_r} |u_t| \left| \frac{\partial u}{\partial r} \right| \leq \int_{B_{r_2} \setminus B_{r_1}} r^{-2} \left| \frac{\partial u}{\partial r} \right|^2 + \int_{B_{r_2}} |u_t|^2.$$

Putting these inequalities into (2.11), we obtain (2.9).

*Step 2* (estimate on good  $\Lambda$ -slices). For any  $\Lambda \geq 1$ , define the set of good  $\Lambda$ -slices by

$$G^\Lambda = \left\{ t \in \left[ -\frac{1}{4}, 0 \right] \mid \int_{B_{\frac{1}{2}}} |u_t|^2 \leq \Lambda^2 \int_{P_{\frac{1}{2}}} |u_t|^2 \right\}, \quad (2.12)$$



and the set of bad  $\Lambda$ -slices  $B^\Lambda = [-\frac{1}{4}, 0] \setminus G^\Lambda$ . By Fubini's theorem, we have

$$|B^\Lambda| \leq \frac{1}{\Lambda^2}. \quad (2.13)$$

For any  $t \in G^\Lambda$ , by (2.3) and (2.12), we have

$$\int_{B_{\frac{1}{2}}} |\nabla u(t)|^2 + \int_{B_{\frac{1}{2}}} |u_t(t)|^2 \leq C\Lambda^2 \int_{P_1} |\nabla u|^2. \quad (2.14)$$

This and (2.9) imply that for any  $t \in G^\Lambda$ , we have

$$\begin{aligned} \sup \left\{ s^{-2} \int_{B_s(x)} |\nabla u(t)|^2 \mid x \in B_{\frac{1}{4}}, 0 < s \leq \frac{1}{4} \right\} &\leq C \int_{B_{\frac{1}{2}}} (|\nabla u(t)|^2 + |u_t(t)|^2) \\ &\leq C\Lambda^2 \int_{P_1} |\nabla u|^2. \end{aligned} \quad (2.15)$$

Let  $\eta \in C_0^\infty(B_1)$  be such that  $0 \leq \eta \leq 1$ ,  $\eta = 1$  on  $B_{\frac{1}{8}}$ , and  $|\nabla \eta| \leq 16$ . For any  $t \in G^\Lambda$  fixed, observe

$$\int_{B_{\frac{1}{8}}} |\nabla u(t)|^2 \leq \int_{\mathbb{R}^4} \eta^2 |\nabla u(t)|^2 = \int_{\mathbb{R}^4} \eta^2 |\nabla u(t) \wedge u(t)|^2, \quad (2.16)$$

and (1.2) can be written as

$$\nabla \cdot (\nabla u(t) \wedge u(t)) = R(u(t))u_t(t) \wedge u(t), \text{ in } B_1 \quad (2.17)$$

where  $\nabla \cdot$  is the divergence operator.

Now we need to apply the duality theorem (cf. [23]) between Hardy and BMO space to estimate (2.16). First, by the Poincaré inequality and (2.15), we have

$$\begin{aligned} [u(t)]_{\text{BMO}(B_{\frac{1}{4}})} &= \sup \left\{ \inf_{c \in \mathbb{R}^3} s^{-4} \int_{B_s(x)} |u(t) - c| \mid B_s(x) \subset B_{\frac{1}{4}} \right\} \\ &\leq C \sup \left\{ \left( s^{-2} \int_{B_s(x)} |\nabla u(t)|^2 \right)^{\frac{1}{2}} \mid B_s(x) \subset B_{\frac{1}{4}} \right\} \\ &\leq C\Lambda \left( \int_{P_1} |\nabla u|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (2.18)$$

By integration by parts, we have

$$\begin{aligned} \int_{\mathbf{R}^4} \eta^2 |\nabla \mathbf{u}(t) \wedge \mathbf{u}(t)|^2 &= - \int_{\mathbf{R}^4} \nabla \cdot (\eta^2 \nabla \mathbf{u}(t) \wedge \mathbf{u}(t)) \cdot [(\mathbf{u}(t) - \mathbf{c}(t)) \wedge \mathbf{u}(t)] \\ &\quad + \int_{\mathbf{R}^4} \eta^2 [(\nabla \mathbf{u}(t) \wedge \mathbf{u}(t)) \wedge \nabla \mathbf{u}(t) - \lambda] \cdot (\mathbf{u}(t) - \mathbf{c}(t)) \\ &\quad + \lambda \cdot \int_{\mathbf{R}^4} \eta^2 (\mathbf{u}(t) - \mathbf{c}(t)) = I + II + III \end{aligned}$$

where

$$\mathbf{c}(t) = \frac{1}{|B_{\frac{1}{2}}|} \int_{B_{\frac{1}{2}}} \mathbf{u}(t), \quad \lambda = \frac{\int_{\mathbf{R}^4} \eta^2 [\nabla \mathbf{u}(t) \wedge \mathbf{u}(t)] \wedge \nabla \mathbf{u}(t)}{\int_{\mathbf{R}^4} \eta^2}.$$

Direct calculations, (2.9), (2.14), and (2.17) imply

$$\begin{aligned} \int_{\mathbf{R}^4} |\nabla \cdot (\eta^2 \nabla \mathbf{u}(t) \wedge \mathbf{u}(t))|^2 &\leq 2 \left( \int_{\mathbf{R}^4} |\nabla \eta|^2 |\nabla \mathbf{u}(t)|^2 + \int_{\mathbf{R}^4} \eta^2 |\mathbf{u}_t(t)|^2 \right) \\ &\leq C \int_{B_{\frac{1}{2}}} (|\nabla \mathbf{u}(t)|^2 + |\mathbf{u}_t(t)|^2). \end{aligned} \quad (2.19)$$

This implies

$$\begin{aligned} |I| &\leq \|\nabla \cdot (\eta^2 \nabla \mathbf{u}(t) \wedge \mathbf{u}(t))\|_{L^2(\mathbf{R}^4)} \|\mathbf{u}(t) - \mathbf{c}(t)\|_{L^2(B_{\frac{1}{2}})} \\ &\leq C \left( \int_{B_{\frac{1}{2}}} (|\nabla \mathbf{u}(t)|^2 + |\mathbf{u}_t(t)|^2) \right)^{\frac{1}{2}} \left( \int_{B_{\frac{1}{2}}} |\mathbf{u}(t) - \mathbf{c}(t)|^2 \right)^{\frac{1}{2}} \\ &\leq \delta \int_{B_{\frac{1}{2}}} (|\nabla \mathbf{u}(t)|^2 + |\mathbf{u}_t(t)|^2) + \frac{C}{\delta} \int_{B_{\frac{1}{2}}} |\mathbf{u}(t) - \mathbf{c}(t)|^2. \end{aligned} \quad (2.20)$$

For  $III$ , by the Poincaré inequality and (2.3), we have

$$\begin{aligned} |III| &\leq |\lambda| \|\mathbf{u}(t) - \mathbf{c}(t)\|_{L^2(B_{\frac{1}{2}})} \leq C \left( \int_{B_{\frac{1}{2}}} |\nabla \mathbf{u}(t)|^2 \right) \|\nabla \mathbf{u}\|_{L^2(B_{\frac{1}{2}})} \\ &\leq C \left( \int_{B_{\frac{1}{2}}} |\nabla \mathbf{u}(t)|^2 \right) \left( \int_{P_1} |\nabla \mathbf{u}|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (2.21)$$

To estimate  $II$ , we need to recall the definition of local Hardy spaces and the relationship between local Hardy space and  $\mathcal{H}^1(\mathbf{R}^n)$ , due to Semmes [24], and a local Hardy space estimate.

Definition 2.5. For any domain  $U \subset \mathbf{R}^n$ ,  $f \in L^1_{\text{loc}}(U)$  is in the local Hardy space  $\mathcal{H}^1_{\text{loc}}(U)$ , if for any  $K \subset\subset U$ ,  $\exists \epsilon = \epsilon(K, U) > 0$  such that

$$\|f\|_{\mathcal{H}^1_{\text{loc}}(K)} := \int_K \sup_{0 < r \leq \epsilon} \sup_{\eta \in J} |\eta_r * f|(x) < +\infty, \quad (2.22)$$

where  $J = \{\eta \in C_0^\infty(\mathbf{R}^n) \mid \text{supp}(\eta) \subset B_1, \|\nabla \eta\|_{L^\infty} \leq 1\}$ ,  $\eta_r * f(x) = r^{-n} \int_{\mathbf{R}^n} \eta(\frac{x-y}{r}) f(y)$ . We would point out that the norm defined in (2.22) depends on the choice of  $\epsilon$ , although the space  $\mathcal{H}^1_{\text{loc}}(U)$  is independent of  $\epsilon$ . For  $U = \mathbf{R}^n$ , if we choose  $\epsilon = +\infty$  in (2.22), then we get the Hardy space  $\mathcal{H}^1(\mathbf{R}^n)$ :

$$\mathcal{H}^1(\mathbf{R}^n) = \{f \in L^1_{\text{loc}}(\mathbf{R}^n) \mid \|f\|_{\mathcal{H}^1(\mathbf{R}^n)} = \sup_{r > 0, \eta \in J} \|\eta_r * f\|_{L^1(\mathbf{R}^n)} < +\infty\}.$$

**Lemma 2.6** ([24]).

- (a) For any bounded domain  $U \subset \mathbf{R}^n$ , if  $f \in \mathcal{H}^1_{\text{loc}}(U)$ , then for any  $\eta \in C_0^\infty(U)$ , with  $\int \eta \neq 0$ ,  $\eta(f - \lambda) \in \mathcal{H}^1(\mathbf{R}^n)$  and

$$\|\eta(f - \lambda)\|_{\mathcal{H}^1(\mathbf{R}^n)} \leq C(U) \|f\|_{\mathcal{H}^1_{\text{loc}}(\text{supp}(\eta))} \quad (2.23)$$

where  $\lambda = \frac{\int \eta f}{\int \eta}$ .

- (b) If  $f \in H^1(U)$ ,  $g \in L^2(U)$ , and  $\nabla \cdot g \in L^2(U)$ , then  $\nabla f \cdot g \in \mathcal{H}^1_{\text{loc}}(U)$ . Moreover,

$$\|\nabla f \cdot g\|_{\mathcal{H}^1_{\text{loc}}(U)} \leq C(U) (\|\nabla f\|_{L^2(U)}^2 + \|g\|_{L^2(U)}^2 + \|\nabla \cdot g\|_{L^2(U)}^2). \quad (2.24)$$

By Lemma 2.6, (2.18), (2.19), and the duality theorem between  $\mathcal{H}^1(\mathbf{R}^4)$  and  $\text{BMO}(\mathbf{R}^4)$  (cf. [23]), we have

$$\begin{aligned} |II| &\leq C \|\eta^2 [(\nabla u(t) \wedge u(t)) \wedge \nabla u(t) - \lambda]\|_{\mathcal{H}^1(\mathbf{R}^4)} [u(t)]_{\text{BMO}(B_{\frac{1}{4}})} \\ &\leq C \|(\nabla u(t) \wedge u(t)) \wedge \nabla u(t)\|_{\mathcal{H}^1_{\text{loc}}(B_{\frac{1}{4}})} [u(t)]_{\text{BMO}(B_{\frac{1}{4}})} \\ &\leq C [\|\nabla u(t)\|_{L^2(B_{\frac{1}{2}})}^2 + \|\nabla \cdot (\nabla u(t) \wedge u(t))\|_{L^2(B_{\frac{1}{2}})}^2] [u(t)]_{\text{BMO}(B_{\frac{1}{4}})} \\ &\leq C \Lambda \left( \int_{P_1} |\nabla u|^2 \right)^{\frac{1}{2}} \int_{B_{\frac{1}{2}}} (|\nabla u|^2 + |u_t|^2). \end{aligned} \quad (2.25)$$

Putting all estimates together, we obtain, for any  $t \in G^\wedge$ ,

$$\begin{aligned} \int_{B_{\frac{1}{8}}} |\nabla u(t)|^2 &\leq \left[ \delta + C\Lambda \left( \int_{P_1} |\nabla u|^2 \right)^{\frac{1}{2}} \right] \int_{B_{\frac{1}{2}}} (|\nabla u(t)|^2 + |u_t(t)|^2) \\ &\quad + \frac{C}{\delta} \int_{B_{\frac{1}{2}}} |u(t) - c(t)|^2. \end{aligned} \quad (2.26)$$

Integrating (2.26) over  $t \in [-(\frac{1}{8})^2, 0] \cap G^\wedge$  and applying (2.3), we obtain

$$\begin{aligned} \left( \frac{1}{8} \right)^{-4} \int_{B_{\frac{1}{8}} \times [-(\frac{1}{8})^2, 0] \cap G^\wedge} |\nabla u|^2 &\leq \left[ \delta + C\Lambda \left( \int_{P_1} |\nabla u|^2 \right)^{\frac{1}{2}} \right] \int_{P_1} |\nabla u|^2 \\ &\quad + \frac{C}{\delta} \int_{P_1} |u - u_{P_1}|^2. \end{aligned} \quad (2.27)$$

On the other hand, by (2.3) and (2.13), we have

$$\left( \frac{1}{8} \right)^{-4} \int_{B_{\frac{1}{8}} \times [-(\frac{1}{8})^2, 0] \cap B^\wedge} |\nabla u|^2 \leq \frac{C}{\Lambda^2} \int_{P_1} |\nabla u|^2. \quad (2.28)$$

Adding (2.27) and (2.28) together, we have

$$\begin{aligned} \left( \frac{1}{8} \right)^{-4} \int_{P_{\frac{1}{8}}} |\nabla u|^2 &\leq \frac{C}{\delta} \int_{P_1} |u - u_{P_1}|^2 \\ &\quad + [\delta + C\Lambda \left( \int_{P_1} |\nabla u|^2 \right)^{\frac{1}{2}} + \Lambda^{-2}] \int_{P_1} |\nabla u|^2. \end{aligned} \quad (2.29)$$

Therefore, by choosing  $\Lambda = \delta^{-\frac{1}{2}}$  and  $\epsilon_2^2(\delta) = \int_{P_1} |\nabla u|^2 \leq \frac{\delta^3}{C^2}$ , we have

$$\left( \frac{1}{8} \right)^{-4} \int_{P_{\frac{1}{8}}} |\nabla u|^2 \leq \frac{C(\delta)}{\delta} \int_{P_1} |u - u_{P_1}|^2 + 3\delta \int_{P_1} |\nabla u|^2. \quad (2.30)$$

Returning to the original scale, (2.30) implies (2.8).

*Step 3. (decay estimate).* We follow the iteration argument by Melcher [16] closely here. For simplicity, assume  $z_0 = 0$  and  $r = 1$ . For any  $\gamma \in (0, 1)$ , let  $\delta = 8^{-3}$ ,  $\theta = \theta(\gamma) \leq (\frac{\delta^2}{2C_0C(\delta)})^{\frac{1}{2-2\gamma}}$ , and  $k \geq 1$  be such that  $8^k\theta = 1$ , here  $C(\delta) > 0$  is given by (2.8) and  $C_0 > 0$  is given by Lemma 2.4. For  $\rho \in (0, 1)$ , set  $E(u, \rho) = \rho^{-n} \int_{P_\rho} |\nabla u|^2$ . For  $0 \leq i \leq k-1$ , if  $E(u, 8^{i+1}\theta) \leq \epsilon_2(\delta)^2$  and  $E(u, 1) \leq \epsilon_1(8^{i+1}\theta)^2$ , then (2.8) and Lemma 2.4 imply

$$E(u, 8^i\theta) \leq \delta E(u, 8^{i+1}\theta) + \frac{C_0C(\delta)}{\delta} E(u, 1). \quad (2.31)$$

Hence if we choose

$$E(u, 1) \leq \min\{\epsilon_1(8\theta)^2, \dots, \epsilon_1(8^k\theta)^2, \frac{\delta}{2C_0C(\delta)}\epsilon_2(\delta)^2\},$$

then by induction we have  $E(u, 8^i\theta) \leq \epsilon_2(\delta)^2$  for  $0 \leq i \leq k$ . Hence by iteration, (2.8) and Lemma 2.4 yield

$$\begin{aligned} E(u, \theta) &\leq \delta E(u, 8\theta) + C_0C(\delta) \left(\frac{\theta}{\delta}\right)^2 8^2 \delta E(u, 1) \\ &\leq \dots\dots \\ &\leq \delta^k E(u, 8^k\theta) + \frac{C_0C(\delta)}{1-64\delta} \left(\frac{\theta}{\delta}\right)^2 E(u, 1). \end{aligned} \quad (2.32)$$

According to the definition, we have  $\delta^k = \theta^3$ ,  $E(u, 8^k\theta) = E(u, 1)$ , and  $\theta^{2-2\gamma} \leq \frac{\delta^2}{2C_0C(\delta)}$ . Therefore (2.32) implies

$$E(u, \theta) \leq \left(\theta^{3-2\gamma} + \frac{4}{7}\right) \theta^{2\gamma} E(u, 1) \leq \theta^{2\gamma} E(u, 1). \quad (2.33)$$

This and (2.3) imply that

$$s^{-4} \int_{P_s(z)} |\nabla u|^2 + s^2 |u_t|^2 \leq C_\alpha \left(\frac{s}{r}\right)^{2\gamma} r^{-4} \int_{P_r(z_0)} |\nabla u|^2, \quad \forall z \in P_{\frac{r}{2}}(z_0), \quad 0 < s \leq \frac{r}{2}. \quad (2.34)$$

Hence, by the parabolic Morrey's Lemma (cf. [13]), we conclude that  $u \in C^\gamma(P_{\frac{r}{2}}(z_0), \mathbf{S}^2)$  and (2.6) holds. This completes the proof of Lemma 2.3.  $\square$

## 2.2 Proof of Theorem 1.1.

It is similar to [7]. We argue by contradiction. Suppose it were false. Then for any small  $\epsilon > 0$ , we can find a map  $u_0 \in C^\infty(M, \mathbf{S}^2)$  that is not homotopic to a constant, and  $E(u_0) < \epsilon^2$ , and (1.2)–(1.3) has a smooth solution  $u \in C^\infty(M \times [0, i_M^2], \mathbf{S}^2)$ . Denote  $T_0 = i_M$ . By (2.1), we have

$$\int_M |\nabla u(t)|^2 \leq \epsilon^2, \quad \forall 0 \leq t \leq T_0^2. \quad (2.35)$$

Letting  $\epsilon_0$  be given by Lemma 2.3 and  $\epsilon \leq T_0\epsilon_0$ , implies

$$T_0^{-4} \int_{P_{T_0}((x, T_0^2))} |\nabla u|^2 \leq T_0^{-2} \max_{0 \leq t \leq T_0^2} \int_M |\nabla u(t)|^2 \leq T_0^{-2} \epsilon^2 \leq \epsilon_0^2, \quad \forall x \in M. \quad (2.36)$$

Therefore, by Lemma 2.3, we have

$$\text{osc}_{B_{\frac{T_0}{2}}(x)} u(T_0) \leq C_\alpha (T_0^{-4} \int_{P_{T_0}(x, T_0)} |\nabla u|^2)^{\frac{1}{2}} \leq C_\alpha T_0^{-1} \epsilon, \quad \forall x \in M. \quad (2.37)$$

Since  $M$  is compact, there exist  $N_0 = N_0(M) \geq 1$  such that

$$|u(x, T_0) - u(y, T_0)| \leq N_0 \max_{z \in M} \text{osc}_{B_{\frac{T_0}{2}}(z)} u(T_0) \leq C_\alpha N_0 T_0^{-1} \epsilon, \quad \forall x, y \in M. \quad (2.38)$$

Therefore, by choosing  $\epsilon = \epsilon(M, \alpha)$  sufficiently small, we see that  $u(T_0)(M)$  is contained in a convex, hence contractable, coordinate neighborhood in  $\mathbf{S}^2$  and  $u(T_0)$  is homotopic to a constant. But this implies  $u_0$ , through  $u(\cdot, t)$ ,  $0 \leq t \leq T_0$ , is homotopic to a constant. This contradicts with the choice of  $u_0$ . Hence the solution  $u$  to (1.2)–(1.3) must blow up before  $i_M^2$ . The proof is complete.

### 3 Estimate for Dirichlet and Neumann boundary conditions

In this section, we prove boundary a priori estimates for smooth solutions of (1.2), with either the Dirichlet or Neumann conditions, under a small energy condition.

Denoting  $\overline{M} = M \cup \partial M$ , we have

**Lemma 3.1** (Energy inequality). For  $T > 0$  and  $u_0 \in C^\infty(\overline{M}, \mathbf{S}^2)$ , let  $u \in C^\infty(\overline{M} \times [0, T], \mathbf{S}^2)$  solve (1.2)–(1.3), and satisfy either (i)  $u|_{\partial M \times [0, T]} = u_0$  or (ii)  $\frac{\partial u}{\partial \nu}|_{\partial M \times [0, T]} = 0$ . Then  $u$  satisfies the energy equality (2.1), and the local energy inequality: for any  $\phi \in C^\infty(\overline{M})$  and  $0 \leq s < t < T$ ,

$$\alpha \int_s^t \int_M |u_t|^2 \phi^2 + \int_M |\nabla u(t)|^2 \phi^2 \leq \int_M |\nabla u(s)|^2 \phi^2 + \frac{4}{\alpha} \int_s^t \int_M |\nabla u|^2 |\nabla \phi|^2. \quad (3.1) \quad \square$$

*Proof.* The same proof of Lemma 2.1 works, except that we need to show the boundary term  $\int_{\partial M} \frac{\partial u}{\partial \nu} u_t \phi^2$ , appearing in the integration by parts, vanishes for  $\phi \in C^\infty(\overline{M})$ . This is evident, for (i) the Dirichlet condition implies  $u_t = 0$  on  $\partial M$ , and (ii) the Neumann condition implies  $\frac{\partial u}{\partial \nu} = 0$  on  $\partial M$ . (2.1) follows by choosing  $\phi \equiv 1$ .

To simplify the presentation, we assume from now on that  $(M, g) = (\Omega, g_0)$  with  $\Omega \subset \mathbf{R}^n$  a bounded, smooth domain and  $g_0$  the Euclidean metric on  $\mathbf{R}^n$ .

For  $x_0 \in \partial\Omega$ , denote  $B_r^+(x_0) = B_r(x_0) \cap \Omega$  and  $P_r^+(x_0, t_0) = B_r^+(x_0) \times [t_0 - r^2, t_0]$ . Now we have ■

**Lemma 3.2.** Under the same assumptions as in Lemma 3.1, for  $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ ,  $0 < r < \sqrt{t_0}$  and  $t \in (t_0 - \frac{r^2}{4}, t_0)$ , there is  $C_\alpha > 0$  such that

$$r^{2-n} \int_{B_{\frac{r}{2}}(x_0)} |\nabla u(t)|^2 + r^{2-n} \int_{P_{\frac{r}{2}}^+(z_0)} |u_t|^2 \leq C_\alpha r^{-n} \int_{P_r^+(z_0)} |\nabla u|^2. \quad (3.2)$$

Proof. Applying (3.1), (3.2) can be proven by the same argument as in Lemma 2.2. We omit the details.

Now we are ready to state the boundary decay estimate under a small energy assumption. ■

**Lemma 3.3.** For  $1 \leq n \leq 4$ ,  $T > 0$ ,  $\delta \in (0, 1)$ , there exist  $r_0 = r_0(\Omega)$ ,  $\epsilon_3 = \epsilon_3(\delta) > 0$ ,  $C(\delta) > 0$  such that if  $u \in C^\infty(\overline{\Omega} \times [0, T], \mathbf{S}^2)$  solves (1.2), with either

- (1) there is  $p_0 \in \mathbf{S}^2$  such that  $u(x, t) = u_0(x) = p_0$  for  $(x, t) \in \partial\Omega \times [0, T]$ , or
- (2)  $\frac{\partial u}{\partial \nu}(x, t) = 0$  for  $(x, t) \in \partial\Omega \times [0, T]$ , and satisfies, for  $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$  and some  $0 < r \leq \min\{r_0, \sqrt{t_0}\}$ ,

$$r^{-n} \int_{P_r^+(z_0)} |\nabla u|^2 \leq \epsilon_3^2, \quad (3.3)$$

then

$$\left(\frac{r}{8}\right)^{-n} \int_{P_{\frac{r}{8}}^+(z_0)} |\nabla u|^2 \leq \delta r^{-n} \int_{P_r^+(z_0)} |\nabla u|^2 + \frac{C(\delta)}{\delta} r^{-(n+2)} \int_{P_r^+(z_0)} |u - c_r|^2, \quad (3.4)$$

where either (1)  $c_r = p_0$  or (2)  $c_r = \frac{1}{|P_r^+(z_0)|} \int_{P_r^+(z_0)} u$ . □

Proof. We adapt the proof of Lemma 2.3. First, by the same dimension reduction as in Lemma 2.3, we assume  $n = 4$ . Next, observe that we can reduce the case (b) into the interior case. In fact, for any smooth metric  $g$  on  $\Omega$ , there is a sufficiently small  $\rho = \rho(\Omega, g) > 0$ , such that the nearest point projection map  $\Pi_{\partial\Omega} : \Omega_\rho \rightarrow \partial\Omega$  is smooth, where  $\Omega_\rho = \{x \in \Omega : d_g(x, \partial\Omega) < \rho\}$ . Define the reflection map  $R_{\partial\Omega}$  along  $\partial\Omega$  by  $R_{\partial\Omega}(x) = 2\Pi_{\partial\Omega}(x) - x$  for  $x \in \Omega_\rho$ ,  $\Omega_\rho^* = R_{\partial\Omega}(\Omega_\rho)$ , and  $\Omega^* = \overline{\Omega} \cup \Omega_\rho^*$ . It is easy to see that for a sufficiently small  $\rho > 0$ , the nearest point projection, still denoted by  $\Pi_{\partial\Omega}$ , from  $\Omega_\rho^*$  to  $\partial\Omega$  is well-defined and smooth. Hence the inverse map  $R_{\partial\Omega}^- (y) = 2\Pi_{\partial\Omega}(y) - y : \Omega_\rho^* \rightarrow \Omega_\rho$  can be identified with  $R_{\partial\Omega}$ . Hence we can extend  $u_0$ ,  $u$  and the metric  $g$  from  $\Omega$  to  $\Omega^*$  by letting  $u_0(x) = u_0(R_{\partial\Omega}(x))$ ,  $u^*(x, t) = u(R_{\partial\Omega}(x), t)$ , and  $g^*(x) = (R_{\partial\Omega}^* g)(x)$ , the pullback of  $g$  by  $R_{\partial\Omega}^- = R_{\partial\Omega}$ , for  $x \in \Omega_\rho^*$  and  $t \in [0, T]$ . Then one can check that if  $u \in C^1(\Omega \times [0, T], \mathbf{S}^2)$  solves (1.2), (1.3) and (1.8) for  $(\Omega, g)$ , then  $u^*$  is in  $C^1(\Omega^* \times [0, T], \mathbf{S}^2)$

and solves (1.2) for  $(\Omega^*, g^*)$ , and (1.3) with  $u_0$  replaced by  $u_0^*$ . Hence Lemma 3.3 for the case (b) follows from Lemma 2.3. We will only sketch the proof of Lemma 3.3 for the case (a). It follows from the smoothness of  $\Omega$  and the standard boundary flatten argument that we may assume for simplicity that there exists  $r_0 = r_0(\Omega) > 0$  such that  $B_{r_0}^+(x_0) := \Omega \cap B_{r_0}(x_0) = \{x = (x', x_4) \in \mathbf{R}^4 : |x - x_0| \leq r_0, x_4 \geq 0\}$  (the general case can be handled by slight modifications of the argument given below; for example, see [5]).

By translation and rescaling, we assume  $r_0 \geq 1$ ,  $r = 1$ ,  $z_0 = (0, 0)$ ,  $u \in C^\infty(\overline{\mathbf{R}}_+^4 \times (-1, 0], \mathbf{S}^2)$  solves (1.2) and  $u|_{\partial\mathbf{R}_+^4 \times (-1, 0]} = p_0$ . As in Lemma 2.3, we divide the proof into two steps. First, we need

*Step 1* (slice boundary monotonicity inequality). For any  $t \in (-1, 0]$ ,  $x_1 \in \partial\mathbf{R}_+^4$ ,  $0 < r_1 \leq r_2 < +\infty$ , it holds

$$r_1^{-2} \int_{B_{r_1}^+(x_1)} |\nabla u|^2 \leq 2r_2^{-2} \int_{B_{r_2}^+(x_1)} |\nabla u|^2 + 2 \int_{B_{r_2}^+(x_1)} |u_t|^2. \quad (3.5)$$

To prove (3.5), assume  $x_1 = 0$ , write  $B_r^+$  for  $B_r^+(0)$  and define  $S_r^+ = \partial B_r^+ \cap \{x \in \mathbf{R}^4 \mid x_4 > 0\}$ ,  $T_r = \partial B_r^+ \cap \{x \in \mathbf{R}^4 \mid x_4 = 0\}$ . By the Pohozaev argument, we multiply (1.2) by  $x \cdot \nabla u$  and integrate over  $B_r^+$  to get

$$\begin{aligned} \int_{B_r^+} R(u)(u_t)x \cdot \nabla u &= \int_{B_r^+} \Delta u x \cdot \nabla u \\ &= \int_{B_r^+} |\nabla u|^2 + \int_{\partial B_r^+} x \cdot \nabla u \frac{\partial u}{\partial \nu} - \frac{1}{2} \int_{\partial B_r^+} |\nabla u|^2 x \cdot \nu \\ &= \int_{B_r^+} |\nabla u|^2 + \int_{S_r^+} x \cdot \nabla u \frac{\partial u}{\partial r} - \frac{1}{2} \int_{S_r^+} |\nabla u|^2 x \cdot \frac{x}{|x|} \\ &\quad + \int_{T_r} x \cdot \nabla u \frac{\partial u}{\partial x_4} - \frac{1}{2} \int_{T_r} |\nabla u|^2 x_4. \end{aligned}$$

Since  $x_4 = 0$  on  $T_r$ , we have  $\int_{T_r} |\nabla u|^2 x_4 = 0$ .  $\int_{T_r} x \cdot \nabla u \frac{\partial u}{\partial x_4}$  also equals to zero, since  $u(t) = p_0$  on  $T_r$  implies  $x \cdot \nabla u = \sum_{i=1}^3 x_i \frac{\partial u}{\partial x_i} = 0$  on  $T_r$ . Therefore we have

$$\int_{B_r^+} R(u)(u_t)x \cdot \nabla u = \int_{B_r^+} |\nabla u|^2 + r \int_{S_r^+} \left| \frac{\partial u}{\partial r} \right|^2 - \frac{r}{2} \int_{S_r^+} |\nabla u|^2. \quad (3.6)$$

This implies

$$\frac{d}{dr} \left( r^{-2} \int_{B_r^+} \frac{|\nabla u|^2}{2} - r^{-1} \int_{B_r^+} |u_t| \left| \frac{\partial u}{\partial r} \right| \right) \geq r^{-2} \int_{S_r^+} \left| \frac{\partial u}{\partial r} \right|^2 - r^{-1} \int_{S_r^+} |u_t| \left| \frac{\partial u}{\partial r} \right|. \quad (3.7)$$



(3.5) follows by integrating (3.7) from  $r_1$  to  $r_2$ , and

$$\begin{aligned} 2r_1^{-1} \int_{B_{r_1}^+} |u_t| \left| \frac{\partial u}{\partial r} \right| &\leq \frac{1}{2} r_1^{-2} \int_{B_{r_1}^+} |\nabla u|^2 + 2 \int_{B_{r_2}^+} |u_t|^2, \\ 2 \int_{r_1}^{r_2} r^{-1} \int_{S_r^+} |u_t| \left| \frac{\partial u}{\partial r} \right| &\leq \int_{B_{r_2}^+ \setminus B_{r_1}^+} r^{-2} \left| \frac{\partial u}{\partial r} \right|^2 + \int_{B_{r_2}^+} |u_t|^2. \end{aligned}$$

*Step 2* (estimate on good slices). For any  $\Lambda \geq 1$ , denote  $P_r^+(0, 0)$  by  $P_r^+$  and define

$$G_+^\Lambda = \left\{ t \in \left[ -\frac{1}{4}, 0 \right] \mid \int_{B_{\frac{1}{2}}^+} |u_t(t)|^2 \leq \Lambda^2 \int_{P_{\frac{1}{2}}^+} |u_t|^2 \right\}, \quad B_+^\Lambda = \left[ -\frac{1}{4}, 0 \right] \setminus G_+^\Lambda. \quad (3.8)$$

Then we have

$$|B_+^\Lambda| \leq \Lambda^{-2}. \quad (3.9)$$

For any  $t \in G_+^\Lambda$ , by (3.2) and (3.8), we have

$$\int_{B_{\frac{1}{2}}^+} (|\nabla u(t)|^2 + |u_t(t)|^2) \leq C\Lambda^2 \int_{P_{\frac{1}{2}}^+} |\nabla u|^2. \quad (3.10)$$

This, combined with (3.5) and (2.9), implies

$$\begin{aligned} [u(t)]_{\text{BMO}(B_{\frac{1}{2}}^+)}^2 &\leq \sup \left\{ s^{-2} \int_{B_s(x) \cap B_{\frac{1}{2}}^+} |\nabla u|^2 \mid x \in B_{\frac{1}{4}}^+, 0 < s \leq \frac{1}{4} \right\} \\ &\leq C \int_{B_{\frac{1}{2}}^+} (|\nabla u(t)|^2 + |u_t(t)|^2) \leq C\Lambda^2 \int_{P_{\frac{1}{2}}^+} |\nabla u|^2. \end{aligned} \quad (3.11)$$

Now let  $\eta \in C_0^\infty(B_{\frac{1}{4}})$  be even with respect to  $x_4$ ,  $0 \leq \eta \leq 1$ ,  $\eta = 1$  on  $B_{\frac{1}{8}}$ , and  $|\nabla \eta| \leq 16$ . Let  $v : \mathbf{R}^4 \times (-1, 0] \rightarrow \mathbf{S}^2$  be the extension of  $u$  that is even w.r.t.  $x_4$ . Define  $w : \mathbf{R}^4 \times (-1, 0] \rightarrow \mathbf{R}^3$  by

$$\begin{aligned} w(x, t) &= (u - p_0)((x_1, x_2, x_3, x_4), t), \quad x = (x_1, x_2, x_3, x_4) \in \mathbf{R}_+^4, t \in (-1, 0], \\ &= -(u - p_0)((x_1, x_2, x_3, -x_4), t), \quad x = (x_1, x_2, x_3, x_4) \in \mathbf{R}_-^4, t \in (-1, 0]. \end{aligned} \quad (3.12)$$

Then we have

$$\int_{B_{\frac{1}{8}}^+} |\nabla u(t)|^2 \leq \int_{\mathbf{R}_+^4} \eta^2 |\nabla w(t) \wedge v(t)|^2 = \frac{1}{2} \int_{\mathbf{R}^4} \eta^2 |\nabla w(t) \wedge v(t)|^2. \quad (3.13)$$

Since  $\nabla \cdot (\eta^2 \nabla w(t) \wedge v(t))$  is odd w.r.t.  $x_4$ , we have

$$\begin{aligned}
\int_{\mathbf{R}^4} |\nabla \cdot (\eta^2 \nabla w(t) \wedge v(t))|^2 &= 2 \int_{\mathbf{R}_+^4} |\nabla \cdot (\eta^2 \nabla u(t) \wedge u(t))|^2 \\
&\leq 4 \int_{\mathbf{R}_+^4} (|\nabla \eta|^2 |\nabla u(t)|^2 + |\eta|^2 |u_t(t)|^2) \\
&\leq C \int_{B_{\frac{1}{2}}^+} (|\nabla u(t)|^2 + |u_t(t)|^2) \\
&\leq C \Lambda^2 \int_{P_1^+} |\nabla u|^2.
\end{aligned} \tag{3.14}$$

By (3.11), we also have

$$[w(t)]_{\mathbf{BMO}(B_{\frac{1}{4}}^+)}^2 \leq C [u(t)]_{\mathbf{BMO}(B_{\frac{1}{4}}^+)} \leq C \Lambda^2 \int_{P_1^+} |\nabla u|^2. \tag{3.15}$$

By integration by parts, we have

$$\begin{aligned}
\int_{\mathbf{R}^4} \eta^2 |\nabla w(t) \wedge v(t)|^2 &= - \int_{\mathbf{R}^4} \nabla \cdot (\eta^2 \nabla w(t) \wedge v(t)) \cdot (w(t) \wedge v(t)) \\
&\quad + \int_{\mathbf{R}^4} \eta^2 [(\nabla w(t) \wedge v(t)) \wedge \nabla v(t) - \lambda] \cdot w(t) \\
&\quad + \lambda \int_{\mathbf{R}^4} \eta^2 w(t) = I + II + III
\end{aligned}$$

where

$$\lambda = \frac{\int_{\mathbf{R}^4} \eta^2 (\nabla w(t) \wedge v(t)) \wedge \nabla v(t)}{\int_{\mathbf{R}^4} \eta^2}.$$

It is easy to see

$$|\lambda| \leq C \int_{B_{\frac{1}{2}}^+} |\nabla u(t)|^2.$$

Hence, by the Poincaré inequality and (3.2), we have

$$\begin{aligned}
|III| &\leq |\lambda| \|w(t)\|_{L^2(B_{\frac{1}{2}}^+)} \leq C \int_{B_{\frac{1}{2}}^+} |\nabla u(t)|^2 \|\nabla u(t)\|_{L^2(B_{\frac{1}{2}}^+)} \\
&\leq C \int_{B_{\frac{1}{2}}^+} |\nabla u(t)|^2 \left( \int_{P_1^+} |\nabla u|^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

For  $I$ , similar to (2.20), we have

$$\begin{aligned}
 |I| &\leq \|\nabla \cdot (\eta^2 \nabla w(t) \wedge v(t))\|_{L^2(\mathbb{R}^4)} \|w(t)\|_{L^2(B_1)} \\
 &\leq C \left( \int_{B_{\frac{1}{2}}^+} (|\nabla u(t)|^2 + |u_t(t)|^2) \right)^{\frac{1}{2}} \|u(t) - p_0\|_{L^2(B_1^+)} \\
 &\leq \delta \int_{B_{\frac{1}{2}}^+} (|\nabla u(t)|^2 + |u_t(t)|^2) + \frac{C}{\delta} \int_{B_1^+} |u(t) - p_0|^2.
 \end{aligned} \tag{3.17}$$

It follows from (3.14) and (3.15) that  $II$  can be estimated exactly as same as in (2.25). Namely, we have

$$|II| \leq C\Lambda \left( \int_{P_1^+} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{B_{\frac{1}{2}}^+} (|\nabla u(t)|^2 + |u_t(t)|^2) \right). \tag{3.18}$$

Putting all these estimates together, we obtain, for  $t \in G_+^\wedge$ ,

$$\int_{B_{\frac{1}{8}}^+} |\nabla u(t)|^2 \leq \left[ \delta + C\Lambda \left( \int_{P_1^+} |\nabla u|^2 \right)^{\frac{1}{2}} \right] \int_{B_{\frac{1}{2}}^+} (|\nabla u(t)|^2 + |u_t(t)|^2) + \frac{C}{\delta} \int_{B_{\frac{1}{2}}^+} |u - p_0|^2. \tag{3.19}$$

Finally, by integrating (3.19) over  $t \in [-(\frac{1}{8})^2, 0] \cap G_+^\wedge$ , applying (3.2), (3.8), and the following inequality:

$$\left( \frac{1}{8} \right)^{-4} \int_{B_{\frac{1}{8}}^+ \times [-(\frac{1}{8})^2, 0] \cap G_+^\wedge} |\nabla u|^2 \leq \frac{1}{\Lambda^2} \int_{P_1^+} |\nabla u|^2, \tag{3.20}$$

we obtain

$$\begin{aligned}
 \left( \frac{1}{8} \right)^{-4} \int_{P_{\frac{1}{8}}^+} |\nabla u|^2 &\leq \frac{C}{\delta} \int_{P_1^+} |u - p_0|^2 \\
 &\quad + [\delta + C\Lambda \left( \int_{P_1^+} |\nabla u|^2 \right)^{\frac{1}{2}} + \Lambda^{-2}] \int_{P_1^+} |\nabla u|^2.
 \end{aligned} \tag{3.21}$$

It is clear that, by choosing  $\Lambda^2 = \delta^{-1}$  and  $\epsilon_3^2(\delta) \leq \frac{\delta^3}{C^2}$ , (3.21) implies (3.4).

Now we need the boundary version of Lemma 2.4. For simplicity, we only consider the case  $n = 4$  and  $\Omega = R_+^4$ . ■

**Lemma 3.4.** There exists  $C_0 > 0$  such that for any  $\theta \in (0, \frac{1}{4})$  there is  $\epsilon_4(\theta) > 0$  such that if  $u \in C^\infty(\mathbf{R}_+^4 \times (-1, 0], \mathbf{S}^2)$  solves (1.2), with either (a) there exists  $p_0 \in \mathbf{S}^2$  such that  $u(x, t) = p_0$  for  $(x, t) \in \partial\mathbf{R}_+^4 \times (-1, 0]$  or (b)  $\frac{\partial u}{\partial \nu}(x, t) = 0$  for  $(x, t) \in \partial\mathbf{R}_+^4 \times (-1, 0]$ , and satisfies, for  $z_0 = (x_0, t_0) \in \partial\mathbf{R}_+^4 \times (-1, 0]$  and  $0 < r < \sqrt{t_0}$ ,

$$r^{-4} \int_{P_r^+(z_0)} |\nabla u|^2 \leq \epsilon_4^2(\theta), \quad (3.22)$$

then

$$(\theta r)^{-6} \int_{P_{\theta r}^+(z_0)} |u - c_r|^2 \leq C_0 \theta^2 r^{-4} \int_{P_r^+(z_0)} |\nabla u|^2, \quad (3.23)$$

where

$$\begin{aligned} c_r &= p_0, & \text{for case (a)} \\ &= \frac{1}{|P_r^+(z_0)|} \int_{P_r^+(z_0)} u, & \text{for case (b)}. \end{aligned} \quad \square$$

*Proof.* Without loss of generality, we assume  $r = 1$  and  $z_0 = (0, 0)$ . The case (b) is an easy consequence of Lemma 2.4. In fact, let  $v : \mathbf{R}^4 \times (-1, 0] \rightarrow \mathbf{S}^2$  be the extension of  $u$  that is even w.r.t.  $x_4$ . Then, as in the proof of Lemma 3.3, we have that  $v \in C^\infty(\mathbf{R}^4 \times (-1, 0], \mathbf{S}^2)$  solves (1.2), and satisfies

$$\int_{P_1} |\nabla v|^2 = 2 \int_{P_1^+} |\nabla u|^2 \leq 2\epsilon_4^2(\theta).$$

Hence, by Lemma 2.4, there exists  $\epsilon_4(\theta) > 0$  such that

$$\theta^{-6} \int_{P_\theta} |v - v_{P_\theta}|^2 \leq C_0 \theta^2 \int_{P_1} |\nabla v|^2.$$

This easily implies (3.23).

To prove (3.23) for case (a), we argue by contradiction (cf. also [17]). Suppose it were false. Then there exist  $\theta_0 \in (0, \frac{1}{4})$  and  $\{u_k\} \in C^\infty(\overline{\mathbf{R}_+^4} \times (-1, 0], \mathbf{S}^2)$  solving (1.2) with  $u_k = p_k$  on  $\partial\mathbf{R}_+^4 \times (-1, 0]$ , such that

$$\int_{P_1^+} |\nabla u_k|^2 = \epsilon_k^2 \rightarrow 0,$$

but

$$\theta_0^{-6} \int_{P_{\theta_0}^+} |u_k - p_k|^2 \geq k \theta_0^2 \epsilon_k^2. \quad (3.24)$$

Define  $v_k = \frac{u_k - p_k}{\epsilon_k} : \mathbf{R}_+^4 \times (-1, 0] \rightarrow \mathbf{R}^3$ . Then we have

$$\int_{P_1^+} |\nabla v_k|^2 = 1, \quad v_k|_{\partial \mathbf{R}_+^4 \times (-1, 0]} = 0, \quad (3.25)$$

$$R(u_k) \left( \frac{\partial v_k}{\partial t} \right) - \Delta v_k = \epsilon_k |\nabla v_k|^2 u_k, \quad \text{in } \mathbf{R}_+^4 \times (-1, 0], \quad (3.26)$$

and

$$\theta_0^{-6} \int_{P_{\theta_0}^+} |v_k|^2 \geq k \theta_0^2. \quad (3.27)$$

It follows from (3.25), (3.2), and Ponicaré inequality that  $\{v_k\} \subset H^1(P_{\frac{1}{2}}^+, \mathbf{R}^3)$  is bounded. Hence we may assume that  $v_k \rightarrow v$  weakly in  $H^1(P_{\frac{1}{2}}^+)$ , strongly in  $L^2(P_{\frac{1}{2}}^+)$ , and  $p_k \rightarrow p_0 \in \mathbf{S}^2$  and  $u_k \rightarrow p_0$  a.e.. Then we have

$$\int_{P_{\frac{1}{2}}^+} |\nabla v|^2 \leq 1, \quad v|_{\partial \mathbf{R}_+^4 \times (-1, 0]} = 0, \quad (3.28)$$

$$R(p_0)(v_t) - \Delta v = 0, \quad \text{in } \mathbf{R}_+^4 \times (-1, 0]. \quad (3.29)$$

By the standard theory of linear parabolic equations, there exists  $C > 0$  such that

$$\theta_0^{-6} \int_{P_{\theta_0}^+} |v|^2 \leq C \theta_0^2. \quad (3.30)$$

This contradicts (3.27). Hence (3.23) holds.

With Lemmas 3.3, 3.4, and Lemma 2.3, we are ready to prove the boundary apriori estimate for smooth solutions of (1.2), with either constant Dirichlet or zero Neumann boundary conditions, under a small energy assumption.  $\blacksquare$

**Lemma 3.5.** For  $1 \leq n \leq 4$ ,  $\Omega \subset \mathbf{R}^n$  bounded smooth domain,  $0 < T < +\infty$ , there exists  $r_0 = r_0(\Omega) > 0$  such that for any  $\gamma \in (0, 1)$  there are  $\epsilon_5 > 0$ ,  $C_\alpha > 0$  depending only on  $\gamma, \alpha$  such that if  $u \in C^\infty(\overline{\Omega} \times (0, T), \mathbf{S}^2)$  solves (1.2), with either (a)  $u(x, t) = p_0$  for  $(x, t) \in \partial\Omega \times (0, T)$  or (b)  $\frac{\partial u}{\partial \nu}(x, t) = 0$  for  $(x, t) \in \partial\Omega \times (0, T)$ , and satisfies, for  $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$  and  $0 < r < \min\{r_0, \sqrt{t_0}\}$ ,

$$r^{-n} \int_{P_r^+(z_0)} |\nabla u|^2 \leq \epsilon_5^2, \quad (3.31)$$

then  $u \in C^\gamma(P_{\frac{r}{2}}^+(z_0), \mathbf{S}^2)$  and

$$[u]_{C^\gamma(P_{\frac{r}{2}}^+(z_0))}^2 \leq C_\alpha r^{-(n+2\gamma)} \int_{P_r^+(z_0)} |\nabla u|^2. \quad (3.32)$$

$\square$

Proof. For simplicity, assume  $x_0 = 0 \in \partial\Omega$ ,  $t_0 = 1$ ,  $T = 2$ . Let  $r_0 = r_0(\Omega) > 0$  be so small that we may assume  $\overline{\Omega} \cap B_{r_0}(x_0) = \overline{\mathbf{R}^n} \cap B_{r_0}(x_0) = \overline{B_{r_0}^+}$ . We follow the same iteration scheme as in Lemma 2.3. Let  $\delta = \frac{1}{8^3}$  and  $\theta = \theta(\gamma) \leq (\frac{\delta}{2C_0C(\delta)})^{\frac{1}{2-2\gamma}}$ , here  $C_0$  and  $C(\delta)$  are the constants in Lemma 3.4 and 3.3 respectively. Let  $k \geq 1$  be such that  $8^k\theta = 1$  and denote  $E^+(u, \rho) = \rho^{-n} \int_{P_\rho^+(0,1)} |\nabla u|^2$  for  $0 < \rho \leq r_0$ . For  $0 \leq i \leq k-1$ , if

$$E^+(u, 8^{i+1}\theta r) \leq \epsilon_3(\delta)^2, \quad E^+(u, r) \leq \epsilon_4(8^{i+1}\theta)^2,$$

where  $\epsilon_3, \epsilon_4$  are given by Lemma 3.3 and 3.4, then we have

$$E^+(u, 8^i\theta r) \leq \delta E^+(u, 8^{i+1}\theta r) + \frac{C_0C(\delta)}{\delta} E^+(u, r). \quad (3.33)$$

Hence, if we choose

$$\epsilon_5^2 := E^+(u, r) \leq \min\{\epsilon_4(8\theta)^2, \dots, \epsilon_4(8^k\theta)^2, \frac{\delta}{2C_0C(\delta)}\epsilon_3(\delta)^2\},$$

then we have  $E^+(u, 8^i\theta r) \leq \epsilon_3(\delta)^2$  for all  $0 \leq i \leq k$ . Hence, by iteration, Lemma 3.3 and Lemma 3.4 yield

$$E^+(u, \theta r) \leq \delta^k E^+(u, r) + \frac{C_0C(\delta)}{1-64\delta} \left(\frac{\theta}{\delta}\right)^2 E^+(u, r) \leq \theta^{2\gamma} E^+(u, r). \quad (3.34)$$

This, combined with Lemma 2.3 and (3.2), implies

$$s^{-n} \int_{P_s(z) \cap (\Omega \times (0,2))} |\nabla u|^2 + s^2 |u_t|^2 \leq C_\alpha \left(\frac{s}{r}\right)^{2\gamma} r^{-n} \int_{P_r(z) \cap (\Omega \times (0,2))} |\nabla u|^2, \quad (3.35)$$

for any  $z \in P_{\frac{r}{2}}^+(z_0)$  and  $0 < s \leq \frac{r}{2}$ . Therefore, by the parabolic Morrey's Lemma, we conclude that  $u \in C^\gamma(P_{\frac{r}{2}}^+(z_0), \mathbf{S}^2)$  with the desired estimate (3.32).  $\blacksquare$

## 4 Proof of Theorem 1.3 and Theorem 1.5

In this section, we apply Lemma 2.3 and 3.4 to prove both Theorem 1.3 and 1.5.

### 4.1 Proof of Theorem 1.3.

It is similar to the proof of Theorem 1.1. Suppose it were false. Then for any  $\epsilon > 0$  we can find  $u_0 \in C^\infty(\overline{M}, \mathbf{S}^2)$  with  $u_0|_{\partial M} = p_0 \in \mathbf{S}^2$ ,  $E(u_0) \leq \epsilon^2$ , and  $u_0$  homotopically nontrivial

rel.  $\partial M$ , such that (1.2)–(1.3)–(1.6) has a smooth solution  $u \in C^\infty(\overline{M} \times [0, i_M^2], \mathbf{S}^2)$ . Denote  $T_0 = i_M^2$ . By (2.1), we have

$$\int_M |\nabla u(t)|^2 \leq \int_M |\nabla u_0|^2 \leq \epsilon^2, \quad \forall t \in (0, T_0). \quad (4.1)$$

Let  $\epsilon_6 = \min\{\epsilon_0, \epsilon_5\}$ , where  $\epsilon_0$  and  $\epsilon_5$  are given by Lemma 2.3 and Lemma 3.5 respectively. Choosing  $\epsilon \leq T_0 \epsilon_6$ , we have

$$T_0^{-4} \int_{(B_{T_0}(x) \cap M) \times [0, T_0^2]} |\nabla u|^2 \leq T_0^{-2} \max_{0 \leq t \leq T_0^2} \int_M |\nabla u(t)|^2 \leq T_0^{-2} \epsilon^2 \leq \epsilon_6^2, \quad \forall x \in \overline{M}. \quad (4.2)$$

Therefore, by Lemma 2.3 and 3.5, we have

$$\text{osc}_{B_{\frac{T_0}{2}}(x)} u(T_0) \leq C_\alpha (T_0^{-4} \int_{(B_{T_0}(x) \cap M) \times [0, T_0^2]} |\nabla u|^2)^{\frac{1}{2}} \leq C_\alpha T_0^{-1} \epsilon, \quad \forall x \in \overline{M}. \quad (4.3)$$

Since  $M$  is compact, there exists  $N_0 = N_0(M) \geq 1$  such that

$$\text{osc}_{\overline{M}} u(T_0) \leq N_0 \max_{x \in \overline{M}} \text{osc}_{B_{\frac{T_0}{2}}(x)} u(T_0) \leq N_0 C_\alpha T_0^{-1} \epsilon. \quad (4.4)$$

By choosing  $\epsilon$  sufficiently small, this implies that  $u(T_0)(\overline{M})$  is contained in a contractible, coordinate neighborhood in  $\mathbf{S}^2$  and hence  $u(T_0)$  is homotopic to  $p_0$  (rel.  $\partial M$ ). In particular,  $u_0$  is homotopic to  $p_0$  (rel.  $\partial M$ ). This contradicts with the choice of  $u_0$ . Therefore, any smooth solution of (1.2)–(1.3)–(1.6) has to blow up before  $T_0$ .

#### 4.2 Proof of Theorem 1.5.

Suppose it were false. Then there exists  $T_0 > 0$  such that for  $\lambda_k \rightarrow \infty$ , there are smooth solutions  $u_k \in C^\infty(\{x \in \mathbf{R}^4 : 1 \leq |x| \leq 2\} \times [0, T_0], \mathbf{S}^2)$  to (1.8),  $u_k(x, 0) = (H \circ \Phi_{\lambda_k})(\frac{x}{|x|})$  for  $x \in \mathbf{R}^4$  with  $1 \leq |x| \leq 2$ , and  $\frac{\partial u_k}{\partial \nu}(x, t) = 0$  for  $x \in \partial\{x \in \mathbf{R}^4 : 1 \leq |x| \leq 2\}$  and  $t \in [0, T_0]$ . Since

$$\lim_{\lambda_k \rightarrow \infty} \int_{\{x \in \mathbf{R}^4 : 1 \leq |x| \leq 2\}} |\nabla u_k(0)|^2 = 0,$$

we have, by (2.1), Lemma 2.3, Lemma 3.5, and the same argument as in the proof of Theorem 1.3, that for  $\lambda_k$  sufficiently large,  $u_k(T_0)(\{x \in \mathbf{R}^4 : 1 \leq |x| \leq 2\})$  is contained in a contractible, coordinate neighborhood in  $\mathbf{S}^2$ . Let  $F(x, t) = u_k(x, t) : \{x \in \mathbf{R}^4 : |x| = 1.5\} \times$

$[0, T_0] \rightarrow \mathbf{S}^2$ . Then  $F$  deforms the Hopf map  $H(x) = u_k(x, 0)$  into a contractable coordinate neighborhood of  $\mathbf{S}^2$ , which yields that the Hopf map  $H : \mathbf{S}^3 \rightarrow \mathbf{S}^2$  is homotopically trivial. We get the desired contradiction. This completes the proof.

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