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Asymptotic Behavior of Infinity Harmonic Functions Near an Isolated Singularity

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In this paper, we prove that if $n \geq 2$ and x_0 is an isolated singularity of a non-negative infinity harmonic function u, then either x_0 is a removable singularity of u or $u(x) = u(x_0) + c|x - x_0| + o(|x - x_0|)$ near x_0 for some fixed constant $c \neq 0$. In particular, if x_0 is nonremovable, then u has a local maximum or a local minimum at x_0. We also prove a Bernstein-type theorem, which asserts that if u is a uniformly Lipschitz continuous, one-side bounded infinity harmonic function in $\mathbb{R}^n \setminus \{0\}$, then it must be a cone function with center at 0.

1 Introduction

Throughout this paper, we assume that $n \geq 2$. Let Ω be an open subset of $\mathbb{R}^n$. Recall that a function $u \in C(\Omega)$ is an *infinity harmonic function* in Ω if it is a viscosity solution of the infinity Laplace equation

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 \quad \text{in } \Omega. \tag{1.1}$$

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See Appendix A for the definition of viscosity supersolution, subsolution, and solution of elliptic equations. It is well known that an infinity harmonic function in Ω is a local minimizer of the supremum norm of the gradient, in the sense that for any open set $V \subset \Omega$, if $v \in W^{1,\infty}(V)$ satisfies

$$u|_{\partial V} = v|_{\partial V},$$

then

$$\mathrm{esssup_V}|Du| \leq \mathrm{esssup_V}|Dv|.$$

An infinity harmonic function is also an *absolutely minimal Lipschitz extension* in Ω, i.e. for any open set $V \subset \overline{V} \subset \Omega$,

$$\sup_{x \neq y \in \partial V} \frac{|u(x) - u(y)|}{|x - y|} = \sup_{x \neq y \in \overline{V}} \frac{|u(x) - u(y)|}{|x - y|}. \tag{1.2}$$

The infinity Laplace equation can be viewed as the limiting equation of p-Laplace equations as $p \uparrow +\infty$. More precisely, for $p \geq 1$, let u_p be a p-harmonic function in Ω, i.e. u_p solves the p-Laplace equation

$$\Delta_p u_p = \mathrm{div}(|Du_p|^{p-2} Du_p) = 0 \quad \text{in } \Omega. \tag{1.3}$$

If $u_p \to u_\infty$ uniformly in Ω as $p \uparrow +\infty$, then u_∞ is an infinity harmonic function in Ω. We refer to a recent survey article by Crandall [3] for more backgrounds and information of equation (1.1). For $1 \leq p \leq +\infty$, if u_p is a p-harmonic function in $\Omega \backslash \{x_0\}$, then x_0 is called an *isolated singularity* of u_p. x_0 is called a *removable singularity,* if u_p can be extended to be a p-harmonic function in Ω. Otherwise x_0 is called a *nonremovable singularity*. When $1 < p \leq n$, the classical theorem of Serrin [10] says that a non-negative p-harmonic function u_p is comparable to the fundamental solution of p-Laplace equation near its nonremovable isolated singularity. When $n = 2$ and $2 < p < +\infty$, Manfredi [8] derived an asymptotic representation of u_p near its nonremovable isolated singularity. In this paper, we will show that a non-negative infinity harmonic function is asymptotically a cone function near its nonremovable isolated singularity. In particular, an infinity harmonic function has a local maximum or minimum value at a nonremovable isolated singularity. This is surprising and is largel related to the highly degenerate ellipticity of the infinity Laplace equation. Note that cone functions are fundamental solutions of the infinity Laplace equation. The following is our first main theorem.

Theorem 1.1. Suppose that $n \geq 2$ and $u \in C(B_1(x_0)\backslash\{x_0\})$ is a non-negative infinity harmonic function in $B_1(x_0)\backslash\{x_0\}$. Then, $u \in W^{1,\infty}_{\text{loc}}(B_1(x_0))$ and one of the following holds: either

(i) x_0 is a removable singularity;

or

(ii) there exists a fixed constant $c \neq 0$, such that

$$u(x) = u(x_0) + c|x - x_0| + o(|x - x_0|),$$

i.e.

$$\lim_{x \to x_0} \frac{|u(x) - u(x_0) - c|x - x_0||}{|x - x_0|} = 0.$$

In particular, in case (ii), u has either a local maximum or a local minimum at x_0, and

$$|c| = \text{esssup}_V |Du|,$$

where V is some neighborhood of x_0. □

We want to note here that the above theorem is not correct when $n = 1$. For example, for any $t \in (0, 1]$, $u_t = t(-|x|) + (1 - t)x$ is an infinity harmonic function on $(-1, 1)\backslash\{0\}$ and 0 is an isolated singularity. When $t \neq 1$, Theorem 1.1 (ii) is not satisfied.

Also, the assumption that u is one-sided bounded near its isolated singularity is necessary for the above theorem to hold. Otherwise u may oscillate between $-\infty$ and $+\infty$. See such an example in Bhattacharya [2].

As an application of Theorem 1.1, we can construct a family of nonclassical infinity harmonic functions in $\mathbb{R}^2$.

Corollary 1.2. Suppose that Ω is a bounded domain in $\mathbb{R}^2$ and $x_0 \in \Omega$. Assume that $u \in C(\overline{\Omega})$ is an infinity harmonic function in $\Omega\backslash\{x_0\}$ and satisfies $u|_{\partial\Omega} = 0$ and $u(x_0) = 1$. Then $u \in C^2(\Omega\backslash\{x_0\})$, if and only if $\Omega = B_r(x_0)$ for some $r > 0$, and $u(x) = 1 - \frac{|x-x_0|}{r}$ for $x \in B_r(x_0)$. □

We also prove a Bernstein-type theorem on uniformly Lipschitz continuous infinity harmonic functions in $\mathbb{R}^n\backslash\{0\}$.

Theorem 1.3. If u satisfies the following:

(i) $\operatorname{esssup}_{\mathbb{R}^n}|Du| = 1$;
(ii) for some $M \in \mathbb{R}$ and $\epsilon > 0$, $u(x) \leq M + (1-\epsilon)|x|$ for all $x \in \mathbb{R}^n$;
(iii) u is an infinity harmonic function in $\mathbb{R}^n \backslash \{0\}$.

Then

$$u(x) = u(0) - |x|.$$

□

The first author has proved in [9] that if u is a uniformly Lipschitz continuous infinity harmonic function in $\mathbb{R}^2$, then u must be linear, i.e. $u = p \cdot x + c$ for some $p \in \mathbb{R}^2$ and $c \in \mathbb{R}$. In general, a uniformly Lipschitz continuous infinity harmonic function in $\mathbb{R}^n\backslash\{0\}$ might be neither linear nor a cone. The following is a family of such functions. For $R > 0$ and $0 < \alpha < 1$, let $u_{R,\alpha}$ be the solution of the following equation:

$$\Delta_\infty u_{R,\alpha} = 0 \quad \text{on} B_R(0)\backslash\{0\},$$
$$u_{R,\alpha}(0) = 0 \ \text{ and } u_{R,\alpha}|_{\partial B_R(0)} = \alpha x_n - (1-\alpha)R.$$

It is clear that

$$u_{R,0}(x) = -|x|$$

and

$$u_{R,1}(x) = x_n.$$

Hence for each R, there exists $0 < \alpha(R) < 1$, such that

$$u_{R,\alpha(R)}(0, \ldots, 0, 1) = 0.$$

Suppose now that $u = \lim_{R\to+\infty} u_{R,\alpha(R)}$. Then u is an infinity harmonic function in $\mathbb{R}^n\backslash\{0\}$ and $\operatorname{esssup}_{\mathbb{R}^n}|Du| = 1$. Moreover, u is neither a linear nor a cone function, since $u(0, \ldots, 0, 1) = 0$ and $u(0, \ldots, 0, t) = t$ for $t \leq 0$. Using Theorem 1.1 and the fact that $u(x', x_n) = u(-x', x_n)$, it is not hard to see that the u constructed as above is not C^2 in $\mathbb{R}^n\backslash\{0\}$. See Corollary 3.2 for the proof. When $n = 2$, using some techniques developed by [9] we can show that any uniformly Lipschitz continuous infinity harmonic function in

$\mathbb{R}^2\backslash\{0\}$ must be bounded by a linear function and a cone. We conjecture that if u is C^2 in $\mathbb{R}^n\backslash\{0\}$, then it must be either a linear or a cone function. We say that u is an *entire infinity harmonic function* if it is a viscosity solution of equation (1.1) in $\mathbb{R}^n$. Here, we want to mention that Aronsson [1] has proved that a C^2 entire infinity harmonic function must be linear when $n = 2$. Estimates derived by Evans [5] imply that this conclusion is true for a C^4 entire infinity harmonic function in all dimensions. It remains an interesting question whether the C^4 assumption in [5] can be relaxed to C^2.

This paper is organized as follows. In Section 2, we will review some preliminary facts of infinity harmonic functions. In Section 3, we will prove our theorems from the introduction. In Appendix A, we will prove a simple lemma of isolated singularities of viscosity solutions of fully nonlinear elliptic equations. Similar arguments can also be found in [7]. In Appendix B, we will present the tightness argument.

2 Preliminary

For $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$, we denote

$$B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}.$$

and

$$\partial B_1 = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

Also, we set

$$S^+_{u,r}(x_0) = \frac{\max_{x\in\partial B_1(x_0)} u(x) - u(x_0)}{r}$$

and

$$S^-_{u,r}(x_0) = \frac{u(x_0) - \min_{x\in\partial B_1(x_0)} u(x)}{r}.$$

It is obvious that

$$\max\left\{S^+_{u,r}(x_0), S^-_{u,r}(x_0)\right\} \leq \operatorname{esssup}_{B_r(x_0)}|Du|. \tag{2.1}$$

The following theorem is due to Crandall–Evans–Gariepy [4].

Theorem 2.1. ([4]). If $u \in C(\Omega)$ is a viscosity subsolution of equation (1.1), $S^+_{u,r}(x_0)$ is monotonically increasing with respect to r. We denote

$$S^+_u(x_0) = \lim_{r\downarrow 0^+} S^+_{u,r}(x_0).$$

For $x_r \in \partial B_r(x_0)$ such that $u(x_r) = \max_{\partial B_r(x_0)} u$, the following endpoint estimate holds:

$$S^+_u(x_r) \geq S^+_{u,r}(x_0) \geq S^+_u(x_0). \tag{2.2}$$

If $u \in C(\Omega)$ is a viscosity supersolution of equation (1.1), then $S^-_{u,r}(x_0)$ is monotonically increasing with respect to r. We denote

$$S^-_u(x_0) = \lim_{r\downarrow 0^+} S^-_{u,r}(x_0).$$

For $x_r \in \partial B_r(x_0)$ such that $u(x_r) = \min_{\partial B_r(x_0)} u$, the following endpoint estimate holds:

$$S^-_u(x_r) \geq S^-_{u,r}(x_0) \geq S^-_u(x_0). \tag{2.3}$$

If $u \in C(\Omega)$ is a viscosity solution of equation (1.1), then $S^+_u(x_0) = S^-_u(x_0)$. We denote

$$S_u(x_0) = S^+_u(x_0) = S^-_u(x_0).$$

If u is differentiable at x_0, then

$$|Du(x_0)| = S^+_u(x_0) = S^-_u(x_0). \tag{2.4}$$

□

By the above theorem, if u is a viscosity subsolution, then $S^+_u(x)$ is upper-semicontinuous. Combining equations (2.1) and (2.4), we have that

$$S^+_u(x) = \lim_{r\downarrow 0^+} \operatorname{esssup}_{B_r(x)}|Du|. \tag{2.5}$$

Similar conclusion holds for $S_u^-(x)$ when u is a viscosity supersolution. Moreover, the following theorem is well known.

Theorem 2.2. Suppose that $u \in W^{1,\infty}(B_1(0))$ is a viscosity subsolution of equation (1.1) in $B_1(0)$. Assume that

$$S_u^+(0) = \text{esssup}_{B_1(0)}|Du|.$$

Then there exists an $e \in \partial B_1$ such that

$$u(te) = u(0) + tS_u^+(0) \ \text{ for all } t \in [0, 1].$$

Also $Du(te)$ exists and $Du(te) = eS_u^+(0)$ for $t \in (0, 1)$. □

When $n = 2$, Savin has proved in [9] that any infinity harmonic function is C^1. More recently, Evans–Savin [6] have shown $C^{1,\alpha}$-regularity for infinity harmonic functions in $\mathbb{R}^2$. Moreover, the following uniform estimate holds.

Theorem 2.3. ([9]). For $n = 2$ if u is an infinity harmonic function in $B_1(0)$ and for some $e \in B_1(0)$,

$$\max_{B_1(0)} |u - e \cdot x| \leq \epsilon.$$

Then for any $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that if $\epsilon < \epsilon(\delta)$, then

$$|Du(0) - e| \leq \delta.$$

□

From Theorem 2.1, it is easy to see that if $u \in C(B_1(0))$ is a viscosity subsolution or supersolution of equation (1.1) in $B_1(0)$, then, for $0 < r < 1$,

$$\text{esssup}_{B_r(0)}|Du| \leq \frac{2}{1-r} \sup_{B_1(0)} |u|.$$

Applying Theorem 2.3, a simple compactness argument implies the following result which will be needed in the proof of Corollary 1.2.

Theorem 2.4. For $n = 2$, suppose that u and v are two infinity harmonic functions in $B_1(0)$ satisfying $|u|, |v| \le 1$ and

$$\max_{B_1(0)} |u - v| \le \epsilon.$$

Then for any $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that if $\epsilon < \epsilon(\delta)$, then

$$|Du(0) - Dv(0)| \le \delta.$$

□

3 Proofs of Theorems

The following lemma is the crucial step in the proofs of our theorems.

Lemma 3.1. Assume that $u : \mathbb{R}^n \to \mathbb{R}$ satisfies the following:

(i) $\operatorname{esssup}_{\mathbb{R}^n} |Du| \le 1$.
(ii) $u(0) = 0$ and for some $\epsilon > 0$, $u(x) \le (1 - \epsilon)|x|$ for all $x \in \mathbb{R}^n$.
(iii) u is a viscosity subsolution of equation (1.1) in $\mathbb{R}^n \backslash \{0\}$.
(iv) There exists $e \in S_1$ such that

$$u(-te) = -t \quad \text{for all } t \ge 0.$$

Then

$$u(x) = -|x|.$$

□

Proof. Note that (i) and (iv) imply that $\operatorname{esssup}_{\mathbb{R}^n} |Du| = 1$. Without loss of generality, we assume that $e = (0, \dots, 0, 1)$. For a fixed $\epsilon > 0$, let

$$\mathcal{S} = \{u : \mathbb{R}^n \to \mathbb{R} \mid u \text{ satisfies (i)–(iv)}\}.$$

Let us denote

$$w = \sup_{v \in \mathcal{S}} v.$$

It is clear that $w \in \mathcal{S}$. For any $\lambda > 0$, we have $w_\lambda = \frac{w(\lambda x)}{\lambda} \in \mathcal{S}$. Hence for all $\lambda > 0$,

$$w \geq w_\lambda.$$

This implies that $w = w_\lambda$ for all $\lambda > 0$, i.e. w is homogeneous of degree 1. By (i), (iv), and Lemma B.1 in Appendix B, we have that for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$w(x', x_n) \leq x_n. \tag{3.1}$$

If there exists a point $(x_1, 0)$ with $x_1 \in \mathbb{R}^{n-1}$ and $|x_1| = 1$ such that $w(x_1, 0) = 0$, then (i) and equation (3.1) imply that for all $t \geq 0$,

$$w(x_1, -t) = -t.$$

We claim that

$$w(x_1, t) = t \text{ for all } t \in \mathbb{R}.$$

In fact, denote $T = \sup\{t \mid w(x_1, s) = s \quad \text{for all } s \leq t\}$. It is clear that $T \geq 0$. If $T < +\infty$, then (i) and equation (2.5) imply that

$$S_w^+\left(x_1, T - \frac{1}{2}\right) = 1.$$

Hence by (i), Theorem 2.2, and the triangle inequality

$$w(x_1, t) = t \quad \text{for } T - \frac{1}{2} \leq t \leq T + \frac{1}{2}.$$

This contradicts the definition of T. Thus, $T = +\infty$. Therefore, applying Lemma B.1 again, (i) implies that

$$w(x', x_n) = x_n \text{ for all } (x', x_n) \in \mathbb{R}^n.$$

This contradicts (ii). Hence,

$$\max_{\{(x',0)|\ |x'|=1\}} w < 0.$$

This, combined with the homogeneity of w, implies that there exists $\delta > 0$ such that

$$w \le 0 \quad \text{on } \Gamma_\delta, \tag{3.2}$$

where

$$\Gamma_\delta = \left\{ \left(x', x_n\right) \mid x_n \le 0 \text{ or } x_n^2 \le \delta |x'|^2 \right\}.$$

Let us denote

$$C_\delta = \left\{ (x', x_n) \mid x_n \le 0 \text{ and } x_n^2 > \frac{1}{\delta} |x'|^2 \right\}.$$

Geometrically, it is clear that for all $x \in C_\delta$, $B_{|x|}(x) \subset \Gamma_\delta$. Therefore, for $x \in C_\delta$, equation (3.2) and Theorem 2.1 imply that

$$S_w^+(x) \le -\frac{w(x)}{|x|}. \tag{3.3}$$

Suppose that u is differentiable at $x \in C_\delta$. By the homogeneity of w,

$$Dw(x) \cdot x = w(x).$$

Hence, by equations (2.4) and (3.3)

$$Dw(x) = \frac{w(x)}{|x|^2} x.$$

Since w is Lipschitz continuous, this easily implies that $w \in C^\infty(C_\delta)$ and

$$Dw(x) = \frac{w(x)}{|x|^2} x. \tag{3.4}$$

Hence,

$$|Dw(x)| = -\frac{w(x)}{|x|} \quad \text{in } C_\delta. \tag{3.5}$$

Therefore, by equation (3.4), we have

$$D(|Dw(x)|) = \frac{1}{|x|}\left(-Dw(x) + \frac{w}{|x|^2}x\right) = 0 \quad \text{in } C_{\delta.}$$

By (i) and (iv), this yields that $|Dw(x)| \equiv 1$ in $C_{\delta,}$ and hence by (3.5),

$$w(x) = -|x| \quad \text{in } C_{\delta.}$$

Since

$$-|x| \leq u(x) \leq w(x),$$

we have that

$$u(x) = -|x| \quad \text{in } C_\delta.$$

Now we denote

$$\mathcal{A} = \{v \in \partial B_1 \mid u(-tv) = -t \quad \text{for all } t \geq 0\}.$$

It is obvious that $\mathcal{A}$ is closed and nonempty. Moreover, the above proof implies that $\mathcal{A}$ is also an open set of ∂B_1. Since ∂B_1 is connected for $n \geq 2$, we conclude that $\mathcal{A} = \partial B_1$. Thus, we obtain $u(x) = -|x|$ for all $x \in \mathbb{R}^n$. ■

Proof of Theorem 1.1. It follows from [2] that $\lim_{x\to x_0} u(x)$ exists. Hence, by defining $u(x_0) = \lim_{x\to x_0} u(x)$, $u \in C(B_1)$. Suppose that x_0 is a nonremovable singularity. Then by Lemma A.2 in Appendix A, we may assume that u is viscosity supersolution of equation (1.1) in $B_1(x_0)$. Hence, $u \in W^{1,\infty}_{\text{loc}}(B_1(0))$. Without loss of generality, we assume that $x_0 = 0$ and $u(0) = 0$. Since u is not a subsolution of equation (1.1) at x_0, (A.4) and Remark A.3 in the Appendix A imply that there exist $0 \neq p \in \mathbb{R}^n$ and $\epsilon > 0$, such that in a neighborhood of 0,

$$u(x) \leq p \cdot x - \epsilon |x|.$$

Hence, there exist $\delta > 0$ and another smaller neighborhood $V \subset \overline{V} \subset B_1(0)$ of 0, such that

$$u(x) \geq p \cdot x - \delta \quad \text{in } V$$

and

$$u(x) = p \cdot x - \delta \quad \text{on } \partial V.$$

Let us denote

$$\bar{t} = \sup \{ t \geq 0 | \, [0, -tp] \subset V \},$$

where $[0, -tp]$ denote the line segment between 0 and $-tp$. Hence,

$$c = \text{esssup}_V |Du| \geq \frac{u(0) - u(-\bar{t}p)}{|\bar{t}p|} = \frac{\delta}{|\bar{t}p|} + |p| > |p|.$$

Let us denote

$$K = \sup_{x \neq y \in \partial(V \setminus \{0\})} \frac{|u(x) - u(y)|}{|x - y|}.$$

Since u is an absolutely minimal Lipschitz extension in $B_1(0) \setminus \{0\}$, then, by equation (1.2), we have that

$$K \geq \text{esssup}_V |Du| = c > |p|. \tag{3.6}$$

Also

$$u(x) \leq p \cdot x - \epsilon |x| \leq |p||x| \quad \text{in V}. \tag{3.7}$$

Combining equations (3.6) and (3.7) gives

$$K = \max_{x \in \partial V} \frac{u(0) - u(x)}{|x|}.$$

Choose $\bar{x} \in \partial V$, such that

$$-u(\bar{x}) = u(0) - u(\bar{x}) = K|\bar{x}|.$$

Since $K > |p|$, we have by the triangle inequality and the definition of K,

$$\{t\bar{x}|\ 0 \le t < 1\} \subset V.$$

This implies that $K \le c$. Therefore, $K = c$ and

$$u(t\bar{x}) = tu(\bar{x}) = -tc|\bar{x}| \quad \text{for } 0 \le t \le 1. \tag{3.8}$$

We now choose $\lambda_m \to 0^+$ as $m \uparrow +\infty$. Then we may assume that

$$\lim_{m \to +\infty} \frac{u(\lambda_m x)}{\lambda_m} = w(x) \text{ uniformly in } C^0_{\mathrm{loc}}(\mathbb{R}^n).$$

It follows from equations (3.7) and (3.8) that $\frac{w(x)}{c}$ satisfies all the assumptions of Lemma 3.1 with $e = -\frac{\bar{x}}{|\bar{x}|}$. Hence, Lemma 3.1 implies that

$$w(x) = -c|x| \quad \text{for all } x \in \mathbb{R}^n.$$

Since this is true for any sequence $\{\lambda_m\}$, we have that

$$\lim_{\lambda \to 0^+} \frac{u(\lambda x)}{\lambda} = -c|x|.$$

Hence, Theorem 1.1 holds. ■

Proof of Theorem 1.3. It is clear that when R is sufficiently large, by (ii) we have that

$$u(x) \le M + (1-\epsilon)R \le u(0) + \left(1 - \frac{\epsilon}{2}\right) R \quad \text{for all } x \in \partial B_R.$$

Hence, by the comparison principle with cones (cf. [4]), we have

$$u(x) \le u(0) + \left(1 - \frac{\epsilon}{2}\right)|x| \quad \text{on } B_R(0).$$

Sending $R \to +\infty$ implies that

$$u(x) \le u(0) + \left(1 - \frac{\epsilon}{2}\right)|x| \quad \text{for all } x \in \mathbb{R}^n.$$

Without loss of generality, we may assume that $u(0) = 0$. Therefore, by Lemma 3.1, it suffices to show that there exists $e \in \partial B_{1,}$ such that

$$u(-te) = -t \ \text{ for all } t \ge 0, \tag{3.9}$$

We first claim that

$$\lim_{r\to 0} \operatorname{esssup}_{B_r(0)}|Du| = 1. \tag{3.10}$$

If the claim were false, then there exist $r > 0$ and $\delta \in (0, \epsilon)$ such that

$$\operatorname{esssup}_{B_r(0)}|Du| \le 1 - \delta. \tag{3.11}$$

Choose $x_0 \in \mathbb{R}^n$ such that

$$S_u^+(x_0) \ge 1 - \frac{\delta}{2}.$$

By equations (2.5) and (3.11), we have that $x_0 \notin \overline{B_{\frac{r}{2}}(0)}$. Hence, by the endpoint estimate (2.2), there exists a sequence $\{x_m\}_{m\ge 0}$ such that

$$\begin{aligned} |x_m - x_{m-1}| &= \frac{r}{2}, \\ u(x_m) - u(x_{m-1}) &\ge S_u(x_0)\frac{r}{2}, \end{aligned}$$

and

$$S_u(x_m) \ge 1 - \frac{\delta}{2}.$$

Therefore, it is easy to show that

$$\lim_{m\to+\infty} |x_m| = +\infty,$$

and

$$u(x_m) \geq u(x_0) + \left(1 - \frac{\delta}{2}\right)|x_m - x_0| \geq u(x_0) + \left(1 - \frac{\epsilon}{2}\right)|x_m - x_0|.$$

This contradicts (ii) when m is sufficiently large. Hence equation (3.10) holds.

It follows from Lemma A.2 in Appendix A that u is either a viscosity supersolution or viscosity subsolution of equation (1.1) in $\mathbb{R}^n$. If u is a viscosity subsolution of equation (1.1) in $\mathbb{R}^n$, then

$$S_u^+(0) = 1.$$

Hence by Theorem 2.2, there exists $e \in \partial B_1$ such that

$$u(te) = t \quad \text{for all } t \geq 0.$$

This contradicts (ii) when t is sufficiently large. Thus, u must be a viscosity supersolution of equation (1.1) in $\mathbb{R}^n$. Then

$$S_u^-(0) = 1.$$

Hence, by considering $-u$ and applying Theorem 2.2, there exists $e \in \partial B_1$ such that $u(-te) = -t$ for all $t \geq 0$. Now the conclusion follows from Lemma 3.1. ∎

Proof of Corollary 1.2. Without loss of generality, we assume that $x_0 = 0$. Choose $\bar{x} \in \partial\Omega$, such that $|\bar{x}| = d(0, \partial\Omega) = r$. Then by equation (1.2), it is not hard to see that

$$\text{esssup}_{\Omega}|Du| = \frac{1}{r}$$

and for $0 \leq t \leq 1$,

$$u(t\bar{x}) = 1 - t.$$

Hence by Theorem 1.1, we conclude that

$$\lim_{\lambda \to 0^+} \frac{u(\lambda x) - u(0)}{\lambda} = -\frac{|x|}{r}.$$

This and Theorem 2.4 then imply that

$$\lim_{x \to 0} \left| Du(x) - \frac{x}{r|x|} \right| = 0.$$

In particular, we have

$$\lim_{x \to 0} |Du(x)| = \frac{1}{r}. \tag{3.12}$$

Let us choose $y_0 \in \Omega \backslash \{0\}$ such that $Du(y_0) \neq 0$. Then there exist $\delta > 0$ and $\xi : [0, \delta] \to \Omega$ such that

$$\dot{\xi}(t) = Du(\xi(t)) \quad \text{and} \quad \xi(0) = y_0.$$

Then if $\delta > 0$ is the maximal time interval, then $\xi(\delta) \in \partial\Omega \cup \{0\}$. On the other hand, it is easy to see that

$$u(\xi(\delta)) = u(y_0) + |Du(y_0)|\delta > 0,$$

so that

$$\xi(\delta) = 0.$$

Since $|Du(\xi(t))| \equiv |Du(y_0)|$, equation (3.12) implies that

$$|Du(y_0)| = \frac{1}{r}.$$

Hence, we have

$$|Du(x)| \equiv \frac{1}{r} \quad \text{for all } x \in \Omega \backslash \{0\},$$

so that $u(x) = 1 - \frac{|x|}{r}$. Since $u|_{\partial\Omega} = 0$, this forces $\Omega = B_r(0)$. ■

Corollary 3.2. The uniformly Lipschitz continuous function constructed in the introduction is not $C^2(\mathbb{R}^n\backslash\{0\})$. □

Proof. By the construction and the comparison principle with cones, u

$$\operatorname{esssup}_{\mathbb{R}^n}|Du| = 1, \tag{3.13}$$

$$u(0,-t) = -t \ \forall t \geq 0, \tag{3.14}$$

and

$$u(x', x_n) = u(-x', x_n) \ \forall (x', x_n) \in \mathbb{R}^n. \tag{3.15}$$

It follows from equations (3.13), (3.14), and Theorem 1.1 that

$$u(x) = -|x| + o(|x|), \ \ x \text{ near } 0. \tag{3.16}$$

Assume that $u \in C^2(\mathbb{R}^n\backslash\{0\})$. Then $|Du|$ is constant along its gradient follows, u is linear on the half line $\{(0,\ldots,0,t)|\ t \geq 0\}$. Since $u(0,\ldots,0,1) = 0$, we have that for $t > 0$,

$$u(0,\ldots,0,t) \equiv 0.$$

This contradicts equation (3.16). ■

Definition 3.3. Let F be a closed set and g a uniformly Lipschitz continuous function on F, we say that $u : \mathbb{R}^n \to \mathbb{R}$ is an absolutely minimal Lipschitz extension of (F, g), if $u|_F = g$ and for any open subset $U \subset \mathbb{R}^n\backslash F$,

$$\sup_{x,y\in\overline{U},x\neq y} \frac{u(x) - u(y)}{|x-y|} = \sup_{x,y\in\partial U,x\neq y} \frac{u(x) - u(y)}{|x-y|}. \tag{3.17}$$

□

In general, the uniqueness of absolutely minimal Lipschitz extensions is an open problem. In the following, as an application of Lemma 3.1, we will prove the uniqueness of absolutely minimal Lipschitz extensions for a special pair of (F, g). Fix $e \in \partial B_1$, we choose $F = \{te||t \leq 0\}$ and $g(x) = e \cdot x$. When $n \geq 2$, we can see that $u(x) = -|x|$ is an absolutely minimal Lipschitz extension of (F, g). Moreover, Definition 3.3 implies that any absolutely minimal Lipschitz extension u of (F, g) satisfies

(i) $\text{esssup}_{\mathbb{R}^n}|Du| = 1$;

(ii) $u \leq 0$;

(iii) u is an infinity harmonic function in $\mathbb{R}^n \backslash \{0\}$;

(iv) $u(-te) = -t$ for $t \geq 0$.

Here (i) and (iv) are obvious and (ii) follows if we apply equation (3.17) to the open set $U_\epsilon = \{x \in \mathbb{R}^n |\ u(x) > \epsilon\}$ for any $\epsilon > 0$. Suppose that $U_\epsilon \neq \emptyset$. Then we would have that $u|_{U_\epsilon} \equiv \epsilon$, which is impossible. We want to say a little bit about (iii). It is clear that u is an infinity harmonic function in $\mathbb{R}^n \backslash F$. By Lemma B.2 in Appendix B, we have that, for $x \in F \backslash \{0\}$, $Du(x) = e$. Hence, by the definition of viscosity solutions and (iv), u is an infinity harmonic function on $F \backslash \{0\}$.

The following corollary is an immediate result of Lemma 3.1.

Corollary 3.4. $u(x) = -|x|$ is the unique absolutely minimal Lipschitz extension of (F, g). □

A Appendix A: Simple Lemma of Isolated Singularities of Fully Nonlinear Elliptic Equations

Let $S^{n \times n}$ denote the set of all $n \times n$ symmetric matrices. Suppose that $F \in C(S^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \Omega)$ and satisfies that

$$F(M_1, p, z, x) \geq F(M_2, p, z, x),$$

if all the eigenvalues of $M_1 - M_2$ are non-negative.

Definition A.1. We say that $u \in C(\Omega)$ is a *viscosity supersolution (subsolution)* of

$$F(D^2u, Du, u, x) = 0,$$

if for any $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$,

$$\phi(x) - u(x) \leq (\geq) \phi(x_0) - u(x_0) = 0$$

implies that

$$F(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0) \leq (\geq) 0.$$

u is *viscosity solution* if it is both a supersolution and a subsolution. □

The following is a simple lemma. Similar argument can be found in [7].

Lemma A.2. Suppose that $u \in C(B_1)$ and is a viscosity solution of the equation

$$F(D^2u, Du, u, x) = 0 \quad \text{in } B_1(0)\backslash\{0\}.$$

Then u is either a viscosity supersolution or a viscosity subsolution in the entire ball. In particular, if u is differentiable at 0, then u is a solution in the entire ball, i.e. 0 is a removable singularity. □

Proof. Without loss of generality, we assume that $u(0) = 0$. We claim that if u is not a viscosity supersolution, then there exist $\epsilon > 0$ and $p \in \mathbb{R}^n$ such that

$$u(x) \geq p \cdot x + \epsilon|x| \quad \text{in } \overline{B_\epsilon(0)}. \tag{A.1}$$

In fact, if u is not a viscosity supersolution in the entire ball, then there exists $\phi \in C^2(B_1(0))$ such that

$$\phi(x) - u(x) < \phi(0) - u(0) = 0 \quad \text{for } x \in B_1(0)\backslash\{0\} \tag{A.2}$$

and

$$F(D^2\phi(0), D\phi(0), u(0), 0) > 0.$$

Let us choose $p = D\phi(0)$. If equation (A.1) is not true, then for any $m \in \mathbb{N}$, there exists $x_m \in B_{\frac{1}{m}}(0)$ such that

$$u(x_m) < \phi(x_m) + \frac{1}{m}|x_m|. \tag{A.3}$$

It is clear that $x_m \neq 0$. Let usdenote

$$\phi_m(x) = \phi(x) + \frac{x_m}{m|x_m|} \cdot x.$$

Let us choose $y_m \in \overline{B_1(0)}$ such that

$$u(y_m) - \phi_m(y_m) = \min_{\overline{B_1(0)}}(u - \phi_m).$$

It follows from equations (A.2) and (A.3), that $y_m \neq 0$ and $\lim_{m\to+\infty} y_m = 0$. Hence,

$$F\left(D^2\phi(y_m), D\phi(y_m) + \frac{x_m}{m|x_m|}, u(y_m), y_m\right) \leq 0.$$

Sending $m \to +\infty$, we obtain

$$F(D^2\phi(0), D\phi(0), u(0), 0) \leq 0.$$

This is a contradiction. Hence, equation (A.1) holds. Similarly, we can show that if u is not a viscosity subsolution at 0, then there exist $\epsilon > 0$ and $p \in \mathbb{R}^n$ such that

$$u(x) \leq p \cdot x - \epsilon|x| \quad \text{in } \overline{B_\epsilon(0)}. \tag{A.4}$$

Note that equations (A.1) and (A.4) cannot happen simultaneously. In particular, if u is differentiable at 0 neither can happen. Hence, Lemma A.2 holds. ■

Remark A.4. If F is the infinity Laplace operator, i.e. $F(p, M) = p \cdot M \cdot p$, then the vector $p \in \mathbb{R}^n$ in equations (A.1) and (A.4) is not 0, since $p = D\phi(0)$. □

B Appendix B: Tightness Argument and Conclusions

The results in this section are well known. We present here for reader's convenience.

Lemma B.1. Suppose that $u \in W^{1,\infty}(\mathbb{R}^n)$ and satisfies that

(i) $\operatorname{esssup}_{\mathbb{R}^n}|Du| \leq 1$;

(ii) for $t \geq 0$,

$$u(0, \dots, 0, -t) = -t.$$

Then, for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$u(x) \le x_n. \tag{B.1}$$

In particular, if (ii) is true for all $t \in \mathbb{R}$, then $u = x_n$ for $x \in \mathbb{R}^n$. □

Proof. Note that (i) and (ii) imply that $\text{esssup}_{\mathbb{R}^n}|Du| = 1$. By (i) and (ii), we have that for $t > 0$,

$$|u(x) + t| = |u(x) - u\,(0, \ldots, 0, -t)| \le \sqrt{|x'|^2 + (x_n + t)^2}.$$

Hence

$$u(x)^2 + 2tu(x) \le |x'|^2 + x_n^2 + 2tx_n.$$

So

$$\frac{u(x)^2}{2t} + u(x) \le \frac{|x'|^2 + x_n^2}{2t} + x_n.$$

Sending $t \to +\infty$, we derive equation (B.1). ■

Lemma B.2. Suppose that $u \in W^{1,\infty}(B_1(0))$ and satisfies that

(i)

$$\text{esssup}_{B_1(0)}|Du| \le 1; \tag{B.2}$$

(ii) for some $e \in \partial B_1$,

$$u(e) - u(0) = 1. \tag{B.3}$$

Then for $0 < t < 1$,

$$u(te) = u(0) + t \tag{B.4}$$

and

$$Du(te) = e. \tag{B.5}$$

□

Proof. By equation (B.2), we have that, for any $x, y \in B_1(0)$,

$$|u(x) - u(y)| \leq |x - y|.$$

Hence, equation (B.4) follows from equation (B.3) and the triangle inequality. Choose $x_0 \in \{te|\ 0 < t < 1\}$. Suppose that $\lambda_m \to 0$ as $m \to +\infty$ and

$$\lim_{m\to+\infty} \frac{u(\lambda_m x + x_0) - u(x_0)}{\lambda_m} = w(x).$$

By equations (B.2) and (B.4), $w(x)$ satisfies that

$$\mathrm{esssup}_{\mathbb{R}^n} |Dw| \leq 1$$

and

$$w(te) = t \ \text{ for all } t \in \mathbb{R}.$$

Hence, by Lemma B.1, $w(x) = e \cdot x$. Since this is true for any sequence $\{\lambda_m\}$, we get that

$$\lim_{\lambda\to 0} \frac{u(\lambda x + x_0) - u(x_0)}{\lambda} = e \cdot x.$$

Therefore, equation (B.5) holds. ■

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