P1

Solution to Review problems

1. $f_1, \cdots f_n$ is linearly dependent

$\Leftrightarrow \exists k_1 \cdots k_n \in \mathbb{R}$ not all zero, s.t. $k_1 f_1 + \cdots + k_n f_n \equiv 0$.

$$W[e^t, e^{2t}, e^{3t}] = \begin{vmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{vmatrix}$$

$$= e^t \cdot e^{2t} \cdot e^{3t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6t} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix}$$

$= 2e^{6t} \neq 0 \Rightarrow \{e^t, e^{2t}, e^{3t}\}$ is linearly independent

2. $\lambda^4 + 1 = 0 \Rightarrow \lambda^4 = e^{i\pi}$

$\Rightarrow \lambda = e^{i\frac{\pi + 2k\pi}{4}} \quad k = 0, 1, 2, 3$

$\Rightarrow \lambda_{1,2} = \cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4} \qquad \lambda_{3,4} = \cos\frac{3\pi}{4} \pm i \sin\frac{3\pi}{4}$

$= \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i \qquad\qquad = -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$

$$\Rightarrow y_c = e^{\frac{\sqrt{2}}{2}t}\left(C_1 \cos\left(\frac{\sqrt{2}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{2}}{2}t\right)\right)$$
$$+ e^{-\frac{\sqrt{2}}{2}t}\left(C_3 \cos\left(\frac{\sqrt{2}}{2}t\right) + C_4 \sin\left(\frac{\sqrt{2}}{2}t\right)\right).$$

P2

3. $5\lambda^3+6\lambda^2+\lambda=0 \rightarrow \lambda(5\lambda+1)(\lambda+1)=0$

$\rightarrow \lambda_1=0, \lambda_2=-\frac{1}{5}, \lambda_2=-1$

$\rightarrow y(t)=c_1+c_2e^{-\frac{1}{5}t}+c_3e^{-t}$

$y'(t)=-\frac{1}{5}c_2e^{-\frac{1}{5}t}-c_3e^{-t}$

$y''(t)=\frac{1}{25}c_2e^{-\frac{1}{5}t}+c_3e^{-t}$

$$\begin{cases} c_1+c_2+c_3 = -1 \\ -\frac{1}{5}c_2-c_3 = 1 \\ \frac{1}{25}c_2+c_3 = 2 \end{cases} \Rightarrow \frac{4}{25}c_2=3$$

$\Rightarrow c_2=\frac{75}{4}$

$c_3=2-\frac{1}{25}\cdot\frac{75}{4}=2-\frac{3}{4}=\frac{5}{4}$

$c_1=-1-\frac{75}{4}-\frac{5}{4}=-1-20=-21$

$\Rightarrow y(t)=-21+\frac{75}{4}e^{-\frac{1}{5}t}+\frac{5}{4}e^{-t}$

$\square$

1^{st} method

4) $$(D^2-4)^2 y = e^{2t}$$

$$\Rightarrow \quad (D-2)(D-2)^2(D+2)^2 y = 0$$

$$\Rightarrow \quad (D-2)^3(D+2)^2 y = 0$$

$\Rightarrow$ y can be obtain as:

$$(\lambda-2)^3(\lambda+2)^2 = 0$$

$$\lambda = 2 \ (m=3)$$

$$\lambda = -2 \ (m=2)$$

$$\Rightarrow y = (C_1 + C_2 t + C_3 t^2)e^{2t} + (C_4 + C_5 t)e^{-2t} \quad \square$$

It turns out a particular solution is

$$Y_p = C_3 t^2 e^{2t}$$ for some coefficient C_3.

4. (2nd method) $y = At^2 e^{2t}$.

$y' = A(2t + 2t^2)e^{2t}$

$y'' = A(4t^2 + 8t + 2)e^{2t}$

$y^{(3)} = A(8t^2 + 24t + 12)e^{2t}$

$y^{(4)} = A(16t^2 + 64t + 48)e^{2t}$

$y^{(4)} - 8y^{(2)} + 16y$

$= A[16t^2 + 64t + 48 - 8(4t^2 + 8t + 2) + 16t^2]e^{2t}$

$= 32Ae^{2t} = e^{2t} \Rightarrow \underline{A = \frac{1}{32}}$

$\boxed{Y_p = \frac{1}{32}t^2 e^{2t}}$

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5. $\lambda^4+2\lambda^3+2\lambda^2=0 \Rightarrow \lambda=0 \quad \lambda=-1\pm i$

$y_c = c_1+c_2t+c_3e^{-t}\cos t+c_4e^{-t}\sin t$

$Y_p = Ae^{t}+(Bt^2+Ct+D)e^{-t}+\left[(\tilde{A}t+\tilde{B})e^{-t}\cos t + (\tilde{\tilde{A}}t+\tilde{\tilde{B}})e^{-t}\sin t\right]t$

6. $W[x,x^2,x^3] = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 12x^3+2x^3-6x^3-6x^3 = 2x^3$

$\Rightarrow \{x,x^2,x^3\}$ is L.I.

8. $y = u_1x+u_2x^2+u_3x^3$

$$\begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$\Rightarrow u_1' = \frac{W_1 g}{W}$ $\quad W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$

$u_2' = \frac{W_2 g}{W}$ $\quad W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$

$u_3' = \frac{W_3 g}{W}$ $\quad W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$

$$\Rightarrow \quad u_1' = \frac{x^4}{2x^3}\,g = \frac{x}{2}\,g$$

$$u_2' = \frac{-2x^3}{2x^3}\,g = -g$$

$$u_3' = \frac{x^2}{2x^3}\,g = \frac{1}{2x}\,g$$

$$\Rightarrow \quad Y(x) = x\left(\int \frac{x}{2}\,g(x)\,dx\right) \cancel{+}\ x^2\left(\int g(x)\,dx\right) + \left(\int \frac{1}{2x}\,g\right) x$$

7. $$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$x y'' + y' + x y = 0$$

$$\rightarrow (x-1+1)\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + (x-1+1)\sum_{n=0}^{\infty} a_n (x-1)^{n} =$$

$$\rightsquigarrow \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

$$+ \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\rightsquigarrow \sum_{n=1}^{\infty} (n+1)n a_{n+1}(x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n$$

$$+ \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$\Rightarrow$ For $n \geq 1$,

$$\sum_{n\geq 1}^{\infty} \left\{ (n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_{n-1} + a_n \right\}(x-1)^n$$

$$+ (2a_2 + a_1 + a_0) = 0$$

$$\Rightarrow \quad 2a_2 + a_1 + a_0 = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_{n-1} + a_n = 0 \quad \checkmark$$

8. $y = \sum a_n x^n$, $y' = \sum n a_n x^{n-1}$, $y'' = \sum n(n-1) a_n x^{n-2}$

$$\Rightarrow \left(2+x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$+ 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$a_0 = -1$

$\underline{a_1 = -3}$

$$\Rightarrow \sum_{n=0}^{\infty} \left\{ 2(n+2)(n+1) a_{n+2} + 4 a_n \right\} x^n$$

$$+ \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n = 0$$

$\Rightarrow$ $\underline{n \geq 2}$:

$$2(n+2)(n+1) a_{n+2} + \left(n^2 - n + 4\right) a_n - n a_n = 0$$

$$\Rightarrow 2(n+2)(n+1) a_{n+2} + \left(n^2 - 2n + 4\right) a_n = 0.$$

$\square$

9. $\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$

$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$

$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

$\mathcal{L}\{t^{\frac{1}{2}}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} = \frac{\Gamma(3/2)}{s^{3/2}}$

10. $\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$\Rightarrow A=3,\ B=5,\ C=-4$

$\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\} = 3\cdot 1 + \mathcal{L}^{-1}\left\{\frac{5s-4}{s^2+4}\right\}$

$= 3 + 5\cos 2t - 2\sin 2t$

$$\frac{2s-3}{s^2+2s+10} = \frac{2(s+1)-5}{(s+1)^2+3^2} = 2\frac{s+1}{(s+1)^2+3^2} - \frac{5}{3}\frac{3}{(s+1)^2+3^2}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+2s+10}\right\} = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t.$$

11. $s^2Y - s\cdot 1 + 1 + \omega^2 Y = \dfrac{s}{s^2+4}$

$$\Rightarrow \quad Y = \frac{s-1}{s^2+\omega^2} + \frac{s}{(s^2+4)(s^2+\omega^2)}$$

$$y = \cos\omega t - \frac{1}{\omega}\sin\omega t + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+\omega^2)}\right\}$$

$$\frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+\omega^2}$$

$$\Rightarrow (As+B)(s^2+\omega^2) + (Cs+D)(s^2+4) = s$$

$$\Rightarrow (A+C) = 0, \quad B+D=0, \quad B\omega^2 + 4D = 0$$

$$A\omega^2 + 4C = 1 \qquad B = D = 0$$

$$A = \frac{1}{\omega^2-4} \quad C = -\frac{1}{\omega^2-4}.$$

$$\Rightarrow y(t) = \cos\omega t - \frac{1}{\omega}\sin\omega t + \frac{1}{\omega^2-4}\cos 2t$$

$$- \frac{1}{\omega^2-4}\sin\omega t$$

12. $s^4 Y - s^3\cdot 0 - s^2\cdot 1 - s\cdot 0 - 1$

$$-4(s^3 Y - s^2\cdot 0 - s\cdot 1 - 0) + 6(s^2 Y - s\cdot 0 - 1)$$

$$-4(sY - 0) + Y = 0$$

$$\Rightarrow Y = \frac{s^2+1-4s+6}{s^4-4s^3+6s^2-4s+1}$$

$$= \frac{s^2-4s+7}{s^4-4s^3+6s^2-4s+1}$$

$$= \frac{s^2-4s+7}{(s-1)^4}$$

$$= \frac{1}{(s-1)^2} - 2\frac{1}{(s-1)^3} + \frac{4}{(s-1)^4}$$

$$y(t) = te^t - t^2 e^t - 4\frac{t^3}{3!}e^t$$