Solutions for problems of Review I, Fall 2013

1. $||\mathbf{AB}|| = \sqrt{(2-1)^2 + (-1+3)^2 + (7-5)^2} = \sqrt{1+4+4} = 3.$

2.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2\cdot 1 + (-3)\cdot 6 + 1\cdot 2}{\sqrt{2^2+(-3)^2+1^2}\sqrt{1^2+6^2+(-2)^2}} = \frac{-14}{\sqrt{14}\sqrt{41}} = -\sqrt{\frac{14}{41}}.$$

3.

$$||\mathbf{AB}\times\mathbf{AC}|| = ||(1,4,-9)\times(-2,-3,-1)|| = ||(-31,19,5)|| = \sqrt{(31)^2+(19)^2+5^2}.$$

4.

$$||(\mathbf{PQ}\times\mathbf{PR})\cdot\mathbf{PS}|| = ||(2,1,1)\times(1,-1,2)\cdot(0,-2,3)|| = 3.$$

5. Recall that the distance from $P(x_0, y_0, z_0)$ to a plane $ax+by+cz+d=0$ is

$$\text{dist} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

Hence the answer is

$$\frac{|4\cdot 1+5\cdot 3+6\cdot 2+7|}{\sqrt{4^2+5^2+6^2}} = \frac{38}{\sqrt{77}}.$$

6.

$$\mathbf{r}(t) = \mathbf{P} + t(\mathbf{Q}-\mathbf{P}) = (1,2,3) + t(-3,-3,-6) = (1-3t, 2-3t, 3-6t).$$

7. The problem is not correctly formula. An alternative question is to determine the normal vector $\mathbf{n}$ that is perpendicular to both lines L1 and L2. Hence

$$\mathbf{n} = (1,2,3)\times(-2,3,-1) = (-11,-5,7).$$

8.

a) $r = 2, \theta = \frac{\pi}{3}, z = 2\sqrt{3}$; $\rho = 4, \theta = \frac{\pi}{3}, \phi = \frac{\pi}{6}$.

b) $(x,y,z) = (\sqrt{3}, \sqrt{3}, \sqrt{2})$ and $(r,\theta,z) = (\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$.

9.

a) $\mathbf{v}(t) = (1, 2t.3t^2)$ and $\mathbf{a}(t) = (0,2,6t)$.

b) $\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{a}(s)\,ds = \langle -2-t^2, \frac{3}{4}t^4, 3+e^t\rangle$;

$\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(s)\,ds = \langle 1,-1,2\rangle + \langle -2t-\frac{1}{3}t^3, \frac{3}{20}t^5, 3t+e^t-1\rangle = \langle 1-2t-\frac{1}{3}t^3, -1+\frac{3}{20}t^5, 1+3t+e^t\rangle.$

10.

$$L = \int_0^{2\pi} \sqrt{e^{2t} + e^{2t}(\sin t+\cos t)^2 + e^{2t}(\cos t - \sin t)^2}\,dt = \sqrt{3}\int_0^{2\pi} e^t\,dt = \sqrt{3}(e^{2\pi}-1).$$

11. The formula for curvature is

$$k(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}.$$

While $\mathbf{r}'(t) = (-e^{-t}, 2, \frac{2t}{1+t^2})$ and $\mathbf{r}''(t) = (e^{-t}, 0, \frac{2(1-t^2)}{(1+t^2)^2})$. Now we can follow the forumla to figure out the answer.

12.

$$\mathbf{T}(t) = \langle \sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \cos t \rangle,$$

and

$$\mathbf{N}(t) = \langle \cos t, -\frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \sin t \rangle.$$

13. Since $|f(x, y)| \leq |x| + |y|$, it is easy to see that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

14.

$$f_x = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}};\ f_y = -\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}}.$$

15. Since

$$f_x(x, y) = -\frac{4x}{\sqrt{9 - 4(x^2 + y^2)}},\ f_y(x, y) = -\frac{4y}{\sqrt{9 - 4(x^2 + y^2)}},$$

we have

$$f_x(1, 1) = f_y(1, 1) = -4$$

and hence

$$f(1.01, 0.99) \approx f(1, 1) + f_x(1, 1)(0.01) + f_y(1, 1)(-0.01) = 1 + (-4)(0.01) + (-4)(-0.01) = 1.$$

16.

$$\begin{aligned} f(1.02, 0.01, -0.03) &\approx f(1, 0, 0) + f_x(1, 0, 0)(0.02) + f_y(1, 0, 0)(0.01) + f_z(1, 0, 0)(-0.03) \\ &= -3 - 0.04 + 0.04 - 0.06 = -3.06. \end{aligned}$$