P0

Solution to Review problems #2

1) $\lambda^2-2\lambda-2=0 \Rightarrow \lambda = 1 \pm \sqrt{3} \Rightarrow y_c = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}$.

2) $\lambda^2+\frac{1}{4}\lambda+\frac{5}{32}=0 \Rightarrow \left(\lambda+\frac{1}{8}\right)^2+\left(\frac{3}{8}\right)^2=0$

$\Rightarrow \lambda = -\frac{1}{8} \pm \frac{3}{8} i \Rightarrow y_c(t) = e^{-\frac{1}{8}t}\left(c_1 \cos\left(\frac{3}{8}t\right)+c_2 \sin\left(\frac{3}{8}t\right)\right)$

3) $9\lambda^2-30\lambda+25=0 \Rightarrow (3\lambda-5)^2=0 \Rightarrow \lambda = 5/3$

$\Rightarrow y_c(t) = (c_1+c_2 t)\, e^{\frac{5}{3}t}$

$\Rightarrow y_c'(t) = c_2 e^{\frac{5}{3}t} + (c_1+c_2 t)\frac{5}{3}e^{\frac{5}{3}t}$

$$\left.\begin{aligned} \Rightarrow\ 1 &= y_c(0) = c_1 \\ -1 &= y_c'(0) = c_2 + \tfrac{5}{3}c_1 \end{aligned}\right\} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -8/3 \end{cases}.$$

$\Rightarrow \boxed{y_c(t) = \left(1-\frac{8}{3}t\right)e^{\frac{5}{3}t}}$

4). $W[e^{5t}\cos 6t,\, e^{5t}\sin 6t] = 6e^{10t}$

5) $t\,g'(t) - 1\,g(t) = t^2 e^t$ $\mu(t) = e^{\int -\frac{1}{t}dt} = \frac{1}{t}$

$\Rightarrow g' - \frac{1}{t}g = t e^t$ / $\left(\frac{1}{t}g\right)' = e^t \Rightarrow \boxed{g(t) = t e^t}$.

#

P1

6). $y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2} = 0$

$$\frac{dW}{dt} - \frac{2x}{1-x^2}W = 0 \Rightarrow W = ce^{+\int \frac{2x}{1-x^2}dx}$$

$$= ce^{-\ln(1-x^2)} = \frac{c}{1-x^2}$$ □

7) $\cancel{\text{Try } y_2(t) = \frac{u}{t} \text{ Then}}$

$\cancel{y_2' = \frac{u't-u}{t^2}}$

Try $y_2(t) = u\left(\frac{1}{t}\right) \Rightarrow y_2' = u'\left(\frac{1}{t}\right) + u\left(\frac{1}{t}\right)'$

$y_2'' = u''\left(\frac{1}{t}\right) + 2u'\left(\frac{1}{t}\right) + u\left(\frac{1}{t}\right)''$

$$\Rightarrow 2t^2\left[u''\left(\frac{1}{t}\right) + 2u'\left(\frac{1}{t}\right) + \cancel{u\left(\frac{1}{t}\right)''}\right]$$

$$+3t\left[u'\left(\frac{1}{t}\right) + \cancel{u\left(\frac{1}{t}\right)'}\right] - \cancel{u\frac{1}{t}} = 0$$

$$\cancel{\Rightarrow 2tu'' + 4tu' + 3u' = 0}$$

$$\cancel{\Rightarrow u'' + \frac{3+4t}{2t}u' = 0 \Rightarrow u' = e^{-\int\frac{3+4t}{2t}dt}}$$

$$\cancel{= e^{-(2t+\frac{3}{2}\ln t)} = e^{-2t}t^{-3/2} \Rightarrow u = \int t^{-\frac{3}{2}}}$$

P2

$\Rightarrow 2t\,u'' + 3u' - 4u' = 0$

$\Rightarrow u'' = \frac{1}{2t} u' \Rightarrow u' = e^{\int \frac{1}{2t} dt} = t^{\frac{1}{2}}$

$\Rightarrow u = \frac{1}{3/2} t^{3/2} = \frac{2}{3} t^{3/2}$ $\square$

8) • $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$

• $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$

• $Y_p(t) = A\cos t + B\sin t + Cte^{3t} = Y_p^1(t) + Y_p^2(t)$

$(A\cos t + B\sin t)'' - 5(A\cos t + B\sin t)' + 6(A\cos t + B\sin t) = \cos t$

$\begin{cases} -A - 5B + 6A = 1 \\ -B + 5A + 6B = 0 \end{cases} \Rightarrow \begin{cases} 5(A-B) = 1 \\ 5(A+B) = 0 \end{cases}$

$\Rightarrow A = \frac{1}{10}, \quad B = -\frac{1}{10},$

$(cte^{3t})'' - 5(cte^{3t})' + 6(cte^{3t}) = e^{3t}$

$(9ct + 6c)e^{3t} - 5(3ct + c)e^{3t} + 6cte^{3t} = e^{3t}$

$ce^{3t} = e^{3t} \Rightarrow c = 1$

P3

$$Y_p(t) = te^{3t} + \frac{1}{10}\cos t + \frac{1}{10}\sin t \qquad \square$$

9). $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow y_c(t) = c_1\cos 2t + c_2 \sin 2t$.

$$\cancel{Y_p(t) = t\left[\cos 2t\,(At^2+Bt+C) + \sin 2t\,(Dt^2+Et+F)\right]}$$

$$+\,t\left[\cos 2t\,(A_1 t + B_1) + \sin 2t\,(A_2 t + B_2)\right]$$

10). $y'' - \frac{1+t}{t} y' + \frac{1}{t} y = te^{2t}$.

$$y_p = u_1 y_1 + u_2 y_2$$

$$\begin{cases} (1+t)\, u_1' + e^t u_2' = 0 \\ 1\ u_1' + e^t u_2^2 = te^{2t} \end{cases}$$

$$\Rightarrow \begin{cases} u_1' = -\dfrac{e^t\ te^{2t}}{\begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix}} = -e^{2t} \\[2ex] u_2' = \dfrac{(1+t)te^{2t}}{te^t} = (1+t)e^t \end{cases}$$

$$\Rightarrow u_1 = -\frac{1}{2}e^{2t}, \quad u_2 = \int e^t(1+t) = (1+t)e^t - e^t = te^t$$

P4

$$Y_p(t) = -\frac{1}{2}e^{2t}(1+t) + te^{t}e^{t}$$

$$= -\frac{1}{2}e^{2t} + \frac{1}{2}te^{2t} = \underline{\frac{1}{2}(t-1)e^{2t}} \qquad \square$$

ii) $m = \frac{16}{32} = \frac{1}{2}$

$k = \frac{16}{\frac{1}{4}} = 64$

$\gamma = 2$

$u(0) = 0$

$u'(0) = \frac{1}{4}$

$$\Rightarrow \left.\begin{aligned} \frac{1}{2}u'' + 2u' + 64u &= 0 \\ u(0) &= 0 \\ u'(0) &= \frac{1}{4} \end{aligned}\right\}$$

The rest leaves to my proud students!